STRESS INTENSITY FACTORS FOR TWO COPLANAR GRIFFITH CRACKS IN AN INFINITELY LONG ELASTIC STRIP, PERPENDICULAR TO THE EDGES OF THE STRIP, FOR ANY VARIABLE DISTRIBUTION OF PRESSURE

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(Communicated by F. C. Auluck, F. N. A.)

(*Received 13 July 1974*)

In this paper we consider the problem of determining the distribution of stress and displacement in an infinitely long elastic strip of finite thickness containing two coplanar Griffith cracks which are perpendicular to the edges of the strip and then find the stress intensity factors at the crack tips and the crack energy. Here it is assumed that the cracks are opened up by a variable internal pressure varying along the length of the cracks and the edges of the strip are stress free. The ease in which the pressure is constant is also discussed. Firstly, the problem is reduced to solving a set of triple integral equations by the use of Fourier transforms. These triple integral equations are in turn reduced to solving a Fredholm integral equation by the application of Finite Hilbert transfroms techniques. Finally, the Fredholm integral equation containing one unknown function is solved by a numerical technique and the simple formulae for determining stress intensity factors and crack energy have been derived assuming that the thickness 2δ of the strip is much greater than the length of each crack arm. Numerical results for the quantities of physical interest have been tabulated taking a suitable value for $\delta >> 1$.

1. INTRODUCTION

Barenblat (1962, p. 55) and Irwin (1958) have shown the importance of determining the stress in the neighbourhood of cracks in the theory of fracture mechanics. The theory of crack problems in two dimensional elastic medium was first developed by Griffith (1921). Sneddon and Elliot (1946) consider the case in which the Griffith crack is situated in a semi-infinite two-dimensional medium where the crack is opened up by internal variable pressure

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varying along the length of the crack. Tranter (1961), using Fourier transform method, considers the problem of determining the stress distribution in the neighbourhood of two coplanar Griffith cracks of equal length, which are opened up by a constant internal pressure when the cracks are in an infinite elastic medium. Lowengrub and Srivastava (1968*a*) solve the same problem with a variable internal pressure by the application of finite Hilbert transform technique developed by Lowengrub and Srivastava (1970). Recently England and Green (1963) and Lowengrub (1966*b*) discuss some crack problems in which a single crack lies in two dimensional elastic strip of infinite length but of finite thickness. Lowengrub and Srivastava (1968*b*) consider the Problem of determining the stress distribution in an infinitely long elastic strip containing two coplanar Griffith cracks parallel to the edges of the strip. They also derived the expressions for the stress intensity factors, crack shape and crack energy. Sneddon and Lowengrub (1969) consider the case of single crack perpendicular to the edges of the strip. In the present paper, the similar problem as given by Sneddon and Lowengrub (1969) with two Griffith cracks lying symmetrically and perpendicularly to the edges of the strip, has been discussed. In deriving the solution of the problem, a set of triple integral equations has been obtained and these are finally reduced to a single Fredholm integral equation by using Finite Hilbert transform technique developed by Lowengrub and Srivastava (1970).

2. Formulation of the Problem and Derivation of Integral Equations

It is considered that the elastic strip of material $-\infty < y < +\infty$, $-\delta \leq x \leq \delta$ contains two cracks located in the interior of the material, on the line $y=0$, $-1 \leq x \leq -a$, $a \leq x \leq 1$, the y and x axes being assumed to be drawn along and perpendicular respectively to the edges of the strip and the origin is the middle point of the thickness of the strip. It is assumed that each of the cracks is opened up by the application of the variable pressure $\sqrt{(\pi/2)}\, P(x)$ acting normally to the face of the crack where the edges of the strip are free from the shearing and normal stresses. It is assumed that the equations of classical theory of elasticity hold.

The boundary conditions may be stated as follows :

$$\sigma_{yy}(x, 0) = -\sqrt{(\pi/2)}\, P(x), \qquad -1<x<-a, \quad a<x<1 \tag{2.1}$$

$$u_y\,(x, 0) = 0 \qquad , \qquad 1 \leqslant |x| \leqslant \delta, \quad 0 < |x| \leqslant a \tag{2.2}$$

$$\sigma_{xy}(x, 0) = 0 \qquad , \qquad |x| \leqslant \delta \tag{2:3}$$

$$\sigma_{xy}(\pm\delta, |y|) = \sigma_{xx}(\pm\delta, |y|) = 0, \quad |y| \geqslant 0. \tag{2.4}$$

The appropriate solutions of the equations of elastic equilibrium are obtained by adopting the method of Sneddon and Lowengrub (1969). In this way, we obtain

$$\mu u_x(x,|y|) = -\tfrac{1}{2}\mathbf{F}_c[\xi^{-1}\{f(\xi)-(1-2\eta)g(\xi)\}\sinh(\xi x)+ +xg(\xi)\cosh(\xi x);\xi\to|y|]\ \tfrac{1}{2}\mathbf{F}_s(\zeta^{-1}\phi(\zeta)(1-2\eta- -\zeta|y|)e^{-\zeta|y|};\zeta\to x] \tag{2.5}$$

$$\mu u_y(x,|y|) = \tfrac{1}{2}\mathbf{F}_s[\xi^{-1}\{f(\xi)+2(1-\eta)g(\xi)\}\cosh(\xi x)+xg(\xi)\sinh(\xi x);\xi\to|y|] +\tfrac{1}{2}\mathbf{F}_c[\zeta^{-1}\phi(\zeta)(2-2\eta+\zeta|y|)e^{-\zeta|y|};\zeta\to x] \tag{2.6}$$

