AN ELECTROSTATIC POTENTIAL PROBLEM INVOLVING TWO COAXIAL CIRCULAR DISKS IN A FREE SPACE

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We present here the solution of an electrostatic problem involving two coaxial circular disks charged to any prescribed potentials in a free space. An integral equation technique is used to reduce the solution of this problem to that of four Volterra integral equations of the first kind and two simultaneous Fredholm integral equations of the second kind. We have solved the resulting integral equations when the two disks are charged to two different constant potentials and are lying far apart. From these solutions approximate expressions for various physical quantities of interest are readily derived.

INTRODUCTION

Nicholson (1924), Love (1949), Cooke (1956), Noble (1958) and others discussed the boundary value problems of electrostatic potential involving two coaxial circular disks charged to like or unlike constant potentials in a free space. Some boundary value problems in electrostatics and perfect fluid theory for two spherical caps or two coaxial circular disks in a free space were studied by Collins (1960). The reason for the failure of the above authors to solve these interesting boundary value problems completely is that they used very complicated formulations as well as techniques to solve these problems. Vaid and Jain (1974) have recently developed an integral equation technique by modifying an integral equation method presented by Jain and Kanwal (1971) and this technique has been effectively used by the authors in solving various boundary value problems involving two coaxial circular disks or two concentric coaxial spherical caps in electrostatic potential, generalized axially symmetric potential theory, hydrodynamics and diffraction theory.

We present here the solution of a boundary value problem of electrostatic potential involving two coaxial circular disks charged to any prescribed

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potentials in a free space. The starting point in such a boundary value problem is to formulate two simultaneous Fredholm integral equations of the first kind which embody the Laplace equation as well as boundary conditions of the problem. This is achieved by following the usual Green's function approach. It is rather difficult to solve these two governing Fredholm integral equations of the first kind simultaneously. By the help of an integral equation technique of Vaid and Jain (1974), the two governing simultaneous Fredholm integral equations of the first kind of this boundary value problem are easily converted to four Volterra integral equations of the first kind and two Fredholm integral equations of the second kind. The four Volterra integral equations have a simple kernel and therefore can be readily inverted, whereas the two Fredholm integral equations of the second kind can be easily solved simultaneously by the method of successive approximations. Approximate solutions of these integral equations are presented in terms of small perturbation parameters $\sigma_i^{-1} = \frac{a_i}{f}$, $(i = 1, 2)$, where a_i are the radii of the two disks and f is the distance between their centres. Formulae for the physical quantities of interest are expressed in terms of these parameters σ_i^{-1}, $(i = 1, 2)$.

2. Problem

We discuss here the boundary value problem of electrostatic potential for two unequal coaxial circular disks of radii a_1 and a_2 charged to any prescribed potentials. When we take the centre of one of these disks and their common axis as origin and z-axis of cylindrical polar coordinates (ρ, φ, z), the disks are defined by $0 \leqslant \rho \leqslant a_1$, $z = 0$, for all φ and $0 \leqslant \rho \leqslant a_2$, $z = f$ for all φ, where f is the distance between their centres. Without loss of generality, the potential $V(\rho, \varphi, z)$ is assumed to take prescribed values $f_i^{(n)}(\rho) \cos n(\varphi + \alpha)$, $(i = 1, 2)$ at the two disks, because a general solution can be obtained by appealing to the principle of Fourier superposition, n being an arbitrary non-negative integer. Thus we have the following boundary value problem of electrostatic potential:

$$\nabla^2 V(\rho, \varphi, z) = 0 \quad \text{in } D, \tag{2.1}$$

$$V(\rho, \varphi, 0) = f_1^{(n)}(\rho) \cos n(\varphi + \alpha), \; 0 \leqslant \rho \leqslant a_1, \text{ for all } \varphi, \tag{2.2}$$

$$V(\rho, \varphi, f) = f_2^{(n)}(\rho) \cos n(\varphi + \alpha), \; 0 \leqslant \rho \leqslant a_2, \text{ for all } \varphi, \tag{2.3}$$

