CHARGED BODIES IN GENERAL RELATIVITY

by K. D. KRORI *and* JAYANTIMALA CHAKRABARTY, *Mathematical Physics Forum, Cotton College, Gauhati 781 001*

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We have presented here some results on test particles in the fields of charged spherical and infinitely long cylindrical bodies. These results include one putting lower limits to the size of an electron and a proton.

1. INTRODUCTION

We have recently obtained some results on the escape of photons from charged bodies (Krori and Barua 1975). We have further pursued the study of charged bodies in this paper. We have presented here an investigation on test particles in the fields of charged spherical and infinitely long cylindrical bodies. We have found from our study that the attractive force on a test particle is effective at all distances from a sphere and an infinitely long cylinder if they are uncharged but it is so only at sufficiently large distances if they are charged. In the latter case, the radius of the charged sphere must be equal to or greater than a critical value depending upon the charge and mass of the sphere (this puts lower limits to the size of an electron and a proton) and that of the charged infinitely long cylinder must be equal to or greater than a critical value depending on the charge per unit length and the mass per unit length.

We have also found that a test particle may move around a sphere in circles only with radii larger than a critical value equal to 3/2 times the Schwarzschild radius if the sphere is uncharged but only with radii larger than a lesser critical value (depending on charge and mass) if the sphere is charged. On the other hand, a test particle may move around an infinitely long cylinder in a circle of any radius if it is uncharged provided the mass per unit length is less than a certian critical limit, but in a circle with a radius smaller than a critical value (depending upon the charge per unit length and the mass per unit length) if it is charged provided the mass per unit length is less than the same critical value.

2. DETAILED CALCULATIONS

(I) *Charged Sphere*

The line element we use here is the spherically symmetric one for a charged sphere given by (Tolman 1934)

$$ds^2 = -e^{\lambda} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 + e^{\nu} dt \qquad ...(1)$$

with

$$e^{-\lambda} = e^{\nu} = \left(1 - \frac{2m}{r} + \frac{4\pi\epsilon^2}{r^2}\right) \qquad ...(2)$$

where m represents the mass and ϵ the charge of the body.

(i) When a test particle is at rest in the field,

$$\frac{dr}{ds} = \frac{d\theta}{ds} = \frac{d\phi}{ds} = 0.$$

Then, from the geodesic equations

$$\frac{d^2 x^\sigma}{ds^2} + \Gamma^\sigma_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0$$

we get

$$\frac{d^2\theta}{ds^2} = \frac{d^2\phi}{ds^2} = \frac{d^2 t}{ds^2} = 0. \qquad ...(3)$$

From the line element (1)

$$\frac{dt}{ds} = \left[1 - \frac{2m}{r} + \frac{4\pi\epsilon^2}{r^2}\right]^{-1/2}.$$

Then, from (3) we get the force experienced by a test particle at rest

$$\frac{d^2 r}{dt^2} = -\frac{\frac{m}{r^2} - \frac{4\pi\epsilon^2}{r^3}}{1 - \frac{2m}{r} + \frac{4\pi\epsilon^2}{r^2}}. \qquad ...(4)$$

(ii) When a test particle is in radial motion,

$$\frac{d\theta}{ds} = \frac{d\phi}{ds} = 0.$$

From (3)

$$\frac{dt}{ds} = \frac{\chi}{1 - \frac{2m}{r} + \frac{4\pi\epsilon^2}{r^2}}$$

where χ is a constant. Using the line element (1) we get the radial velocity

$$\frac{dr}{dt} = \pm\left[\left(1 - \frac{2m}{r} + \frac{4\pi\epsilon^2}{r^2}\right)^2 - \frac{\left(1 - \frac{2m}{r} + \frac{4\pi\epsilon^2}{r^2}\right)^3}{\chi^2}\right]^{1/2}.$$

This shows that

$$\frac{dr}{dt} = 0$$

when

$$1 - \frac{2m}{r_0} + \frac{4\pi\epsilon^2}{r_0^2} = \chi^2$$

and then the particle will experience an inward force according to eqn. (4) and it will begin its return journey.