$$\sigma_{xx}(x,|y|) = -\mathbf{F}_c[f(\xi)\cosh(\xi x)+\xi x\,g(\xi)\sinh(\xi x);\xi\to|y|]- -\mathbf{F}_c[\phi(\zeta)(1-\zeta|y|)e^{-\zeta|y|};\zeta\to x] \tag{2.7}$$

$$\sigma_{yy}(x,|y|) = \mathbf{F}_c[\{f(\xi)+2g(\xi)\}\cosh(\xi x)+\xi x\,g(\xi)\sinh(\xi x);\xi\to|y|]- -\mathbf{F}_c[\phi(\zeta)(1+\zeta|y|)e^{-\zeta|y|};\zeta\to x] \tag{2.8}$$

$$\sigma_{xy}(x,|y|) = \mathbf{F}_s[\{f(\xi)+g(\xi)\}\sinh(\xi x)+\xi xg(\xi)\cosh(\xi x);\xi\to|y|]- -|y|\mathbf{F}_s[\zeta\phi(\zeta)^{-\zeta|y|};\zeta\to x] \tag{2.9}$$

where $f(\xi)$ and $g(\xi)$ are unknown functions of ξ and μ and η are respectively the modulus of rigidity and Poission's ratio of the elastic material and

$$\mathbf{F}_c[f(\xi,y),\xi\to x]=\sqrt{\frac{2}{\pi}}\int_0^\infty f(\xi,y)\cos(\xi x)\,d\xi \tag{2.10}$$

$$\mathbf{F}_s[f(\xi,y),\xi\to x]=\sqrt{\frac{2}{\pi}}\int_0^\infty f(\xi,y)\sin(\xi x)\,d\xi \tag{2.11}$$

we obtain

$$\mu u_y(x,0)=(1-\eta)\mathbf{F}_c[\zeta^{-1}\phi(\zeta);\zeta\to x] \tag{2.12}$$

$$\sigma_{yy}(x,0) = -\frac{d}{dx}[\mathbf{F}_s\{\zeta^{-1}\phi(\zeta);\zeta\to x]+ +\sqrt{\frac{2}{\pi}}\int_0^\infty\{f(\xi)+2g(\xi)\}\cosh(\xi x)+\xi xg(\xi)\,\mathrm{sh}(\xi x)]d\xi \tag{2.13}$$

$$\sigma_{xy}(x,0)=0. \tag{2.14}$$

Thus we see that condition (2.3) is automatically satisfied.

Now eqns. (2.7) and (2.9) can be written in the form

$$\mathbf{F}_c[\sigma_{xx}(x,|y|);|y|\to\xi]= -f(\xi)\cosh(\xi x)-\xi xg(\xi)\sinh(\xi x)- -\frac{4\xi^2}{\pi}\int_0^\infty\frac{\zeta\phi(\zeta)\cos(\zeta x)\,d\zeta}{(\xi^2+\zeta^2)^2} \tag{2.15}$$

$$\mathbf{F}_s[\sigma_{xy}(x,|y|);|y|\to\xi]=\{f(\xi)+g(\xi)\}\sinh(\xi x)+\xi xg(\xi)\cosh(\xi x)- -\frac{4\xi}{\pi}\int_0^\infty\frac{\zeta^2\phi(\zeta)\sin(\zeta x)\,d\zeta}{(\xi^2+\zeta^2)^2} \tag{2.16}$$

and eqn. (2.4) in the form

$$\mathbf{F}_s[\sigma_{xy}(\pm\delta,|y|);|y|\to\xi]=0,\quad \mathbf{F}_c[\sigma_{xx}(\pm\delta,|y|);|y|\to\xi]=0. \tag{2.17}$$

we can write from (2.15), (2.16) and (2.17)

$$f(\xi)\cosh(\xi\delta)+\xi\delta\sinh(\xi\delta)\,g(\xi)=-\frac{4\xi^2}{\pi}\int_0^\infty\frac{\zeta\phi(\zeta)\cos(\zeta\delta)}{(\xi^2+\zeta)^2}\,d\zeta \tag{2.18}$$

$$f(\xi)\sinh(\xi\delta)+g(\xi)\{\sinh(\xi\delta)+\xi\delta\cosh(\xi\delta)\}=\frac{4\xi}{\pi}\int_0^\infty\frac{\zeta^2\phi(\zeta)\sin(\zeta\delta)\,d\zeta}{(\xi^2+\zeta^2)^2} \tag{2.19}$$

(Sneddon and Lowengrub 1969, pp. 62-72).

From the boundary conditions (2.2) and (2.1) and from eqns. (2.13) and (2.12), we can write

$$\frac{d}{dx}\left[\mathbf{F}_s\{\zeta^{-1}\phi(\zeta);\zeta\to x\}-\sqrt{\frac{\pi}{2}}\int_0^\infty[\{f(\xi)+2g(\xi)\}\cosh(\xi x)+\xi x g(\xi)\sinh(\xi x)]\,d\xi\right]=\sqrt{\frac{\pi}{2}}P(x),\quad a<|x|<1 \tag{2.20}$$

$$\int_0^\infty\zeta^{-1}\phi(\zeta)\cos(\zeta x)\,d\zeta=0,\qquad 0\le|x|\le a,\quad 1\le x|\le\delta. \tag{2.21}$$