V and $\frac{\partial V}{\partial z}$ are continuous across the plane regions

$$z = 0, \rho > a_1 \text{ and } z = f, \rho > a_2, \tag{2.4}$$

where D is the region exterior to the circular disks. It is obvious that the charge densities of the disks are also of the form $g_i^{(n)}(\rho) \cos n(\varphi + \alpha)$,

$(i=1,2)$. Indeed, if we denote (ρ, φ, z) and (t, φ_1, z_1) as the field and source points respectively, we have

$$4\pi g_i^{(n)}(t)\cos n(\varphi_1+\alpha)=\left\{\left[\frac{\partial V(t,\varphi_1,z_1)}{\partial z_1}\right]_{z_1=0-,\,f-}-\left[\frac{\partial V(t,\varphi_1,z_1)}{\partial z_1}\right]_{z_1=0+,\,f+}\right\}. \tag{2.5}$$

Following the usual Green's function approach, we obtain the integral representation formula

$$V(\rho,\varphi,z)=\int_0^{a_1} tg_1^{(n)}(t)\,K^{(n)}(t,\rho\,;\varphi\,;0,z)\,dt+\int_0^{a_2} tg_2^{(n)}(t)\,K^{(n)}(t,\rho\,;\varphi\,;f,z)\,dt, \tag{2.6}$$

where

$$K^{(n)}(t,\rho;\varphi,z_1,z)=\int_0^{2\pi}\frac{\cos n(\varphi_1+\alpha)\,d\varphi_1}{[\rho^2+t^2-2\rho t\cos(\varphi-\varphi_1)+(z-z_1)^2]^{1/2}}, \tag{2.7}$$

and this formula embodies Laplace's equation (2.1) as well as the continuity conditions (2. 4.). When we use the boundary conditions (2.2) and (2.3), we obtain two simultaneous Fredholm integral equations of first kind

$$\int_0^{a_1} t\,g_1^{(n)}(t)\,K_1^{(n)}(t,\rho)\,dt+\int_0^{a_2} t\,g_2^{(n)}(t)\,G_1^{(n)}(t,\rho)\,dt=f_1^{(n)}(\rho),\ 0\leqslant\rho\leqslant a_1, \tag{2.8}$$

$$\int_0^{a_1} t\,g_1^{(n)}(t)\,G_1^{(n)}(t,\rho)\,dt+\int_0^{a_2} t\,g_2^{(n)}(t)\,K_1^{(n)}(t,\rho)\,dt=f_2^{(n)}(\rho),\quad 0\leq\rho\leq a_2, \tag{2.9}$$

where

$$K_1^{(n)}(t,\rho)=\int_0^{2\pi}\frac{\cos n\psi\,d\psi}{[\rho^2+t^2-2\rho t\cos\psi]^{1/2}}=2\pi\int_0^{\infty}J_n(p\rho)\,J_n(pt)\,dp, \tag{2.10}$$

$$G_1^{(n)}(t,\rho)=\int_0^{2\pi}\frac{\cos n\psi\,d\psi}{[\rho^2+t^2-2\rho t\cos\psi+f^2]^{1/2}}=2\pi\int_0^{\infty}e^{-pf}J_n(p\rho)\,J_n(pt)\,dp, \tag{2.11}$$

and J_n is the Bessel function of order n. Furthermore, when we use the well-known identities used by Jain and Kanwal (1972), we obtain from (2.10) and (2.11)

$$K_1^{(n)}(t,\rho)=4(\rho t)^{-n}\int_0^{\min(\rho,\,t)}\frac{w^{2n}\,dw}{(\rho^2-w^2)^{1/2}(t^2-w^2)^{1/2}}, \tag{2.12}$$

$$G_1^{(n)}(t,\rho)=4(\rho t)^{-n}\int_0^{\rho}\int_0^{t}\frac{(w\,v)^n\,L^{(n)}(v,w)\,dv\,dw}{(\rho^2-w^2)^{1/2}(t^2-v^2)^{1/2}},\ 0\leqslant t,\rho\leqslant a_i, \tag{2.13}$$

where

$$L^{(n)}(v, w)=(w\,v)^{1/2}\int_0^\infty p\, e^{-pf} J_{n-1/2}\,(p\,w)\, J_{n-1/2}\,(pv)\, dp. \tag{2.14}$$