(iii) When the test particle rotates in a circular path on $\theta = \pi/2$ plane,

$$\frac{dr}{ds} = \frac{d\theta}{ds} = 0.$$

For the spherically symmetric line element (1)

$$\Gamma^1_{33}\left(\frac{d\phi}{ds}\right)^2 + \Gamma^1_{44}\left(\frac{dt}{ds}\right)^2 = 0. \qquad ...(5)$$

Now, (5) gives

$$r\left(\frac{d\phi}{dt}\right)^2 - \frac{e^\nu \nu'}{2} = 0. \qquad ...(6)$$

Using (6) in the spherically symmetric line element

$$\left(\frac{ds}{dt}\right)^2 = e^\nu\left[1 - \frac{r\nu'}{2}\right]. \qquad ...(7)$$

(7) shows that the condition that the geodesic will be time-like is

$$r\nu' < 2. \qquad ...(8)$$

From (2) and (8), we get the relation

$$\frac{3m}{r} - \frac{8\pi\epsilon^2}{r^2} < 1. \qquad ...(9)$$

(II) *Infinitely Long Charged Cylinder*

The line element we use here is given by (Som 1964)

$$ds^2 = -r^{2c^2}(c_1 r^{-c} + c_2 r^c)^2 \ (dr^2 + dz^2)$$
$$- r^2(c_1 r^{-c} + c_2 r^c)^2 \ d\phi^2 + (c_1 r^{-c} + c_2 r^c)^{-2} \ dt^2. \qquad ...(10)$$

where $\frac{1}{2}c$ is the mass per unit length of the cylinder. The constants c, c_1 and c_2 are related to charge per unit length q and the electric field F_{14} by the equations

$$q = 2c\sqrt{-c_1 c_2}$$

$$F_{14} = \frac{2c\sqrt{-c_1 c_2}}{r(c_1 r^{-c} + c_2 r^c)^2}.$$

Comparing the line element (10) with the uncharged case we see that c_1 should be positive. In order to get real values for q and F_{14} we should take c_2 to be negative. Obviously c_2 refers to the charge of the cylinder.

(i) When a test particle is at rest in the field,

$$\frac{dr}{ds} = \frac{dz}{ds} = \frac{d\phi}{ds} = 0.$$

From the geodesic equations (3), we get

$$\frac{d^2\phi}{ds^2} = \frac{d^2z}{ds^2} = \frac{d^2t}{ds^2} = 0.$$

From the line element (10)

$$\frac{dt}{ds} = c_1 r^{-c} + c_2 r^{c}.$$

Then, from (3) we get the force experienced by the particle at rest

$$\frac{d^2 r}{dt^2} = -\frac{c}{r^{2c^2+1}} \frac{c_1 r^{-c} - c_2 r^{c}}{(c_1 r^{-c} + c_2 r^{c})^5}. \qquad \ldots(11)$$

(ii) When a test particle is in radial motion,

$$\frac{dz}{ds} = \frac{d\phi}{ds} = 0.$$

From (3)

$$\frac{dt}{ds} = \frac{\chi}{(c_1 r^{-c} + c_2 r^{c})^2}$$

where χ is a constant. Using the line element (10) we get the radial velocity

$$\frac{dr}{dt} = \pm \left[\frac{1}{r^{2c^2}(c_1 r^{-c} + c_2 r^{c})^4} - \frac{(c_1 r^{-c} + c_2 r^{c})^2}{r^{2c^2} \chi^2} \right]^{1/2}$$

This shows that

$$\frac{dr}{dt} = 0$$

when

$$(c_1 r_0^{-c} + c_2 r_0^{c})^3 = \chi,$$

and then the particle will experience an inward force [according to (11)] and it will begin its return journey.