If we now assume that,

$$\phi(\zeta)=\int_a^1 h(t^2)\sin(\zeta t)\,dt \tag{2.22}$$

then we see that (2.21) is automatically satisfied if and only if

$$\int_a^1 h(t^2)\,dt=0. \tag{2.23}$$

Putting the values of $\phi(\zeta)$ in (2.20) we obtain

$$\int_a^1\frac{th(t^2)}{t^2-x^2}\,dt=\frac{\pi}{2}\,\phi_1(x),\quad a<|x|<1 \tag{2.24}$$

where

$$\phi^1(x)=P(x)+\frac{2}{\pi}\int_0^\infty[\{f(\xi)+2g(\xi)\}\cosh(\xi x)+\xi x g(\xi)\sinh(\xi x)]\,d\xi. \tag{2.25}$$

Now by using finite Hilbert transform technique adopted by Lowengrub and Srivastava (1970) the solution of (2.24) can be written in the form

$$h(t^2)=-\frac{2}{\pi}\left(\frac{1-t^2}{t^2-a^2}\right)^{\frac{1}{2}}\int_a^1\left(\frac{x^2-a^2}{1-x^2}\right)^{\frac{1}{2}}\frac{x\,\phi_1(x)}{x^2-t^2}\,dx+ +c_1\{(t^2-a^2)(1-t_2)\}^{-\frac{1}{2}} \tag{2.26}$$

where

$$c_1=\frac{1}{\pi F}\int_a^1\left(\frac{x^2-a^2}{1-x^2}\right)^{\frac{1}{2}}2x\,\phi_1(x)\,dx\int_a^1\left(\frac{1-t^2}{t^2-a^2}\right)^{\frac{1}{2}}\frac{dt}{x^2-t^2}. \tag{2.27}$$

The solution of (2.24) can also be written in the form

$$h(t^2) = -\frac{2}{\pi}\left(\frac{t^2-a^2}{1-t^2}\right)^{\frac{1}{2}}\int_a^1\left(\frac{1-x^2}{x^2-a^2}\right)^{\frac{1}{2}}\frac{x\,\phi(x)\,dx}{x^2-t^2}+c_2\{(t^2-a^2)(1-t^2)\}^{\frac{1}{2}} \quad (2.28)$$

where

$$c_2 = \frac{1}{\pi F}\int_a^1\left(\frac{1-x^2}{x^2-a^2}\right)^{\frac{1}{2}}2x\,\phi_1(x)\,dx\int_a^1\left(\frac{t^2-a^2}{1-t^2}\right)^{\frac{1}{2}}\frac{dt}{x^2-t^2} \quad (2.29)$$

and

$$F \equiv F(\pi/2, \sqrt{1-a^2}). \quad (2.30)$$

is the complete elliptical integral of the first kind:

$$F = \int_a^1\frac{dt}{\sqrt{(1-t^2)(t^2-a^2)}}. \quad (2.31)$$

Putting the value of $\phi_1(x)$ from (2.25) in (2.26) we get

$$h(t^2)+\frac{4}{\pi^2}\left(\frac{1-t^2}{t^2-a^2}\right)^{\frac{1}{2}}\left[\int_a^1\left(\frac{x^2-a^2}{1-x^2}\right)^{\frac{1}{2}}x\left\{\frac{1}{x^2-t^2}-\right.\right.$$

$$\left.\left.-\frac{1}{F(1-t^2)}\int_a^1\left(\frac{1-\alpha^2}{\alpha^2-a^2}\right)\frac{d\alpha}{x^2-\alpha^2}\right\}I_1(x)\right]dx$$

$$= S(t^2) \qquad a<t<1 \quad (2.32)$$

where

$$I_1(x) = \int_0^\infty\{f(\xi)+2g(\xi)\}\,\mathrm{ch}\,(\xi x)+g(\xi)\,\xi x\,\mathrm{sh}\,(\xi x)]\,d\xi \quad (2.32a)$$

and

$$S(t^2) = \frac{2}{\pi}\left[-g(t^2)+\frac{1}{F\sqrt{(1-t^2)(t^2-a^2)}}\int_a^1 g(\alpha^2)\,d\alpha\right] \quad (2.32b)$$

$$g(t^2) = \sqrt{\frac{1-t^2}{t^2-a^2}}\int_a^1\left(\frac{x^2-a^2}{1-x^2}\right)^{\frac{1}{2}}\frac{P(x)\,x\,dx}{x^2-t^2}. \quad (2.32c)$$

Now putting the value of $\phi(\zeta)$ from (2.22) in (2.18) and (2.19) and using the results

$$\int_0^\infty\frac{x\cos(bx)\sin(ax)}{(x^2+\beta^2)}\,dx = -\tfrac{1}{2}\pi e^{-\beta b}\,\mathrm{sh}\,(a\beta), \quad 0<a<b$$

$$= \tfrac{1}{2}\pi e^{-\beta a}\cosh(\beta b), \quad 0<b<a$$

(Gradshteyn and Ryzhik 1965, p.429).

We obtain

$$f(\xi)\cosh(\xi\delta)+\xi\delta g(\xi)\sinh(\xi\delta)$$

$$= -\xi e^{-\xi\delta}\int_a^1 h(t^2)\{t\cosh(\xi t)-\delta\sinh(\xi t)\}\,dt \quad (2.33)$$

$$f(\xi)\sinh(\xi\delta)+g(\xi)\{\sinh(\xi\delta)+\xi\delta\cosh(\xi\delta)\}$$

$$=e^{-\xi\delta}\int_a^1 h(t^2)\{\xi t\cosh(\xi t)+(1-\xi\delta)\sinh(\xi t)\}\,dt \tag{2.34}$$