By applying an integral equation technique of Vaid and Jain (1974), we reduce the solution of the governing simultaneous Fredholm integral equations of the first kind (2.8) and (2.9) to that of four Volterra integral equations of the first kind and two simultaneous Fredholm integral equations of the second kind

$$S_1^{(n)}(\rho)+\int_0^{a_2} L^{(n)}(v, \rho)\, S_2^{(n)}(v)\, dv=C_1^{(n)}(\rho),\quad 0\leqslant\rho\leqslant a_1, \tag{2.15}$$

$$S_2^{(n)}(\rho)+\int_0^{a_1} L^{(n)}(v, \rho)\, S_1^{(n)}(v)\, dv=C_2^{(n)}(\rho),\quad 0\leqslant\rho\leqslant a_2, \tag{2.16}$$

where

$$S_i^{(n)}(\rho)=\rho^n\int_\rho^{a_i}\frac{t^{1-n}\, g_i^{(n)}(t)\, dt}{(t^2-\rho^2)^{1/2}},\quad 0\leq\rho\leqslant a_i,\ (i=1,2), \tag{2.17}$$

$$4\,\rho^{-n}\int_0^\rho\frac{w^n\, C_i^{(n)}(w)\, dw}{(\rho^2-w^2)^{1/2}}=f_i^{(n)}(\rho),\quad 0\leqslant\rho\leqslant a_i,\ (i=1, 2). \tag{2.18}$$

Volterra integral equations (2.17) and (2.18) can be readily inverted by using the well known formulae given by Sneddon (1966) and Kanwal (1971) and we have

$$g_i^{(n)}(t)=-(2/\pi)\, t^{n-1}\,(d/dt)\int_t^{a_i}\frac{u^{1-n}\, S_i^{(n)}(u)\, du}{(u^2-t^2)^{1/2}},\ (i=1,2), \tag{2.19}$$

$$C_i^{(n)}(\rho)=\frac{1}{2\pi\,\rho^n}\,\frac{d}{d\rho}\int_0^\rho\frac{u^{n+1}\, f_i^{(n)}(u)\, du}{(\rho^2-u^2)^{1/2}},\ (i=1, 2). \tag{2.20}$$

Substituting these known values of $C_i^{(n)}(\rho)$ in (2.15) and (2.16), we can solve these equations simultaneously for the unknown functions $S_i^{(n)}(\rho)$ by the method of standard approximation or eqns. (2.15) and (2.16) are first reduced to two ordinary Fredholm integral equtions of the second kind :

$$S_i^{(n)}(a_i\,\rho)=C_i^{(n)}(a_i\,\rho)-a_j\int_0^1 L^{(n)}(a_j\, v, a_i\,\rho)\, C_j^{(n)}(a_j\, v)\, dv$$
$$+a_i\int_0^1 L_t^{(n)}(a_i\, v, a_i\,\rho)\, S_i^{(n)}(a_i\, v)\, dv,\quad 0\leqslant\rho\leqslant 1, \tag{2.21}$$

where

$$L_i^{(n)}(a_i\, v, a_i\, \rho) = a_j \int_0^1 L^{(n)}(a_j\, w, a_i\, \rho)\, L^{(n)}(a_i\, v, a_j\, w)\, dw,$$

$$(i \neq j, \quad i, j = 1, 2) \tag{2.22}$$

and (2.21) can be easily solved for $S_i^{(n)}$, $(i=1, 2)$ by the method of iterations. When we substitute these values in (2.19), we obtain the required values of the charge densities. We demonstrate it for the special case when the two circular disks are charged to two different constant potentials V_1 and V_2. We have in this case

$$n=0, f_1^{(o)}(\rho) = V_1 . f_2^{(o)}(\rho) = V_2. \tag{2.23}$$

The relations (2.20) gives rise to

$$C_i^{(o)}(\rho) = \frac{V_i}{2\pi}, \ (i=1,2). \tag{2.24}$$

To solve equations (2.21), we assume that the parameters $\sigma_i^{-1} = \frac{a_i}{f}$, $(i=1, 2)$ are small and $\sigma_2^{-1} = O(\sigma_1^{-1})$. This leads to the expansions of kernels $L^{(o)}(v, \rho)$ and $L_i^{(o)}(v, \rho)$ in powers of small parameters σ_i^{-1}, $(i=1, 2)$

