ON THE REAL ROOTS OF THE RANDOM ALGEBRAIC EQUATION

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(Communicated by F. C. Auluck, F. N. A.)

(*Received 6 March 1974*)

Let $a_0, a_1, \ldots, a_{n-1}$ be dependent random variables following the normal law with mean zero and joint density function

$$[\sqrt{|\Omega|}/(2\pi)^{n/2}]\exp[(-1/2)\bar{a}'\Omega\bar{a}],$$

where Ω^{-1} is the moment matrix with $\sigma_i=1$, $i=0, 1, \ldots, n-1$, $\rho_{ij}=\rho$, $0<\rho<1$, $i, j=0, 1, \ldots, n-1$, and $\bar{a}'$ denoting the transpose of the column vector $\bar{a}$. It is proved that the average number of real zeros of

$$a_0+a_1x+a_2x^2+\ldots+a_{n-1}x^{n-1}=0$$

in $(-\infty, \infty)$ is asymptotically equal to $(2/\pi)(1-\rho^2)^{1/2}\log n$, when n is large.

1. Introduction

Consider the algebraic equation

$$f(x)\equiv a_0+a_1x+a_2x^2+\ldots+a_{n-1}x^{n-1}=0 \tag{1.1}$$

where a_i are dependent random variables assuming real values only and following the normal distribution with mean zero and joint density function

$$[\sqrt{|\Omega|}/(2\pi)^{n/2}]\exp[-(1/2)\bar{a}'\Omega\bar{a}] \tag{1.2}$$

where Ω^{-1} is the moment matrix with $\sigma_i=1$, $i=0, 1, \ldots, n-1$, $\rho_{ij}=\rho$, $0<\rho<1$, $i, j=0, 1, \ldots, n-1$ and $\bar{a}'$ denoting the transpose of the column vector $\bar{a}$.

Let us denote the number of real roots of eqn. (1.1) in the interval (a, b) by $N_n(a, b; f)$. Here we have shown that $E[N_n(-\infty, \infty; f)]$ is asymptotic to $(2/\pi)(1-\rho^2)^{1/2}\log n$. At the end in remark we show that it is unchanged even if $\sigma_i=\sigma$, $i=0, 1, \ldots, n-1$.

Kac (1943) has shown that it is asymptotic to $(2/\pi)\log n$ when the random variables are independent with mean zero and variance one. He

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(Kac 1949) has shown that the result is unchanged if the random variables are uniformly distributed.

Stevens (1965) treated the case when a_i are independent random variables and has shown that the same is $(2/\pi)\log n$ when

$$E(a_i)=0, \quad E(a_i^2)=1, \quad E(a_i^4)<B$$

$$\frac{d}{dy}P(a_1<y)<\frac{B}{1+y^{1+1/B}}, \quad \frac{d}{dy}P(a_n<y)<\frac{B}{1+y^{1+1/B}}$$

for some finite B, and large n.

2. Two Known Lemmas

We now state two lemmas,

Lemma 1—If $M_n^{(1)}$ is the average number of real roots in the interval $(-1, 1)$, $M_n^{(2)}$ the average number of roots outside $(-1, 1)$ and M_n the average number of roots in the interval $(-\infty, \infty)$ then

$$M_n^{(1)}=M_n^{(2)} \tag{2.1}$$

and hence

$$M_n=2M_n^{(1)}. \tag{2.2}$$

Lemma 2—If $f(x)$, continuous for $a\leq x\leq b$ and continuously differentiable for $a<x<b$, has fiinite number of turning points (that is only a finite number of points at which the derivative of $f(x)$ vanishes in (a, b)) then the number of zeros of $f(x)$ in (a, b) is given by

$$M_n(a, b; f)=(2\pi)^{-1}\int_{-\infty}^{\infty}d\xi\int_a^b\cos[\xi f(x)]\,|f'(x)|\,dx \tag{2.3}$$

where $f'(x)$ denotes the derivative of $f(x)$. Multiple zeros are counted once and if a or b is a zero it is counted as 1/2.

The proofs of Lemmas 1 and 2 are given by Kac (1959).

