# ON DISTANCE SETS

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The object of this paper is to study different conditions under which two sets may be distance sets. The spaces considered are metric spaces and metric linear spaces.

## 1 Introduction

In normed linear spaces the conditions under which two sets are distance sets have been studied by Bor-Luh-Lin (1966), Dionisio (1963/64), Tukey (1942), Klee (1955), Pai (1972) and others. An application of distance sets to linear inequalities has been cited by Cheney and Goldstein (1959). The same problem in metric spaces has been studied by Nicolescu (1938) and is also discussed by Singer (1970). In this paper the author has also studied different conditions under which two sets may be distance sets. The study is confined to metric spaces and metric linear spaces.

*Definition*—A subset $G$ of a metric space $(X, d)$ is said to be 'proximinal' if for each $p \in X$, there exists a point $g \in G$ which is nearest to $p$. *i.e.*,

$$d(p, g) = d(p, G).$$

The subset $G$ is said to be proximinal with respect to a subset $H$ of $X$ if to each point $p$ of $H$ there is a point in $G$ nearest to $p$.

*Definition*—Two sets $G_1$ and $G_2$ in a metric space $(X, d)$ are said to be distance sets' if there exist elements $g_1 \in G_1$, $g_2 \in G_2$ such that

$$d(g_1, g_2) = d(G_1, G_2) = \inf_{\substack{x \in G_1 \\ y \in G_2}} d(x, y).$$

The points $g_1$ and $g_2$ are called 'proximinal points' of the sets $G_1$ and $G_2$.

Concerning distance sets *M*. Nicolescu (1938) proved that "any two compact sets in a metric space are distance sets".

Singer (1970) has given its generalization in the following form :

If $G_1$ and $G_2$ are two non-void boundedly compact closed sets in a metric space $(X, d)$ then there exist elements $g_1 \in G_1$ and $g_2 \in G_2$ such that

$$d(g_1, g_2) = d(G_1, G_2).$$

Consider the following example :

Let $(R, \rho)$ be the real line with $\rho$ as the usual metric and let

$$G_1 = \{n : n \in N\} \text{ and } G_2 = \left\{n + \frac{1}{2^n} : n \in N\right\},$$

$N$ denotes the set of natural numbers.

Here $G_1$ and $G_2$ are two non-void boundedly compact closed set in $R$ with $\rho(G_1, G_2) = 0$. Since $G_1 \cap G_2$ is empty there do not exist two points $g_1 \in G_1$, $g_2 \in G_2$ for which $\rho(g_1, g_2) = 0$.

The failure of the theorem in the above case suggests that there is some slip in the statement of the theorem. This difficulty is overcome by taking one of the sets to be bounded. The correct statement of the theorem reads as follows :

"If $G_1$ and $G_2$ are two sets in a metric space $(X, d)$ such that $G_1$ is compact and $G_2$ is boundedly compact closed set then there exist elements $g_1 \in G_1$, $g_2 \in G_2$ such that

$$d(g_1, g_2) = d(G_1, G_2).\text{''}$$

## 2. Distance Sets

*Definition*—A strongly convex (Rolfsen 1967) metric space $(X, d)$ is said to be 'uniformly convex' (Ahuja *et al.*, 1974) if there corresponds to each pair of positive numbers $(\varepsilon, r)$ a positive number $\delta$ such that if $x$ and $y$ lie in $X$ with

$$d(x, y) \geqslant \varepsilon,\ d(x, x_0) < r + \delta,\ d(y, x_0) < r + \delta \text{ then}$$

$d(z, x_0) < r$, $z$ being the mid-point of $x$ and $y$ and $x_0$ is arbitrary but fixed point of $X$.

*Theorem* 1—If $G_1$ is compact and $G_2$ is a complete convex set in a uniformly convex metric space $(X, d)$ then the two sets are distance sets.

Proof : Let $r = \inf \{d(x, y) : x \in G_1, y \in G_2\}$.

By the definition of infimum, there exists a sequence $\langle x_n \rangle$ in $G_1$ and a sequence $\langle y_n \rangle$ in $G_2$ such that

$$\lim_{n \to \infty} d(x_n, y_n) = r.$$

Since $\langle x_n \rangle$ is in $G_1$, which is compact, therefore there exists a subsequence $\langle x_{n_k} \rangle$ of $\langle x_n \rangle$ such that $\langle x_{n_k} \rangle \to x^*$ in $G_1$.

Therefore, $\lim_{k \to \infty} d(x^*, y_{n_k}) = r$.