Putting the values of $f(\xi)$ and $g(\xi)$ obtained from the equations (2.33) and (2.34) in (2.32a) we get $\mathbf{I}_1(x)$ after changing the order of integration and simple calculation as

$$\mathbf{I}_1(x)=\int_a^1 h(\rho^2)\,d\rho\int_0^\infty \frac{\omega_1(ux,u\rho)\,du}{\sinh(2u)+2u} \tag{2.35}$$

where

$$\xi\delta=u \tag{2.36}$$

$$\omega_1(ux,u\rho)=\frac{1}{\delta}\Big[\Big\{\frac{u\rho}{\delta}v_1(u)\cosh(u\rho/\delta)+v_2(u)\sinh(u\rho/\delta)\Big\}\cosh(ux/\delta)+$$

$$+\frac{ux}{\delta}\Big\{\frac{2u\rho}{\delta}\cosh(u\rho/\delta)+v_3(u)\sinh(u\rho/\delta)\Big\}\sinh(ux/\delta)\Big] \tag{2.37}$$

$$v_1(u)=3-2u+e^{-2u} \tag{2.37 a}$$

$$v_2(u)=2-4u+2u^2+2e^{-2u} \tag{2.37 b}$$

$$v_3(u)=1-2u+e^{-2u}. \tag{2.37 c}$$

Thus putting the value of $\mathbf{I}_1(x)$ from (2.35) in (2.32) we have the Fredholm integral equation containing one unknown function given by

$$h(t^2)+\int_a^1 h(\rho^2)\,k(\rho,t)\,d\rho=S(t^2),\qquad a<t<1 \tag{2.38}$$

where $S(t^2)$ is given by (2.32b) and the Karnel $k(\rho,t)$ is given by

$$k(\rho,t)=\frac{4}{\pi^2}\left(\frac{1-t^2}{t^2-a^2}\right)^{\frac{1}{2}}\int_a^1\left\{\omega_2(x,t)\int_0^\infty\frac{\omega_1(ux,u\rho)}{\sinh(2u)+2u}\,du\right\}dx \tag{2.39}$$

where

$$\omega_2(x,t)=\left(\frac{x^2-a^2}{1-x^2}\right)^{\frac{1}{2}}x\left\{\frac{1}{x^2-t^2}-\frac{1}{F(1-t^2)}\right\}\int_a^1\left(\frac{1-\alpha^2}{\alpha^2-a^2}\right)^{\frac{1}{2}}\frac{d\alpha}{x^2-\alpha^2}. \tag{2.40}$$

3. Approximation of the Kernel

Taking into account that $\delta>>1$ and using the results

$$\cosh(u\rho/\delta)=1+(u\rho/\delta)^2\frac{1}{\lfloor 2}+(u\rho/\delta)^4\frac{1}{\lfloor 4}+(u\rho/\delta)^6\frac{1}{\lfloor 6}+\cdots+o\left(\frac{1}{\delta^8}\right) \tag{3.1}$$

$$\sinh(u\rho/\delta)=u\rho/\delta+(u\rho/\delta)^3\frac{1}{\lfloor 3}+(u\rho/\delta)^5\frac{1}{\lfloor 5}+(u\rho/\delta)^7\frac{1}{\lfloor 7}+\cdots+o\left(\frac{1}{\delta^9}\right) \tag{3.2}$$

we have after simplification and retaining terms upto the order δ^{-6}

$$w_1(ux, u\rho) = D_0(u,\rho) + x^2 D_2(u,\rho) + x^4 D_4(u,\rho) \tag{3.3}$$

where D_0, D_2 D_4 are functions of u and ρ given by

$$D_0(u,\rho) = \frac{u\rho}{\delta^2}(v_1+v_2) + \frac{u^3\rho^3}{\delta^4 \lfloor 3}(3v_1+v_2) + \frac{u^5\rho^5}{\lfloor 5 \delta^6}(5v_1+v_2) \tag{3.4}$$

$$D_2(u,\rho) = \frac{u^3\rho}{\delta^4 \lfloor 2}(v_1+v_2+2v_3+4) + \frac{u^5\rho^3}{\delta^6 \lfloor 12}(3v_1+v_2+2v_3+12) \tag{3.5}$$

$$D_4(u,\rho) = \frac{u^5\rho}{\delta^6 \lfloor 4}(v_1+v_2+4v_3+8) \tag{3.6}$$

v_1, v_2, v_3 are functions of u given by (2.37a) to (2.37c). Thus we get

$$\int_0^\infty \frac{w_1(ux, u\rho)\, du}{\sinh(2u)+2u} = \bar{D}_0(\rho) + x^2 \bar{D}_2(\rho) + x_4 \bar{D}_4(\rho) \tag{3.7}$$

where

$$\bar{D}_0(\rho) = \int_0^\infty \frac{D_0(u,\rho)\, du}{\sinh(2u)+2u} \tag{3.7a}$$

$$\bar{D}_2(\rho) = \int_0^\infty \frac{D_2(u,\rho)\, du}{\sinh(2u)+2u} \tag{3.7b}$$

$$\bar{D}_4(\rho) = \int_0^\infty \frac{D_4(u,\rho)\, du}{\sinh(2u)+2u}\,. \tag{3.7c}$$