3. A Transformation

Introduce a nonsingular linear transformation $\bar{a}=C\bar{b}$, where $C=(c_{ij})_{nn}$ and $\bar{b}'=(b_0, b_1,\ldots, b_{n-1})$ such that $C'\Omega C=I$ in (1.2). Then it reduces to

$$(1/|C|)(2\pi)^{-n/2}\exp[(-1/2)(b_0^2+b_1^2+\cdots+b_{n-1}^2)]. \tag{3.1}$$

If $T'=(1, x,\ldots, x^{n-1})$ under this transformation

$$f(x)=T'Cb=\sum_{k=0}^{n-1}(c_{0k}+c_{1k}x+\ldots c_{n-1\,k}x^{n-1})b_k$$

$$=\sum_{k=0}^{n-1}x_k\,b_k \tag{3.2}$$

where

$$X_k = c_{0k} + c_{1k}x + \cdots c_{n-1k}x^{n-1}. \tag{3.3}$$

Since $C'\Omega C = I$, $CC' = \Omega^{-1}$. Equating the corresponding elements

$$c_{i0}c_{j0} + c_{i1}c_{j1} + \ldots + c_{in-1}c_{jn-1} = \begin{cases} 1 & \text{if } i = j \\ \rho & \text{if } i \neq j \\ & i, j = 0, 1, \ldots, n-1. \end{cases} \tag{3.4}$$

4. To Determine $E[N_n(-\infty, \infty; f)]$

The average number of real roots of (1.1) in $(-\infty, \infty)$ is

$$M_n = (2\pi)^{-n/2} \sqrt{|\Omega|} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} N_n(-\infty, \infty; f) \exp[(-1/2)\, \bar{a}'\Omega\bar{a}] \times$$
$$\times da_0\, da_1 \ldots da_{n-1}. \tag{4.1}$$

Using (3.1) we get

$$M_n^{(1)} = (2\pi)^{-n/2} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} N_n^{(1)}(-1, 1; f)$$
$$\exp[(-1/2)(b_0^2 + b_1^2 + \ldots b_{n-1}^2)]\, db_0\, db_1 \ldots db_{n-1}.$$

From Lemma 2 and the identity

$$|y| = (\pi^{-1}) \int_{-\infty}^{\infty} (1 - \cos \eta y)\, \eta^{-2}\, d\eta,$$

we find

$$M_n^{(1)} = (2\pi)^{-1} \int_{-1}^{1} dx \int_{-\infty}^{\infty} R_n(\xi, x)\, d\xi \tag{4.2}$$

where

$$R_n(\xi, x) = \pi^{-1} \int_{-\infty}^{\infty} \eta^{-2}\, d\eta\, (2\pi)^{-n/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}$$
$$\exp[(-1/2)(b_0^2 + b_1^2 + \cdots b_{n-1}^2)]\, [\cos \xi f(x)$$
$$- \cos(\xi f(x)) \cos(\eta f'(x))]\, db_0\, db_1 \ldots db_{n-1}. \tag{4.3}$$

Since

$$\cos[\xi f(x)] \cos[\eta f'(x)] = (1/2)\, \mathrm{Re}\, [\exp i(\xi f(x) + \eta f'(x))$$
$$+ \exp i(\xi f(x) - \eta f'(x))]$$

$$(2\pi)^{-n/2} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \{\exp[(-1/2)(b_0^2 + b_1^2 + \ldots + b_{n-1}^2)] \times$$
$$\times \cos[\xi f(x)] \cos[\eta f'(x)]\}\, db_0\, db_1 \ldots db_{n-1}$$

$$= 1/2\, Re\, \{(2\pi)^{-n/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp[(-1/2)(b_0^2 + b_1^2 + \cdots + b_{n-1}^2)]$$

$$[\exp i \sum_{k=0}^{n-1} (\xi X_k + \eta X_k')\, b_k + \exp i \sum_{k=0}^{n-1} (\xi X_k - \eta X_k')\, b_k] \times$$
$$\times db_0\, db_1 \ldots db_{n-1}\} \tag{4.4}$$

where X_k is as in (3.3) and X_k' is the derivative of X_k with respect to x

$$=(1/2)\,[\exp(-1/2)(U_0^2+U_1^2+\dots U_{n-1}^2)+\exp(-1/2)(V_0^2+V_1^2+\dots+V_{n-1}^2)] \tag{4.5}$$

where

$$U_k = \xi X_k+\eta X_k'\,;\; V_k=\xi X_k-\eta X_k'\,;\quad k=0,1,\dots n-1. \tag{4.6}$$

