A NOTE ON VARIETIES GENERATED BY HILBERT SPACES

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Let $\mathcal{C}$ be a class of Hilbert spaces and $V(\mathcal{C})$ the variety generated by $\mathcal{C}$. In this note we investigate if every closed subspace of a Fréchet space in $V(\mathcal{C})$ can be complemented or not. We also investigate the conditions under which spaces in $V(\mathcal{C})$ are boundedly generated.

All spaces in this note are linear spaces with locally convex Hausdorff topologies.

Following Diestel *et al.* (1972), we call a class V of locally convex Hausdorff spaces a 'variety ' if it is closed under the operations of taking subspaces, separated quotients, (arbitrary) products and isomorphic images. Let $\mathcal{C}$ be a class of locally convex Hausdorff spaces and let $V(\mathcal{C})$ be the intersection of all varieties containing $\mathcal{C}$. Then $V(\mathcal{C})$ is said to be the 'variety generated' by $\mathcal{C}$. Also (*a*) $S(\mathcal{C})$, (*b*) $P_c(\mathcal{C})$, (*c*) $Q(\mathcal{C})$ and (*d*) $P(\mathcal{C})$ denote, respectively, the class of all locally convex spaces isomorphic to (*a*) subspaces of locally convex spaces in $\mathcal{C}$, (*b*) countable products of families of locally convex spaces in $\mathcal{C}$, (*c*) quotient spaces of locally convex spaces in $\mathcal{C}$ and (*d*) products of finite families of locally convex spaces in $\mathcal{C}$. Let l_2 denote the Hilbert space of square summable sequences.

Diestel *et al.* (1972, cor. 4.16) have proved that a Fréchet space in $V(\mathcal{F})$ where $\mathcal{F}$ is a class of separable Fréchet spaces is separable. However we have the following more general result by a simpler proof.

Proposition 1.1—Let $\mathcal{C}$ be a class of separable locally convex Hausdorff spaces. Then a metrizable space in $V(\mathcal{C})$ is separable. In particular every metrizable space in $V(l_2)$ is separable.

PROOF : Let $E \in V(\mathcal{C})$ be metrizable. Then $E \in SP_c\, QP(\mathcal{C})$ (Diestel *et al.* 1972, Th. 4.1). Now finite products of separable space are separable. Also a separated quotient of a separable space is separable and a countable product of separable spaces is separable. So $P_c\, QP(\mathcal{C})$ contains only separable locally convex spaces. Since every metrizable subspace of a separable linear topological space is separable (Lohman and Stiles 1974, Theorem), E is separable.

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Ito and Seidman (1968) have defined and considered 'boundedly generated' (BG) spaces. A locally convex Hausdorff space E is a BG space if and only if E is the closed linear hull of a bounded set in E. They have shown that a Fréchet space has a precompact generating set if and only if it is separable. Also it has been deduced from this result that a separable Fréchet space is a BG space. Their proof works to show the following result.

Proposition 1.2—A separable metrizable space is a BG space. Combining Propositions 1.1 and 1.2 we have the result:

Proposition 1.3—Any metrizable space in $V(\mathcal{C})$ where $\mathcal{C}$ is a class of separable locally convex Hausdorff spaces is a BG space. In particular every metrizable space in $V(l_2)$ is a BG space.

On the other hand, assuming the continuum hypothesis every variety generated by a non-separable Hilbert space contains a non-BG Fréchet space. It can be seen from the following example considered by Ito and Seidman (1968).

Example 1.4—Let $\Lambda = \{\lambda = [\lambda_1, \lambda_2, \ldots] : 1 = \lambda_1 < \lambda_2 < \ldots, \lambda_n \text{ integers}\}$ and let H be a Hilbert space big enough to contain an orthonormal set $\{a_\lambda : \lambda \in \Lambda\}$. Let $X = \prod_{n \in N} H_n$ with $H_n = H$ for each $n \in N$ (the set of natural nos.). So $X \in V(H)$. Also X is a Fréchet BG space. For each $\lambda \in \Lambda$, set $b_\lambda = [a_\lambda, \lambda_2 a_\lambda, \lambda_3 a_\lambda, \ldots]$ $\in X$ and let y be the closed linear hull of $\{b_\lambda : \lambda \in \Lambda\}$. Then the Fréchet space $y \in V(H)$ and is non-BG (see Ito and Seidman 1968, Ex. 2).

However $V(l_2)$ also contains non-BG spaces. Following is an example to show the same.

Example 1.5—Consider the nuclear space $\mathcal{D}$ of test functions for distributions. Since the variety generated by any infinite dimensional Banach space contains the variety $\mathcal{N}$ of nuclear spaces (Diestel *et al.* 1972, Th. 3.1), $\mathcal{D} \in V(l_2)$. But it being the strict inductive limit of Fréchet spaces is non-BG (Chilana 1970, Proposition 3).

Lindenstrauss and Tzafriri (1971) have shown that a Banach space is isomorphic to a Hilbert space provided every closed subspace is complemented. So the result (Diestel *et al.* 1972, Th. 4.4) that for a class $\mathcal{C}$ of Hilbert spaces, a Banach space B in $V(\mathcal{C})$ is isomorphic to a Hilbert space can be interpreted as: every closed subspace of a Banach space in $V(\mathcal{C})$ can be complemented. This property cannot be extended to Fréchet spaces in $V(\mathcal{C})$ as shown in the next two examples.

Example 1.6—The continuous image, under a linear map of a BG space is BG (Ito and Seidman 1968, Th. 2). Thus, if a BG space has a non-BG closed subspace then this cannot be complemented. So the Fréchet space X of Example 1.4 is in $V(H)$ and it has a closed subspace Y which cannot be complemented.

Example 1.7—$V(l_2)$ contains other types of Fréchet spaces in which there are closed subspaces that cannot be complemented. Let $\mathcal{D}_1$ be the space of real-valued infinitely differentiable functions f on the real line vanishing outside the closed interval $I = [0, 1]$, endowed with the topology of uniform convergence on I, in derivatives of order $\geqslant 0$ (Schaefer 1966, page 106, Example II. 8.1). Then $\mathcal{D}_1$ is a non-normable nuclear Fréchet space in $V(l_2)$ on which there exists a continuous norm. So $\mathcal{D}_1$ has a closed subspace y which cannot be complemented (see Schaefer 1966, pp. 192–193, Ex. 13). We just note that both $\mathcal{D}_1$ and y are separable Fréchet spaces and hence BG.

Grothendieck (1955) has proved that the only separable infinite dimensional Fréchet spaces in which every closed subspace is complemented are: s, $s \oplus l_2$ and l_2, where s is the space of all sequences of reals (i.e. the product of a sequence of lines). So we have the following result for separable Fréchet spaces.

Proposition 1.8—All separable Fréchet spaces in which every closed subspace can be complemented are in the variety (Vl_2).

PROOF: We will assume the space to be infinite-dimensional, for otherwise the assertion is trivial. Since for finite index set direct sum is same as product, in view of Grothendieck's result mentioned above, it suffices to show that $s \in V(l_2)$. Now $V(R)$ (R = Real line) is the smallest variety (Diestel *et al.* 1972, Th. 3.6), so $R \in V(l_2)$. Therefore $s (= R^{N_0}) \in V(l_2)$.

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