Now, let $\varepsilon > 0$ be given. Let $\delta$ be taken as in the definition of uniform convexity. Select $N$ such that

$$d(x^*, y_{n_k}) < r + \delta \text{ whenever } n_k \geqslant N.$$

Let

$$n_k, \ m_k \geqslant N$$

so

$$d(x^*, y_{n_k}) < r + \delta, \ d(x^*, y_{m_k}) r < + \delta.$$

Since the space is uniformly convex, it is strongly convex. Therefore, $y_{n_k}$, $y_{m_k}$ have a unique mid-point, let it be $y_{l_k}$. As $G_2$ is convex (Ahuja *et al.* 1974) $y_{l_k} \in G_2$.

$$x^* \in G_1, \ y_{l_k} \in G_2 \text{ imply } d(x^*, y_{l_k}) \geqslant r.$$

Therefore by uniform convexity

$$d(y_{n_k}, y_{m_k}) < \epsilon, n_k, \ m_k \geqslant N$$

consequently $\langle y_{n_k} \rangle$ is a Cauchy sequence in $G_2$.

$G_2$ being complete, $\langle y_{n_k} \rangle \to y^* \in G_2$.

Therefore,

$$\lim_{k \to \infty} d(x^*, y_{n_k}) = d(x^*, y^*) = r.$$

Thus the two sets are distance sets.

*Corollary* 1—If $G_1$ is compact and $G_2$ is a closed convex set in a complete uniformly convex metric space then the two sets are distance sets.

*Theorem* 2—In a metric space $(X, d)$ if $G_1$ is compact and $G_2$ is proximinal with respect to $G_1$ then the two sets are distance sets.

PROOF: Let $x \in G_1$. Since the function $d(x, G_2)$ is a continuous function of $x$ on $G_1$ (Dieudonne 1960) and $G_1$ is compact, therefore it attains its minimum value $d(G_1, G_2)$ at some point of $G_1$, say at $x^*$, *i.e.*,

$$d(x^*, G_2) = \inf_{x' \in G_1} d(x, G_2) = d(G_1, G_2).$$

Since $x^* \in G_1$ and $G_2$ is proximinal with respect to $G_1$, therefore there exists $y^* \in G_2$ such that

$$d(x^*, y^*) = d(x^*, G_2) = d(G_1, G_2).$$

This proves the result.

*Remark* : Ahuja *et al.* (1974) studied some conditions under which a subset of a metric space is proximinal. Theorem 2 says that if $G_1$ is compact and $G_2$ satisfies any one of those conditions under which it is proximinal then the two sets will be distance sets.

*Definition*—A metric linear space $(X, d)$ is said to be 'totally complete' if it has the property that in it all metrically-bounded closed sets are compact.

Finite-dimensional (F)-normed spaces (Kothe 1969) are totally complete.

*Theorem* 3—If $G_1$ is compact and $G_2$ is a closed set in a totally complete metric linear space $(X, d)$ then the two sets are distance sets.

PROOF : Let $r = \inf \{d(x, y): x \in G_1, y \in G_2\}$.

By the definition of $r$, there exist sequences $\langle x_n \rangle$ and $\langle y_n \rangle$ in $G_1$ and $G_2$ respectively such that

$$\lim_{n \to \infty} d(x_n, y_n) = r.$$

$G_1$ being compact, there will exist a subsequence $\langle x_{n_k} \rangle$ of $\langle x_n \rangle$ such that $\langle x_{n_k} \rangle \to x^* \in G_1$.

Therefore

$$\lim_{k \to \infty} d(x^*, y_{n_k}) = r$$

This also implies that the sequence $\langle y_{n_k} \rangle$ is bounded and therefore the set $\{y_{n_k}\} \equiv A$ is bounded.

Now $\bar{A}$ being a bounded closed set in a totally complete metric linear space, is compact.

Let $\langle y_{n_{k_i}} \rangle$ be a subsequence of $\langle y_{n_k} \rangle$ such that $\langle y_{n_{k_i}} \rangle \to y^* \in \bar{A} \subseteq G_2$. Then

$$\lim_{k \to \infty} d(x^*, y_{n_k}) = \lim_{i \to \infty} d(x^*, y_{n_{k_i}}) = d(x^*, y^*) = r.$$

This proves the result.

The above result fails if

(i) both the sets are unbounded

(ii) one of the sets is not closed.

Example in case of real line with the usual metric $d$, let $G_1 = [1, 2]$ $G_2 = [3, 5]$.

Here $d(G_1, G_2) = 1$, but there do not exist two points $g_1 \in G_1$, $g_2 \in G_2$ such that $d(g_1, g_2) = 1$.

*Corollary* 1—If $G_1$ and $G_2$ are closed and one of them is bounded with $d(G_1, G_2) = 0$, then $G_1 \cap G_2$ is non-empty, because by Theorem 3, there exist $g_1 \in G_1, g_2 \in G_2$ such that $d(g_1, g_2) = d(G_1, G_2) = 0$ and therefore $g_1 = g_2 \in G_1 \cap G_2$.

*Corollary* 2—Any closed set $G$ in a totally complete metric linear space $(X, d)$ is proximinal, because for any $p \in X$, the two sets $\{p\}$ and $G$ satisfy conditions of the theorem.

*Corollary* 3—If an element $p$ of $X$ and a closed set $G$ are such that $d(p, G) = 0$ then $p \in G$.

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