Putting $\eta=0$ in (4.4), we get

$$(2\pi)^{-n/2}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp(-1/2)(b_0^2+b_1^2+\cdots+b_{n-1}^2)$$
$$\cos[\,\xi f(x)]\,db_0\,db_1\dots db_{n-1}$$
$$=\exp[(-1/2)(W_0^2+W_1^2+\cdots+W_{n-1}^2)] \tag{4.7}$$

where

$$W_k=\xi X_k,\ k=0,\ 1,\dots,n-1 \tag{4.8}$$

Therefore

$$R_n(\xi,x)=\pi^{-1}\int_{-\infty}^{\infty}\eta^{-2}\,d\eta\,\{\exp[(-\tfrac{1}{2})(W_0^2+W_1^2+\dots+W_{n-1}^2)]$$
$$-\exp[(-\tfrac{1}{2})(V_0^2+V_1^2+\dots+V_{n-1}^2)]\}$$
$$=\pi^{-1}\int_{-\infty}^{\infty}\eta^{-2}\,[\exp((-\tfrac{1}{2})\,\xi^2\sum_{k=0}^{n-1}X_k^2)$$
$$-\exp(-\tfrac{1}{2}\sum_{k=0}^{n-1}(\xi X_k-\eta X_k')^2)]\,d\eta$$
$$=\pi^{-1}\int_{-\infty}^{\infty}\eta^{-2}[\exp((-\tfrac{1}{2})\,\xi^2A)$$
$$-\exp(-\tfrac{1}{2})(\xi^2A-2\xi\eta B+\eta^2C)]\,d\eta \tag{4.9}$$

where

$$A=\sum_{k=0}^{n-1}X_k^2$$

$$B=\sum_{k=0}^{n-1}X_kX_k'$$

$$C=\sum_{k=0}^{n-1}X_k'^2.$$

Using (3.3) and (3.4)

$$A=(1-\rho)\sum_{k=0}^{n-1}x^{2k}+\rho\left(\sum_{k=0}^{n-1}x_k\right)^2$$

$$B=(1-\rho)\sum_{k=0}^{n-1}kx^{2k-1}+\rho\left(\sum_{k=0}^{n-1}kx^{k-1}\right)\left(\sum_{j=0}^{n-1}x^j\right)$$

$$C=(1-\rho)\sum_{k=0}^{n-1}k^2x^{2k-2}+\rho\left(\sum_{k=0}^{n-1}kx^{k-1}\right)^2$$

which can be written for $x<1$ as

$$\left.\begin{aligned} A &= (1-\rho)\frac{1-x^{2n}}{1-x^2}+\rho\left(\frac{1-x^n}{1-x}\right)^2 \\ B &= (1-\rho)\frac{x-x^{2n+1}-n(1-x^2)x^{2n-1}}{(1-x^2)^2} \\ &\quad +\rho\,\frac{1-x^n-n(1-x)x^{n-1}}{(1-x)^2}\cdot\frac{1-x^n}{1-x} \\ C &= (1-\rho)\frac{(1+x^2)(1-x^{2n})-n^2(1-x^2)x^{2n-2}-2n(1-x^2)x^{2n}}{(1-x^2)^3} \\ &\quad +\rho\left[\frac{1-x^n-n(1-x)x^{n-1}}{(1-x)^2}\right]^2; \end{aligned}\right\} \tag{4.10}$$

We assume that (α, β) is an interval in which $(AC-B^2)>0$ (we have always $AC-B^2\geqslant 0$ by Cauchy inequality).

It can be shown

$$\int_{-\infty}^{\infty} d\xi R_n(\xi, x) = \frac{2\sqrt{AC-B^2}}{A}$$

and hence

$$M_n^{(1)} = \pi^{-1}\int_{-1}^{1}\frac{\sqrt{AC-B^2}}{A}\,dx.$$

Therefore, from Lemma 1

$$M_n = 2\pi^{-1}\int_{-1}^{1}\frac{\sqrt{AC-B^2}}{A}\,dx \tag{4.11}$$

5. To Find the Asymptotic Value of M_n

We can write

$$\sqrt{AC-B^2}/A = [G(x)/(1-x^2)] \tag{5.1}$$

where

$$G(x) = [(1-\rho)^2\phi_1(x)+\rho(1-\rho)\phi_2(x)]^{1/2}/\phi_3(x) \tag{5.2}$$

where

$$\left.\begin{aligned} \phi_1(x) &= (1-x)^2[(1-x^{2n})\{(1+x^2)(1-x^{2n})-n^2(1-x^2)^2x^{2n-2} \\ &\quad -2n(1-x^2)x^{2n}\}-(x-x^{2n+1}-n(1-x^2)x^{2n-1})^2] \\ \phi_2(x) &= (1-x^{2n})(1-x^n-n(1-x)x^{n-1})^2(1-x)(1+x)^3 \\ &\quad +(1-x^n)^2(1-x^2)\{(1+x^2)(1-x^{2n})-n^2(1-x^2)^2x^{2n-2} \\ &\qquad -2n(1-x^2)x^{2n} \\ &\quad -2[(1-x^n)(1+x)^2(1-x)(x-x^{2n+1}-n(1-x^2)x^{2n-1}) \\ &\qquad .(1-x^n-n(1-x)x^{n-1})] \\ \text{and}\quad & \\ \phi_3(x) &= (1-\rho)(1-x^{2n})(1-x)+\rho(1-x^n)^2(1+x). \end{aligned}\right\} \tag{5.3}$$