Now putting the values of $v_1(u)$, $v_2(u)$ and $v_3(u)$ from (2.37a) to (2.37c) in (3.7a) to (3.7c) we obtain

$$\bar{D}_0(\rho) = \frac{\rho}{2\delta^2}[5I_1 - 3(2I_2 - I_3 - II_1)] + \frac{\rho^3}{8\delta^4}[11I_3 - 5(4I_4 - 2I_5 - II_3)] + \\ + \frac{\rho^5}{32\delta^6}[17I_5 - 7(6I_6 - 3I_7 - II_5)] \tag{3.8a}$$

$$\bar{D}_2(\rho) = \frac{3\rho}{8\delta^4}\{11I_3 - 5(4I_4 - 2I_5 - II_3)\} + \frac{10\rho^3}{32\delta^6}[25I_5 - 7(6I_6 - 3I_7 - II_5)] \tag{3.8b}$$

$$\bar{D}_4(\rho) = \frac{5\rho}{32\delta^6}[17I_5 - 7(6I_6 - 3I_7 - II_5)] \tag{3.8c}$$

where

$$I_n = \frac{2^n}{\lfloor n}\int_0^\infty \frac{u^n\, du}{\sinh(2u)+2u} \tag{3.9a}$$

$$II_n = \frac{2^n}{\lfloor n}\int_0^\infty \frac{u^n e^{-2u}\, du}{\sinh(2u)+2u} \tag{3.9b}$$

Integrals appearing in (3.9*a*) and (3.9*b*) have been tabulated by Ling (1957). Using (3.7) we get from (2.39)

$$k(\rho, t) = \frac{4}{\pi^2}\left(\frac{1-t^2}{t^2-a^2}\right)^{1/2} \int_a^1 \omega_2(x, t)(\bar{D}_0 + x^2\bar{D}_2 + x_4\bar{D}_4)dx \tag{3.10}$$

which can be written as

$$k(\rho, t) = \frac{4}{\pi^2}\left(\frac{1-t^2}{t^2-a^2}\right)^{1/2}[k_1(\rho, t) + k_2(\rho, t)] \tag{3.11}$$

where

$$k_1(\rho, t) = \int_a^1 \frac{(x^2-a^2)\, x(\bar{D}_0 + x^2\bar{D}_2 + x^4\bar{D}_4)}{\sqrt{(1-x^2)(x^2-a^2)}} \frac{dx}{x^2-t^2} \tag{3.12}$$

and

$$k_2(\rho, t) = -\frac{1}{F(1-t^2)}\int_a^1 \left(\frac{1-\alpha^2}{\alpha^2-a^2}\right)^{1/2} k_1(\rho, \alpha)\, d\alpha \tag{3.13}$$

$\bar{D}_0$, $\bar{D}_2$ and $\bar{D}_4$ are functions of ρ given by (3.8*a*) to (3.8*c*). Putting $z=x^2$, $k_1(\rho, t)$ can be written as

$$k_1(\rho, t) = \tfrac{1}{2}\int_{a^2}^1 \left[z^2\bar{D}_4 + zA + B + \frac{Bt^2 - a^2\bar{D}_0}{z-t^2}\right]\frac{dz}{\sqrt{(1-z)(z-a^2)}} \tag{3.14}$$

where

$$A = \bar{D}_2 + (t^2-a^2)\bar{D}_4. \tag{3.14a}$$

and

$$B = \bar{D}_0 + At^2 - a^2\bar{D}_2.$$

Now using the results

$$\int_a^b \frac{t\, dt}{\sqrt{(t^2-a^2)(b^2-t^2)}}\frac{1}{t^2-y^2} = \pi/2\, \{(a^2-y^2)(b^2-y^2)\}^{-\frac{1}{2}}, \quad 0<y<a$$

$$= 0 \qquad , \quad a<y<b$$

$$= -\pi/2\{(y^2-a^2)(y^2-b^2)\}^{\frac{1}{2}}, \quad y>b \tag{3.15}$$

and the reduction formula

$$J_m = \frac{2m-1}{2m}(1+a^2)\, J_{m-1} - \frac{m-1}{m}\, a^2\, J_{m-2}, \qquad m \geqslant 2 \tag{3.16}$$

where

$$J_m = \int_{a^2}^1 \frac{z^m\, dz}{\sqrt{(1-z)(z-a^2)}} \tag{3.16a}$$

with

$$J_0 = \pi \quad \text{(Gradshteyn and Ryzhik 1965, p. 82)} \tag{3.16b}$$

we obtain

$$k_1(\rho, t) = \frac{\pi}{2}\left[\frac{3+2a^2+3a^4}{8}\bar{D}_4+\frac{1+a^2}{2}A+B\right] \tag{3.17}$$

where A, B are given by (3.14a) and (3.14b). Now using the result (3.17) we can have

$$k_2(\rho, t) = -\frac{\pi}{2}\frac{1}{F(1-t^2)}\left[c_3F+(c_4-c_3)E+(\bar{D}_4-c_4)\tfrac{1}{3}\{2(1+a^2)E-a^2F\}-\right.$$

$$\left.-\frac{\bar{D}_4}{15}\{(8+7a^2+8a^4)E-4a^2(1+a^2)F\}\right] \tag{3.18}$$

where

$$c_3 = \frac{1}{8}(3+2a^2+3a^4)\bar{D}_4+\frac{1+a^2}{2}A_1+B_1 \tag{3.18a}$$

$$c_4 = \frac{1+a^2}{2}\bar{D}_4+A_1 \tag{3.18b}$$

$$A_1 = \bar{D}_2-a^2\bar{D}_2 \tag{3.18c}$$

and

$$B_1 = \bar{D}_2-a^2\bar{D}_2. \tag{3.18d}$$

Here we have also used the reduction formula

$$\mathbf{I}_{2m} = -\frac{2m-3}{2m-1}a^2\mathbf{I}_{2m+4}+\frac{2m-2}{2m-1}(1+a^2)I_{2m-2}, \quad (m=1, 2, 3 \ldots) \tag{3.19}$$