$$a_j\, L_i^{(o)}(a_j\, v, a_i\, \rho) = \frac{2}{\pi} \sigma_j^{-1} [1 - (\sigma_j^{-2} v^2 + \sigma_i^{-2} \rho^2) + (\sigma_j^{-4} v^4 + 6\sigma_i^{-2} \sigma_j^{-2} v^2 \rho^2 + \sigma_i^{-4} \rho^4) + O(\sigma_M^{-6})], \tag{2.25}$$

$$a_i\, L_i^{(o)}(a_i\, v, a_i\, \rho) = \frac{4}{\pi^2} \sigma_1^{-1} \sigma_2^{-1} [1 - (\tfrac{2}{3} \sigma_j^{-2} + (\rho^2 + v^2) \sigma_i^{-2}) + O(\sigma_M^{-4})], \quad (i \neq j\, ; i, j = 1, 2), \tag{2.26}$$

where $\sigma_M^{-1} = \max(\sigma_1^{-1}, \sigma_2^{-1})$.

Ordinary Fredholm integral equations of the second kind (2.21) can be easily solved to obtain approximate values of $S_i^{(o)}(a_i\, \rho)$, $(i=1, 2)$

$$S_1^{(o)}(a_1\, \rho) = \frac{1}{2\pi} \Big\{ V_1 \Big[1 + \frac{4}{\pi^2} \sigma_1^{-1} \sigma_2^{-1} + \frac{16}{\pi^4} \sigma_1^{-2} \sigma_2^{-2} - \frac{4}{3\pi^2} \sigma_1^{-1} \sigma_2^{-1} (\sigma_1^{-2} + 2\sigma_2^{-2}) - \frac{4}{\pi^2} \sigma_1^{-3} \sigma_2^{-1} \rho^2 \Big] - V_2 \Big[2/\pi\, \sigma_2^{-1} + \frac{8}{\pi^3} \sigma_1^{-1} \sigma_2^{-2} - \frac{2}{3\pi} \sigma_2^{-3} - \frac{2}{\pi^2} \sigma_1^{-2} \sigma_2^{-1} \rho^2 \Big] + O(\sigma_M^{-5}) \Big\}, \tag{2.27}$$

$$S_2^{(o)}(a_2\, \rho) = \frac{1}{2\pi} \Big\{ V_2 \Big[1 + \frac{4}{\pi^2} \sigma_1^{-1} \sigma_2^{-1} + \frac{16}{\pi^4} \sigma_1^{-2} \sigma_2^{-2} - \frac{4}{3\pi^2} \sigma_1^{-1} \sigma_2^{-1} (2\sigma_1^{-2} + \sigma_2^{-2}) - \frac{4}{\pi^2} \sigma_1^{-1} \sigma_2^{-3} \rho^2 \Big] - V_1 \Big[\frac{2}{\pi} \sigma_1^{-1} + \frac{8}{\pi^3} \sigma_1^{-2} \sigma_2^{-1} - \frac{2}{3\pi} \sigma_1^{-3} - \frac{2}{\pi} \sigma_1^{-1} \sigma_2^{-2} \rho^2 \Big] + O(\sigma_M^{-5}) \Big\}; \tag{2.28}$$

where the above expansions of the kernels $L^{(0)}$ and $L_i^{(0)}$ have been used. Finally. we can find approximate expressions for the densities $g_i^{(0)}(t)$, $(i=1,2)$ by substituting the above values of $S_i^{(0)}$ in equations (2.19). Total charges $Q_i^{(0)}$, $(i=1, 2)$ on the disks can be easily obtained without finding the values of the densities $g_i^{(0)}$ by applying the results

$$Q_i^{(0)}=2\pi\int_0^{a_i} t\, g_i^{(0)}(t)\, dt=4\, a_i\int_0^1 S_i^{(0)}(a_i\, u)\, du, \quad (i=1, 2,), \tag{2.29}$$

where we have used the relation (2.19) for n=0. On substituting approximate values of $S_i^{(0)}$ from (2.27) and (2.28) in (2.29), we obtain approximate formulae for $Q_i^{(0)}$, $(i=1, 2)$