Integrating (5.1) we get

$$\int_{-1}^{1} \sqrt{AC-B^2}/A\, dx = I_1+I_2+I_3+I_4$$

where

$$\left.\begin{aligned} I_1 &= \int_0^{1-1/n} [G(x)/(1/x^2)]\, dx \\ I_2 &= \int_0^{1-1/n} [G(-x)/(1-x^2)]\, dx \\ I_3 &= \int_{1-1/n}^{1} [G(x)/(1-x^2)]\, dx \\ \text{and}\quad I_4 &= \int_{1-1/n}^{1} [G-x)/(1-x^2)]\, dx \end{aligned}\right\} \qquad (5.4)$$

Using mean value theorem for integrals

$$I_1+I_2 = [G(c_n)+G(-c_n)] \int_0^{1-1/n} [1/1-x^2]\, dx,\ 0\leqslant c_n\leqslant 1-1/n$$

$$= (1/2)[G(c_n)+G(-c_n)][\log(2-1/n)+\log n]. \qquad (5.5)$$

For x in $(1-1/n, 1)$

$$\phi_1(x)<2(1-x)$$

$$\phi_2(x)<(2^3+2^3)(1-x)$$

and

$$\phi_3(x)>\rho.$$

Therefore

$$G(x)<K(1-x)^{1/2}$$

where

$$K = [2(1-\rho)^2+\rho(1-\rho)(2^3+2^3)]^{1/2}/\rho. \qquad (5.6)$$

Hence we get

$$I_3<K\int_{1-1/n}^{1} (1-x)^{-1/2}\, dx = 2K/\sqrt{n} \qquad (5.7)$$

Since

$$\begin{aligned}\phi_1(-x) &= (1+x)^2 [(1-x^{2n})\{(1+x^2)(1-x^{2n})-n^2(1-x^2)x^{2n-2} \\ &\quad -2n(1-x^2)x^{2n}\}-(x-x^{2n+1}-n(1-x^2)x^{2n-1})^2] \\ &<(1+x)^2[(1-x)(1+x+\ldots+x^{2n-1})\{(1+x^2)(1-x^{2n}) \\ &\quad -n^2(1-x^2)x^{2n-2}-2n(1-x^2)x^{2n}\}] \qquad <2^4n(1-x)\end{aligned}$$

$$\phi_2(-x)=(1-x^{2n})(1-x^n+n(1+x)x^{n-1})^2(1+x)(1-x)^3+$$
$$+[(1-x^n)^2(1-x^2)\{(1+x^2)(1-x^{2n})-n^2(1-x^2)^2x^{2n-2}-$$
$$-2n(1-x^2)x^{2n}\}]+2[(1-x^n)(1-x)^2(1+x)(x-x^{2n-1}-$$
$$-n(1-x^2)x^{2n-1})(1-x^n+n(1+x)x^{n-1})]$$
$$=[(1-x)(1+x+\cdots+x^{2n-1})(1-x^n+n(1+x)x^{n-1})^2\times$$
$$\times(1+x)(1-x)^3]+(1-x^n)^2(1-x^2)[(1+x^2)(1-x)(1+x+\cdots+$$
$$+x^{2n-1})-n^2(1-x^2)^2x^{2n-1}-2n(1-x^2)x^{2n}]+2(1-x^n)\times$$
$$\times(1-x)^2(1+x)x[(1-x)(1+x+\cdots+x^{2n-1})-$$
$$-n^2(1-x^2)^2x^{2n-1}-2n(1-x^2)x^{2n}]+2(1-x^n)(1-x)^2\times$$
$$\times(1+x)x[(1-x)(1+x+\cdots+x^{2\ -1})-n(1-x^2)x^{2n-2}]\times$$
$$\times(1-x^n+n(1+x)x^{n-1})$$
$$<2^2n(1-x)+2^3n(1-x)+2^3n(1-x)\qquad<2^5{}_n(1-x)$$