(Gradshteyn and Ryzhik 1965, p. 67) with

$$\mathbf{I}_0 = F \quad \text{given by (2.31)} \tag{3.19a}$$

and $\mathbf{I}_2 = E \equiv E(\pi/2, \sqrt{1-a^2})$ is the complete elliptic integral of the second kind :

$$E=\int_{a^2}^{1}\frac{\alpha^2\,d\alpha}{\sqrt{(1-\alpha^2)(\alpha^2-a^2)}} \tag{3.19b}$$

where

$$\mathbf{I}_{2m}=\int_{a}^{1}\frac{x^{2m}\,dx}{\sqrt{(x^2-a^2)(1-x^2)}} \tag{3.19c}$$

4. Formulae for Stress Intensity Factors and Crack Energy

We have from (2.8) and (2.32a)

$$\sigma_{yy}(x, 0) = \sqrt{2/\pi}\left[\mathbf{I}_1(x)-\int_0^\infty\phi(\zeta)\cos(\zeta x)\,d\zeta\right] \tag{4.1}$$

Putting the value of $\phi(\zeta)$ from (2.22) in (4.1) and using the results (2.35) and (3.7) we can write

$$\sigma_{yy}(x, 0) = \sqrt{2/\pi}\left[\int_a^1 h(t^2)\left\{\bar{D}_0+x^2\bar{D}_2+x^4\bar{D}_4-\frac{t}{t^2-x^2}\right\}dt\right] \tag{4.2}$$

where $\bar{D}_0$, $\bar{D}_2$ and $\bar{D}_4$ are given by (3.8a) to (3.8c).

Now stress-intensity factors at the ends $x=a$ and $x=1$ are defined by the formulae

$$N_a = \underset{x\to a_-}{\mathrm{Lt}} \sqrt{(a-x)}\, \sigma_{yy}(x, 0) \tag{4.3}$$

$$N_1 = \underset{x\to 1_+}{\mathrm{Lt}} \sqrt{(x-1)}\, \sigma_{yy}(x, 0). \tag{4.4}$$

Hence from the result (4.2) and following Das and Poddar (1977) we can write

$$N_a = -(c_1/2)\sqrt{(\pi/a(1-a^2))} \tag{4.5}$$

$$N_1 = (c_2/2)\sqrt{(\pi/1-a^2)} \tag{4.6}$$

where c_2 and c_1 are given by (4.8) and (4.7) respectively.

Putting the value of $\phi_1(x)$ from (2.25) in (2.27) and from (2.32a), (2.35) and the result (3.7) and using (3.15), (3.16) and (3.19) we can write

$$c_1 = \frac{2}{\pi}\cdot\frac{1}{F}\int_a^1 g(\rho^2)+h(\rho^2)\,\{c_3F+(c_4-c_3)E+(D_4-c_4)\,\tfrac{1}{3}\{2(1+a^2)E-a^2F\}-$$

$$-\frac{D_4}{15}\{(8+7a^2+8a^4)E-4a^2(1+a^2)F\}]\,d\rho \tag{4.7}$$

where the values of C_3, C_4 and $\bar{D}_4$ are given in the previous section.

Adopting similar process we can write

$$c_2 = \frac{2}{\pi F}\int_a^1 [g_1(\rho^2)+h(\rho^2)\,\{a^2\bar{c}_3F+(a^2\bar{c}_4-\bar{c}_3)\,E+(a^2\bar{D}_4-\bar{c}_4)\,\tfrac{1}{3}\{2(1+a^2)E-a^2F\}$$

$$-\frac{\bar{D}_4}{15}\{(8+7a^2+8a^4)\,E-4a^2\,(1+a^2)\,F\}]\,d\rho \tag{4.8}$$

where

$$g_1(t^2) = \left(\frac{t^2-a^2}{1-t^2}\right)^{\frac{1}{2}}\int_a^1\left(\frac{1-x^2}{x^2-a^2}\right)^{\frac{1}{2}}\frac{x\,\rho(x)}{x^2-t^2}\,dx \tag{4.8a}$$

and

$$\bar{c}_3 = \frac{3+2a^2+3a^4}{8}D_4+\frac{1+a^2}{2}\bar{A}_1+\bar{B}_1 \tag{4.8b}$$

$$\bar{c}_4 = \frac{1+a^2}{2}\bar{D}_4+\bar{A}_1 \tag{4.8c}$$

$$\bar{A}_1 = \bar{D}_2-\bar{D}_4, \qquad \bar{B}_1 = \bar{D}_0-\bar{D}_2 \tag{4.8d}$$

the values of $\bar{D}_0$, $\bar{D}_2$, $\bar{D}_4$ being obtained previously.

From (2.12) and (2.22) we obtain after interchanging the order of integration

$$2\mu\, u_y(x, 0) = 2(1-\eta)\sqrt{\frac{\pi}{2}}\int_x^1 h(t^2)\,dt. \tag{4.9}$$

Therefore crack energy w is given by

$$w = 2\int_a^1 u_y\,(x,0)\,\sigma_{yy}\,(x, 0)\,dx. \tag{4.10}$$

Putting the value of $\sigma_{yy}(x, 0)$ from (2.1) we can write

$$w = \frac{\pi(1-\eta)}{\mu} \int_a^1 h(t^2)\, dt \int_a^t P(x)\, dx. \tag{4.11}$$

5. Particular Case in which $P(x)$ is a Polynomial or Constant

In this section, we shall discuss two cases.