$$Q_1^{(0)}=\frac{2a_1}{\pi}\Big\{V_1\Big[1+\frac{4}{\pi^2}\sigma_1^{-1}\sigma_2^{-1}+\frac{16}{\pi^4}\sigma_1^{-2}\sigma_2^{-2}$$
$$-8/3\pi^2\,\sigma_1^{-1}\sigma_2^{-1}(\sigma_1^{-2}+\sigma_2^{-2})\Big]-V_2\Big[2/\pi\,\sigma_2^{-1}+8/\pi^3\,\sigma_1^{-1}\sigma_2^{-2}$$
$$-2/3\pi\,\sigma_2^{-1}(\sigma_1^{-2}+\sigma_2^{-2})\Big]+0(\sigma_M^{-5})\Big\}, \tag{2.30}$$

$$Q_2^{(0)}=\frac{2a_2}{\pi}\Big\{V_2\Big[1+4/\pi^2\,\sigma_1^{-1}\sigma_2^{-1}+\frac{16}{\pi^4}\sigma_1^{-2}\sigma_2^{-2}$$
$$-8/3\pi^2\,\sigma_1^{-1}\sigma_2^{-1}(\sigma_1^{-2}+\sigma_2^{-2})\Big]-V_1\Big[2/\pi\,\sigma_1^{-1}+8/\pi^3\,\sigma_1^{-2}\sigma_2^{-1}$$
$$-2/3\pi\,\sigma_1^{-1}(\sigma_1^{-2}+\sigma_2^{-2})\Big]+0(\sigma_M^{-5})\Big\}. \tag{2.31}$$

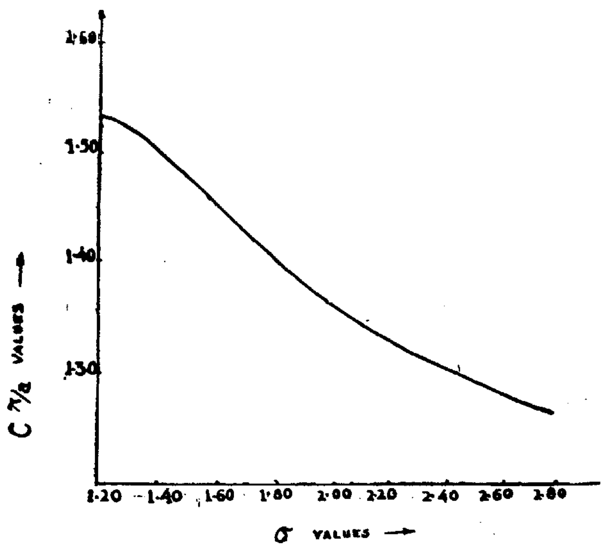

FIG. 1. Graph of $C\,\pi/a$ againsr σ,

From the above formulae, we can readily get approximate values of total charges $Q_i^{(0)}$ $(i = 1, 2)$ on the two circular disks for the following three particular cases of interest :

$$(i) \quad V_1 = V_2 = V; \quad (ii) \quad V_1 = V, V_2 = 0; \quad (iii) \quad V_1 = V, V_2 = -V.$$

In particular case (iii) we find the capacity C of the condenser formed by two equal coaxial circular disks of radius a by using the formula

$$C = [Q_1^{(0)}/2V]_{a_1=a_2=a}$$

and this leads to

$$C = \frac{a}{\pi}\Big[1 + \frac{2}{\pi}\sigma^{-1} + \frac{4}{\pi^2}\sigma^{-2} + \frac{4}{\pi}\Big(-1/3 + \frac{2}{\pi^2}\Big)\sigma^{-3} + \frac{16}{\pi^2}\Big(-\frac{1}{3} + \frac{1}{\pi^2}\Big)\sigma^{-4} + 0(\sigma^{-5})\Big], \qquad (2\ 32)$$

where $\sigma^{-1} = a/f$. As far as the authors know all the above results seem to be new.

We have also solved the electrostatic problems of two coaxial circular disks or two concentric coaxial spherical caps when these are bounded by a grounded vessel and this work shall appear elsewhere.

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