ON THE ORDINARY AND THE ABSOLUTE TAUBERIAN CONSTANTS FOR HAUSDORFF TRANSFORMATIONS

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This paper is concerned with introducing an equivalence relation between a sequence-to-sequence general transformation $t_n = \sum_{k=0}^{\infty} \alpha_{n,k} s_k$ and a series-to-series general transformation $b_n = \sum_{k=0}^{\infty} \gamma_{n,k} a_k$, where $t_n = b_0 + b_1 + \ldots + b_n$ and $s_k = a_0 + a_1 + \ldots + a_k$ in the sence that if t_n converges for all n, then so does b_n and conversely. It has been deduced from this result that the sequence-to-sequence Hausdorff transform $\sigma_n = \sum_{k=0}^{n} \binom{n}{k} (\Delta^{n-k} \mu_k) s_k$, $n = 0, 1, 2, \ldots$ is equivalent to the series-to-series Hausdorff transform $d_n = \frac{1}{n} \sum_{k=1}^{n} \binom{n}{k} k (\Delta^{n-k} \mu_k) a_k$, $n = 1, 2, 3, \ldots$, $d_0 = \mu_0 a_0$ where $\sigma_n = d_0 + d_1 + \ldots + d_n$. It has also been deduced that the ordinary Tauberian constants introduced by various authors for σ_n apply also to d_n and that the absolute Tauberian constants introduced by the author for d_n apply also to σ_n.

§1. Let $\{\mu_n\}_{n=0}^{\infty}$ be a fixed sequence of real or complex numbers. The Hausdorff transform $\{\sigma_n\}$ of a sequence $\{s_n\}$ by means of the fixed sequence $\{\mu_n\}_{n=0}^{\infty}$ [or, in short, the (H, μ_n) transform] is given by

$$\sigma_n = \sum_{k=0}^{n} \binom{n}{k} (\Delta^{n-k} \mu_k) s_k, \; n = 0, 1, 2, \ldots \qquad \ldots(1.1)$$

where for $r, q \geqslant 0$,

$$\Delta^0 \mu_q = \mu_q, \; \Delta\mu_q = \mu_q - \mu_{q+1}; \;\; \Delta^{r+1} \mu_q = \Delta (\Delta^r \mu_q). \qquad \ldots(1.2)$$

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Knopp and Lorentz (1949) have shown (a simpler proof was given by Jakimovski [1960, eqn. (2.1)]) that if (1.1) and (1.2) hold and if

$$\sigma_n = d_0 + d_1 + \ldots + d_n; \; s_k = a_0 + a_1 + \ldots + a_h, \qquad \ldots(1.3)$$

then the series-to-series Hausdorff transform d_n of $\Sigma\, a_n$+ is such that

$$\begin{cases} d_0 = \mu_0 a_0 \\ d_n = \dfrac{1}{n} \displaystyle\sum_{k=1}^{n} \binom{n}{k} k\,(\Delta^{n-k}\,\mu_k)\, a_k,\; n = 1, 2, 3, \ldots \end{cases} \qquad \ldots(1.4)$$

It has been found that theorems of the following type hold. Suppose that $\alpha > 0$, and that P is an integer valued function of n such that $P/n \to \alpha$ as $n \to \infty$. Jakimovski (1961) has shown that if

$$na_n = O(1), \qquad \ldots(1.5)$$

then

$$\limsup_{n\to\infty} |\,\sigma_n - s_p\,| \leqslant A \limsup_{n\to\infty} |\,na_n\,|,$$

where A is a Tauberian constant.

Sherif (1965) has also shown that if

$$\delta_n = O(1), \qquad \ldots(1.6)$$

where

$$\delta_n = \frac{1}{n+1} \sum_{\nu=1}^{n} \nu a_\nu,$$

then

$$\limsup_{n\to\infty} |\,\sigma_n - s_p\,| \leqslant A' \limsup_{n\to\infty} |\,\delta_n\,|, \qquad \ldots(1.7)$$

where A' is a Tauberian constant which has been determined for certain special types of the transformation (1.1).

Meir (1967) has obtained an estimate of a similar type to (1.7) but for the difference between two transforms of certain classes of the transformation (1.1) instead.

Biegert (1968) has also obtained a similar estimate to (1.7) but with (1.6) replaced by a condition of Schmidt's type

$$\limsup_{p\to\infty} \max_{|q-p| \leqslant \lambda p} \left|\, s_q - s_p \,\right| \leqslant \lambda \,.\, L,$$

with fixed L in $0 \leqslant L < \infty$ and any $\lambda > 0$.

+ Unless otherwise indicated, the symbol Σ stands for $\sum_{0}^{\infty}$.