Case 1—When cracks are opened up by a constant internal pressure, i. e. when

$$P(x) = P_0 \qquad \text{(constant)} \tag{5.1}$$

Case 2—When the cracks are opened up by a polynomial distribution of pressure, i. e. when

$$P(x) = \sqrt{\frac{2}{\pi}}\, x^n.$$

Putting the value $P(x) = P_0$ in (2.32*c*), (2.32*b*) and (4.8*a*) and using the results (3.15), (3.16*b*), (3. 19*a*) and (3.19*b*) we obtain

$$g(t^2) = \frac{\pi}{2} P_0 \sqrt{\frac{1-t^2}{t^2-a^2}} \tag{5.3}$$

$$\int_a^1 g(t^2)\, dt = P_0 \frac{\pi}{2}(F - E) \tag{5.4}$$

$$S(t^2) = P_0(t^2 - E/F)/\sqrt{(1-t^2)(t^2-a^2)} \tag{5.4a}$$

$$\int_a^1 g_1(t^2)\, dt = P_0 \frac{\pi}{2}(a^2F - E). \tag{5.5}$$

Thus (2.38) reduces to

$$h(t^2) + \int_a^1 h(\rho^2) K(\rho, t)\, d\rho = P_0 (t^2 - E/F)/\sqrt{(1-t^2)(t^2-a^2)}. \tag{5.6}$$

From (4.5) and (4.6) using (4.8) and (4.7) we can write the expressions for the stress intensity factors at the tips of the crack as

$$N_a = -\frac{1}{2}\sqrt{\frac{\pi}{a(1-a^2)}} \left[P_0 (a^2 - \tau) + \int_a^1 h(\rho^2) R_1(\rho)\, d\rho \right] \tag{5.7}$$

$$N_1 = \frac{1}{2}\sqrt{\frac{\pi}{1-a^2}} \left[P_0 (1 - \tau) + \int_a^1 h(\rho^2) R(\rho)\, d\rho \right] \tag{5.8}$$

where

$$R_1(\rho) = \frac{2}{\pi}\Big[a^2\bar{c}_2 + (a^2\bar{c}_4 - \bar{c}_3)\,\tau + (a^2\bar{D}_4 - \bar{c}_4)\,\tfrac{1}{3}\{2(1+a^2)\tau - a^2\} -$$

$$- \frac{\bar{D}_4}{15}\{(8 + 7a^2 + 8a^4)\,\tau - 4a^2(1+a^2)\}\Big] \tag{5.8a}$$

$$R(\rho) = \frac{2}{\pi}\Big[c_3+(c_4-c_3)\,\tau+(\bar{D}_4-c_4)\,\tfrac{1}{3}\{2(1+a^2)\,\tau-a^2\}-$$

$$-\frac{\bar{D}_4}{15}\,\{(8+7a^2+8a^4)\,\tau-4a^2\,(1+a^2)\}\Big] \tag{5.8b}$$

$$\tau=E/F \tag{5.9}$$

c_3, c_4, $\bar{c}_3$, $\bar{c}_4$ D_4 are functions of P already given in the previous sections.

Thus from (4.11) we obtain the simple expression involving the solution of the Fredholm integral eqn. (5.6) for the energy to open up the crack as

$$w=\frac{\pi}{\mu}(1-\eta)\,P_0\int_a^1(\rho-a)h\,(\rho^2)\,d\rho. \tag{5.10}$$

Here two cases arise according as n is even or odd.

(*i*) Firstly we assume that

$$n=2m,\ (m=1,\,2,\,3\ldots\ldots) \tag{5.11}$$

so that

$$P(x)=\sqrt{\frac{2}{\pi}}\,x^{2m}. \tag{5.11a}$$

Hence from (2.32*c*) and (4i8*a*) $g(t^2)$ and $g_1(t^2)$ are respectively given by

$$g(t^2)=\sqrt{\frac{2}{\pi}}\;\sqrt{\frac{1-t^2}{t^2-a^2}}\int_a^1\frac{(x^2-a^2)x^{2m+1}}{\sqrt{(1-x^2)(x^2-a^2)}}\;\frac{dx}{x^2-t^2} \tag{5.12}$$

$$g_1(t^2)=\sqrt{\frac{2}{\pi}}\left(\frac{t^2-a^2}{1-t^2}\right)^{\frac{1}{2}}\int_a^1\frac{(1-x^2)x^{2m+1}}{\sqrt{(1-x^2)(x^2-a^2)}}\;\frac{dx}{(x^2-t^2)} \tag{5.13}$$

Hence we can evaluate

$$\int_a^1 g\,(t^2)\,dt \text{ and } \int_a^1 g_1\,(t^2)\,dt$$

so also $S(t^2)$ using the procedure adopted by Das and Poddar (1977) and as such stress intensity factors N_a, N_1 given by (4.5) and (4.6) can be evaluated analytically and finally numerically very easily.

(*ii*) Secondly, we assume that

$$n=2m+1,\quad (m=1,\,2,\,3\ldots) \tag{5.14}$$

So that

$$P(x)=\sqrt{\frac{2}{\pi}}\,x^{2m+1},\ (m=1,\,2,\,3\ldots). \tag{5.15}$$

Then $g(t^2)$ and $g_1(t^2)$ are given by

$$g(t^2)=\sqrt{\frac{2}{\pi}}\left(\frac{1-t^2}{t^2-a^2}\right)^{\frac{1}{2}}\int_a^1 \frac{(x^2-a^2)\,x^{2m+2}}{\sqrt{(1-x^2)\,(x^2-a^2)}}\,\frac{dx}{(x^2-t^2)} \qquad (5.16)$$

$$g_1(t^2)=\sqrt{\frac{2}{\pi}}\left(\frac{t^2-a^2}{t^2-a^2}\right)^{\frac{1}{2}}\int_a^1 \frac{(1-x^2)\,x^{2m+2}}{\sqrt{(1-x^2)(x^2-a^2)}}\,\frac{dx}{(x^2-t^2)} \qquad (5.17)$$

which will give integrals of the type (3.19c) and also elliptic integral of the third kind

$$\int_a^1 \frac{1}{\sqrt{(1-x^2)\,(x^2-a^2)}}\,\frac{dx}{x^2-t^2}$$

Hence the integral involved in $S(t^2)$ can be computed numerically as such stress intensity factors N_a, N_1 can be evaluated numerically.