(iii) When a test particle rotates about the z-axis,

$$\frac{dr}{ds} = \frac{dz}{ds} = 0.$$

For Weyl's canonical line-elemeut

$$ds^2 = e^{2\alpha}\, dt^2 - e^{2\beta-2\alpha}\,(dr^2 + dz^2) - r^2 e^{-2\alpha}\, d\phi^2$$

we have [taking 1, 2, 3 and 4 for r, ϕ, z and t respectively]

$$\Gamma^1_{22}\left(\frac{d\phi}{ds}\right)^2 + \Gamma^1_{44}\left(\frac{dt}{ds}\right)^2 = 0. \qquad \text{...(12)}$$

(12) gives

$$r\,(r\alpha' - 1)\left(\frac{d\phi}{dt}\right)^2 + e^{4\alpha}\alpha' = 0. \qquad \text{...(13)}$$

Using (13) in Weyl's line-element

$$\left(\frac{ds}{dt}\right)^2 = e^{2\alpha}\left(\frac{1 - 2r\alpha'}{1 - r\alpha'}\right). \qquad \text{...(14)}$$

(14) shows that the condition that the geodesic will be time-like is either

$$r\alpha' < \tfrac{1}{2}$$

or,

$$r\alpha' > 1$$

of which we accept the condition

$$r\alpha' < \tfrac{1}{2}. \qquad \text{...(15)}$$

The latter condition is rejected as this leads to imaginary values of $d\phi/dt$ from (13).

From (10) and (15) we get the relation

$$(c - \tfrac{1}{2})r^{-2c} < (c + \tfrac{1}{2})\,\frac{c_2}{c_1}. \qquad \text{...(16)}$$

3. Discussion

We further discuss here (A) the case of a test particle at rest [case (i)] and (B) the case of a test particle rotating round the spherical and the cylindrical body in circles [case (iii)].

(A) In the spherical case the condtion for a test particle at rest to experience an attractive force is given from (4) by

$$r > \frac{4\pi\epsilon^2}{m} \qquad \text{...(17)}$$

This means that the radius of the charged sphere should be equal or greater than the critical value

$$r_0 = \frac{4\pi\epsilon^2}{m}. \qquad ...(18)$$

Thus a sphere having a mass m and a radius r_0 cannot possess charge larger than the value given by (18).

If we put the values of the charge and mass of an electron in (18), we find that it cannot have radius less than the value

$$r_0 = 3 \times 10^{-13}\,\text{cm} \qquad ...(19)$$

This is a direct quantitative vindication of Eddington's view (1957) that the electron should not be a point-particle but should have some size. Since a proton has the same charge as that of an electron but mass about 2000 times larger, r_0 will be about 2000 times less for a proton.

If the sphere is uncharged ($\epsilon = 0$), the condition (17) shows that the attractive force will be effective at all distances from it so that there is no restriction to its radius.

In the case of the cylinder the condition for a test particle at rest to experience an attractive force is given from (11) by

$$r > \left(\frac{k}{c_1}\right)^{-1/2c} \qquad ...(20)$$

where $c_2 = -k$. This shows that the radius of the charged cylinder should be equal to or greater than a critical value

$$r_0 = \left(\frac{k}{c_1}\right)^{-1/2c}. \qquad ...(21)$$

But in the uncharged case ($c_2 = 0$) we see from (20) that the attractive force is effective at all distances so that there is no restriction to its radius.

(B) In the spherical case the condition that a test particle will move around the sphere in circles is given from the condition (9) by

$$r > \frac{1}{2}\,[3m + \sqrt{(9m^2 - 32\pi\epsilon^2)}]. \qquad ...(22)$$

This means that there is a critical radius

$$r_0 = \frac{1}{2}\,[3m - \sqrt{(9m^2 - 32\pi\epsilon^2)}] \qquad ...(23)$$

for circular motion. r_0 is apparently less than 3/2 times the Schwarzschild radius ($2m$). Obviously, in the uncharged case this critical radius is higher being equal to 3/2 times the Schwarzschild radius.

The condition for a test particle to move around an infinitely long charged cylinder in circles is given by the condition (16). In order to satisfy this condition, we should have $c < \frac{1}{2}$ and r less than the critical value

$$r_0 = \left(\frac{\frac{1}{2} + c}{\frac{1}{2} - c}\,\frac{k}{c_1}\right)^{-1/2c}. \qquad \ldots(24)$$

If, however, the cylinder is uncharged the particle will move in a circle of any radius provided only $c < \frac{1}{2}$, as can be seen from the condition (16).

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