Another type of estimates of a new feature have been found by Sherif (1974). The estimates were of the forms

$$\sum |d_n - a_n| \leqslant K \sum |\Delta (na_n)|$$

$$\sum |d_n - a_n| \leqslant K' \sum \left|\Delta\left(\frac{1}{n} \sum_{\nu=1}^{n-1} \nu a_\nu\right)\right|$$

where K and K' are absolute Tauberian constants.

In § 2 of this paper we obtain a relation between a sequence-to-sequence and a series-to-series general transformation from which we deduce that the sequence-to-sequence transformation (1.1) is exactly equivalent to the series-to-series transformation (1.4).

We also deduce some results concerning the above mentioned Tauberian theorems of Biegert (1968), Jakimovski (1961), Meir (1967) and Sherif (1965, 1974).

§ 2. *Theorem*—Let any sequence-to-sequence transformation t_n be such that

$$t_n = \sum_{k=0}^{\infty} \alpha_{n,k}\, s_k. \qquad \text{...(2.1)}$$

Suppose that

$$\beta_{n,k}\, s_k \to 0, \quad \text{as} \quad k \to \infty \qquad \text{...(2.2)}$$

where

$$\beta_{n,k} = \sum_{\nu=k}^{\infty} \alpha_{n,\nu}\,. \qquad \text{...(2.3)}$$

Then (2.1) and the transformation

$$b_n = \sum_{k=0}^{\infty} \gamma_{n,k}\, a_k \qquad \text{...(2.4)}$$

where

$$\gamma_{0,k} = \beta_{n,k},$$

$$\gamma_{n,k} = \beta_{n,k} - \beta_{n-1,k},$$

are equivalent (in the sense that if (2.1) converges for all n, then so does (2.4), and conversely, and that the sums are related as in (1.3).

Proof: If the order of summation may be inverted, (2.1) gives

$$t_n = \sum_{\nu=0}^{\infty} \alpha_{n,\nu}\, s_\nu = \sum_{\nu=0}^{\infty} \alpha_{n,\nu} \sum_{k=0}^{\nu} a_k$$

$$= \sum_{k=0}^{\infty} a_k \sum_{\nu=k}^{\infty} \alpha_{n,\nu}$$

$$= \sum_{k=0}^{\infty} \beta_{n,k}\, a_k. \qquad ...(2.5)$$

But although (2.1), (2.5) may be regarded as formally equivalent, they are not necessarily equivalent, since the inversion need not be justified. On the other hand (2.5) is precisely equivalent to the series-to-series transformation (2.4) in the sense that if (2.5) converges for all n then so does (2.4), and conversely, and that the sums are related as in (1.3). For if (2.5) holds for all n we obtain (2.4) by subtracting (2.5) with n replaced by $n-1$ from (2.5) as it stands, if (2.4) holds we obtain (2.5) by adding the equations obtained from (2.4) by replacing n by $0, 1, 2, \ldots, n$. (As we are adding a finite number of convergent series, the result converges). [Of course (2.5) might converge for a particular value of n without (2.4) converging for that n or conversely].

Now assuming only that (2.3) converges, so that $\beta_{n,k}$ is defined, we have

$$\sum_{k=0}^{K} \beta_{n,k}\, a_k = \sum_{k=0}^{K} \beta_{n,k}(s_k - s_{k-1})$$

$$= \sum_{k=0}^{K-1} (\beta_{n,k} - \beta_{n,k+1})\, s_k + \beta_{n,K}\, s_K$$

$$= \sum_{k=0}^{K-1} \alpha_{n,k}\, s_k + \beta_{n,K}\, s_K. \qquad ...(2.6)$$

Using (2.2), we see on making $K \to \infty$ in (2.6) that

$$\sum_{k=0}^{\infty} \beta_{n,k}\, a_k = \sum_{k=0}^{\infty} \alpha_{n,k}\, s_k.$$

Whenever either side exists. Hence (2.1) is precisely equivalent to (2.5), and hence, by what has already been said, to (2.4), for those sequences whch are such that (2.2) holds for every fixed n.

Remark: Now, if the transformation (2.1) is row finite then for any fixed n, $\beta_{n,k} = 0$ for all sufficiently large k, so that (2.2) must hold for every $\{s_k\}$. Thus, in this case (2.1), (2.4) and (2.5) are always exactly equivalent. In the Hausdorff case we do have a row finite transformation [c.f. (1.1)], so that (1.1), (1.4) are always equivalent without any condition.

Thus, we can deduce that the ordinary Tauberian constants determined by Biegert (1968), Jakimovski (1961), Meir (1967) and Sherif (1965) for the sequence-to-sequence transformation (1.1) apply also to the series-to-series transformation (1.4), and that the absolute Tauberian constants determined by Sherif (1974) for the series-to-series transformation (1.4) apply also to the sequence-to-sequence transformation (1.1).

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