6. Numerical Solution of the Integral Equation

Let us seek a solution of the Fredholm integral equation given by (2.38) by Gauss Quadrature Formula when $a\to 0$, i. e. when two cracks merge into a single crack of length 2 units.

Thus when $a\to 0$ eqn. (2.38) reduces to

$$h(t^2) + \int_a^1 h(\rho^2)\,k(\rho, t)\,d\rho=S(t^2) \qquad (6.1)$$

where $k(\rho, t)$ is given by (3.11) (3.17) and (3.18) and $S(t^2)$ is given by (2.32*b*).

When $a\to 0$, $E/F\to 0$. Hence from (3.11) to (3.18*d*) we obtain

$$k(\rho, t)=\frac{1}{\pi}\,\frac{t}{\sqrt{1-t^2}}\,[(\tfrac{1}{4}+t^2-2t^4)\bar{D}_4+(1-2t^2)\,\bar{D}_2-2\bar{D}_0] \qquad (6.2)$$

where $\bar{D}_0$, $\bar{D}_2$, $\bar{D}_4$ are obtained from (3.8*a*) to (3.8*c*) using the values of (3.9*a*) and (3.9*b*) tabulated by Ling (1957) as

$$\bar{D}_0=\frac{\rho}{\delta^2}\times 1.1896415+\frac{\rho^3}{\delta^4}\times 0.1132713-\frac{\rho^5}{\delta^6}\times 0.1197760 \qquad (6.3a)$$

$$\bar{D}_2=0.3398139\times\rho/\delta^4+1.115930\times\rho^3/\delta^6 \qquad (6.3b)$$

$$\bar{D}_4=-0.598880\times\rho/\delta^6. \qquad (6.3c)$$

Now, setting $\rho = t_j$ we can express (6.1) as

$$h(t^2)+\sum_{j=1}^{n} h(t_j^2)\, k(t_j, t)\, w_j = s(t_j^2). \tag{6.4}$$

where w_j $(j = 1, 2, \cdots n)$ are weight functions for n-point Gauss-Quadrature formulae. Again setting $t = t_i$ $(i = 1, 2, \cdots n)$ we can write

$$h(t_i^2)+\sum_{j=1}^{n} h(t_j^2)\, k(t_i, t_j)\, w_j = s(t_j^2). \tag{6.5}$$

Since $\tau = E/F \to 0$ for $a \to 0$ from (4.7) we obtain

$$c_1 = 2/\pi \int_0^1 \left\{ \frac{1}{F} g(\rho^2) + c_3^0 h(\rho^2) \right\} d\rho \tag{6.6}$$

where c_3^0 is obtained from (3.18a) to (3.18d) and can be written as

$$c_3^0 = \tfrac{1}{8}(3\bar{D}_4 + 4\bar{D}_2 + 8\bar{D}_0) \tag{6.7}$$

$\bar{D}_0$, $\bar{D}_2$ and $\bar{D}_4$ are given by (6.3a) to (6.3c).

Thus we get simple expression for the stress intensity factor N_1 as

$$N_1 = \tfrac{1}{2} \sqrt{\frac{\pi}{1-a^2}} \left[2/\pi \left\{ \frac{1}{F}\int_0^1 g(t^2)\, dt + \int_0^1 h(\rho^2)\, c_3^0\, d\rho \right\}\right] \tag{6.8}$$

where $h(\rho^2)$ is the solution of (6.1).

7. Numerical Results

Numerical results for the stress intensity factors and crack energy (as $a \to 0$), when two cracks merge into a single crack of crack length two units have been tabulated in the case in which

$$P(x) = \sqrt{(2/\pi)}\,.\, x^{2m}, \quad \delta = 10 \tag{7.1}$$

for $m = 0, 1, 2, 3, 4$, only for simplicity. These results have been obtained by a high speed computer by using 40 point Gaussian-Quadrature formula.

Values of N_1 and W

m	N_1	$W.E/(1-\eta^2)$
0	0.7113	3.1557
1	0.3546	0.2655
2	0.2657	0.0902
3	6.2213	0.0450
4	0.1936	0 0269

We know that the stress intensity factor N at any tip of a Griffith crack of length 2 units, which is under constant internal pressure. the crack being situated in an infinite body is $(1/\sqrt{2}) \approx 0.7071$ and the crack energy for such a crack is given by $\dfrac{W_0 E}{\pi(1-\eta^2)} = 1$. From the above table we see that

N_1=0.7113 and $\frac{W.E}{\pi(1-\eta^2)} = 3.1557/\pi \approx 1.0045$. This shows that when $\delta \gg 1$, $N \approx N_1$ and $W = W_0$, in other words when $\delta \gg 1$ that is when the length of the crack is very small in comparison with the thickness of the infinite strip, the crack in such a medium behaves approximately like a Griffith crack in an infinite medium. The result is in agreement with what we expected. This verifies the theory of the problem and the numerical results thus obtained.

Acknowledgement

The author is grateful to Dr. S. C. Das for his valuable suggestions and kind discussions during the preparation of this paper.

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