and

$$\phi_3(-x)=(1-\rho)(1-x^{2n})(1+x)+\rho(1-x^n)^2(1-x)<(1-\rho)$$

we get

$$G(-x)<K_1n^{1/2}(1-x)^{1/2}$$

where

$$K_1=[(1-\rho)^2\,2^4+\rho(1-\rho)\,2^5]^{1/2}/(1-\rho). \tag{5.8}$$

Therefore

$$I_4<K_1n^{1/2}\int_{1-1/n}^{1}(1-x)^{-1/2}\,dx=2\,k_1. \tag{5.9}$$

Hence from (5.5), (5.7) and (5.9)

$$\int_{-1}^{1}[G(x)/(1-x^2)]\,dx<(1/2)[G(c_n)+G(-c_n)][\log n+\log(2-1/n)]+$$
$$+2Kn^{-1/2}+2K_1. \tag{5.10}$$

Since $G(x)$ and $G(-x)$ are always greater than zero

$$\int_{-1}^{1}G[(x)/(1-x^2)]\,dx>[G(c_n)+G(-c_n)]\int_{0}^{1-1/n}[1/(1-x^2)]\,dx$$
$$=(1/2)[G(c_n)+G(-c_n)][\log n+\log(2-1/n)]. \tag{5.11}$$

Hence for large n from (5.10) and (5 11) we find

$$2\pi^{-1}\int_{-1}^{1}[G(x)/(1-x^2)]\,dx\sim\pi^{-1}K_2\log n \tag{5.12}$$

where k_2 is the asymptotic value of $[G(c_n)+G(-c_n)]$.

To determine c_n for large n we proceed as follows :

Kac (1943) has shown that when a_i's are independent

$$M_n \sim 2\pi^{-1} \log n$$

Hence when $\rho=0$ we should get for large n

$$[G(c_n)+G(-c_n)] \sim 2. \tag{5.13}$$

Since for large n

$$G(c_n) \sim \frac{[(1-\rho^2)-2c_n(1-\rho)^2+c_n^2(1-\rho)(1-3\rho)]^{1/2}}{(1-\rho)(1-c_n)+\rho(1+c_n)}$$

and

$$G(-c_n) \sim \frac{[(1-\rho^2)+2c_n(1-\rho)^2+c_n^2(1-\rho)(1-3\rho)]^{1/2}}{(1-\rho)(1+c_n)+\rho(1-c_n)}$$

we get

$$\frac{[(1-\rho^2)-2c_n(1-\rho)^2+c_n^2(1-\rho)(1-3\rho)]^{1/2}}{1+3\rho c_n}$$

$$+\frac{[(1-\rho^2)+c_n^2(1-\rho)(1-3\rho)]^{1/2}}{1+c_n}$$

$$< [G(c_n)+G(-c_n)]$$

$$< \frac{[(1-\rho^2)+c_n^2(1-\rho)(1-3\rho)]^{1/2}}{1-c_n}$$

$$+\frac{[(1-\rho^2)+2c_n(1-\rho)^2+c_n^2(1-\rho)(1-3\rho)]^{1/2}}{1-3\rho c_n}$$

If $\rho=0$

$$(1-c_n)+[(1+c_n^2)^{1/2}/(1+c_n)] < [G(c_n)+G(-c_n)]$$
$$< [(1+c_n^2)^{1/2}/(1-c_n)]+(1+c_n)$$

(5.13) shows that we should get

$$(1-c_n)+[(1+c_n^2)/(1+c_n)] \to 2,\ n\to\infty,$$

and

$$[(1+c_n^2)^{1/2}/(1-c_n)]+(1+c_n)\to 2,\ n\to\infty.$$

which will be true only if $c_n\to 0$ as $n\to\infty$.

Hence

$$M_n \sim (2/\pi)(1-\rho^2)^{1/2} \log n,$$

since as $n\to\infty$, K_2, the asymptotic value of $[G(c_n)+G(-c_n)]$ is $2(1-\rho^2)/^{1/2}$.

Remark : When $\sigma_i = \sigma, i = 0, 1, \ldots, n-1$ (3.4) is

$$c_{i0}c_{j0} + c_{i1}c_{j1} + \ldots + c_{in-1}c_{jn-1} = \begin{cases} \sigma^2 & \text{if } i = j \\ \rho\sigma^2 & \text{if } i \neq j \\ & i, j = 0, 1, \ldots, n-1 \end{cases}$$

Here also $\sqrt{AC-B^2}/A$ is same as in (5.1) and hence when $\sigma_1 = \sigma$, M_n is unchanged.

Acknowledgements

The author is deeply indebted to Prof G. Sankaranarayanan for his very valuable and inspiring guidance throughout the preparation of this paper. He thanks U. G. C. for the financial help during the preparation of this paper.

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