ON THE EFFECT OF ROTATION IN THE GENERALIZED BÉNARD PROBLEM

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The present paper investigates the stability of a continuously stratified layer of viscous incompressible fluid confined between two horizontal boundaries of different but uniform temperatures in the presence of a uniform rotation whose axis is parallel to the direction of gravity. The intital stratification which might be produced, for example, by a dissolved solute of negligible diffusivity is assumed to be of the exponential type namely $\rho = \rho_0 e^{-\delta z}$ where δ is a constant which can be non-negative or negative and thus makes the initial stratification a monotone decreasing or increasing function of z respectively, ρ_0 being a positive constant and z the vertical co-ordinate while the temperature of the lower boundary is taken to be greater than that of the upper one. The analysis is carried out for positive as well as negative values of δ and it is shown that the principle of exchange of stabilities is not satisfied. The problem is completely solved at the marginal state for the case of free boundaries and the stabilizing character of rotation is established for all admissible values of the Prandtl number and the initial stratificaton parameter $\delta \geqslant 0$. For $\delta < 0$, the system is shown to be unstable.

1. INTRODUCTION

The problem of the onset of thermal instability in a viscous incompressible liquid layer statically confined between two horizontal boundaries and uniformly heated from below is known as the Bénard problem. The analysis of Rayleigh (1916), Jeffreys (1926), Low (1929), Pellew and Southwell (1940) and others have clearly demonstrated that what decides the stability or otherwise of the above configuration is the numerical value of the non-dmiensional parameter $R = g\alpha\beta d^4/\kappa\nu$, called the Rayleigh number, where g stands for gravity, α the coefficient of volume expansion of the liquid, $\beta > 0$ the uniform adverse temperature gradient maintained between the two boundaries, d the depth of the layer, κ the coefficient of thermometric conductivity and ν the coefficient of kinematic viscosity. Further, instability must set in when R exceeds a certain critical value R_c and when R just exceeds R_c, a stationary pattern of motions must come to prevail. On account of its applications in problems of meteorology and oceanography and various others related fields of practical importance, many authors in the recent years have extended the Bénard model by taking into account various other factors which are relevant to certain real physical problems. Chandrasekhar (1953, 1955) has considered the effect of a uniform rotation, whose axis is parallel to the direction of gravity, on the Bénard problem and established its stabilizing character. It is interesting to note that in Chanrasekhar's model the marginal state could either be stationary or oscillatory for which sufficient conditions are obtained. In particular, for the case when the thermal Prandtl number is greater than one, it is the stationary pattern of motions which manifest at the marginal state. Banerjee (1971, 1972, 1973) investigated the effect of an initial non-homogeneity of the fluid by considering the Zénard problem wherein the liquid has an intial continuous density stratification given by $\rho = \rho_0 e^{-\delta z}$, δ being a constant which can be positive, or negative (this problem will be referred to as the generalized Bénard problem in the subsequent discussions) and showed the stabilizing or destabilizing character of such an initial density distribution in the above respective cases. However, the character of the marginal state in this problem is shown to be definitely oscillatory when δ is positive while for negative values of δ there does not exist any marginal state and the system is unstable through non-oscillatory modes. Thus in some respect the effects of a uniform rotation and an initially stable stratification (*i.e.*, $\delta > 0$) acting separately on the onset of thermal instability in layer of liquid heated from below are remarkably alike: they both inhibit the onset of instability. However, in some other respect they do have dissimilar tendencies. Thus, while in the presence of uniform rotation instability could set in either as overstability or as stationary convection, it definitely sets in as overstability when an initially stable density stratification is present. On these accounts one must not suppose that acting together uniform rotation and an initially stable density distribution will reinforce each other in every respect; on the contrary it becomes desirable to study the Bénard problem in the presence of both the uniform rotation and the initial density stratification especially with a point of view to investigate the manner in which instability sets in and the character of uniform rotation on the generalized Bénard problem. The present paper is precisely in this direction. The analysis is carried out for positive as well as negative values of δ and it is shown that the principle of exchange of stabilities is not satisfied. The problem is completely solved at the marginal state for the case of free boundaries and the stabilizing character of rotation on the generalized Bénard problem is established for all admissible values of the thermal Prandtl number and the initial stratification parameter $\delta > 0$. For $\delta < 0$, the system is shown to be unstable.

2. The Physical Problem and Its Formulaton

An infinite horizontal layer of an initially stratified viscous incompressible fluid is confined between two horizontal boundaries maintained at different but uniform temperatures, with the temperature of the lower boundary being greater than that of the upper one, in the presence of a uniform rotation whose axis is parallel to the direction of gravity. The problem is to investigate the stability of this initial stationary state.

Let the origin be taken on the lower boundary $z=0$ with the positive direction of z-axis along the vertically upward direction. Clearly then, the xy-plane constitutes the horizontal plane $z=0$. Let $z=d$ ($d>0$ is a constant) denote the upper boundary, and T_0 and T_1 ($<T_0$) respectively denote the uniform temperatures of the lower and upper boundaries. Further, let the fluid be kept rotating at a constant rate with angular velocity Ω whose axis is parallel to the direction of gravity.

As is well known (Chandrasekhar 1961), for the problems of the type we are considering, it will suffice to work with the relevant equations of motion and heat conduction in the Boussinesq approximation. The only additional factors which we have to allow for now are the effects of Coriolis acceleration and centrifugal force in the equations of motion. Thus, the relevant equations of momentum, incompressibility, continuity, heat conduction and state (Banerjee 1973) which govern this problem are given by

$$\rho_0\left[\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}\right]$$

$$=-\frac{\partial}{\partial x_i}\left[p-\frac{\rho_0}{2}\,|\Omega\times r|^2\right]+\rho X_i+\mu\nabla^2 u_i+2\rho_0\epsilon_{ijk}u_j\Omega_k \qquad \ldots(1)$$

$$\frac{\partial\rho}{\partial t}+u_j\frac{\partial\rho}{\partial x_j}=0 \qquad \ldots(2)$$

$$\frac{\partial u_j}{\partial x_j}=0 \qquad \ldots(3)$$

$$\frac{\partial T}{\partial t}+u_j\frac{\partial T}{\partial x_j}=\frac{\kappa_1}{\rho_0 c_v}\cdot\frac{\partial^2 T}{\partial x_j\partial x_j}-\frac{p}{\rho_0 c_v}\cdot\frac{\partial u_j}{\partial x_j}+\frac{\Phi}{\rho_0 c_v} \qquad \ldots(4)$$

$$\rho=\rho_0[e^{-\delta z}+\alpha(T_0-T)] \qquad \ldots(5)$$

where x_j ($j=1,2,3$) respectively denote the x, y and z coordinates, u_j, x_i respectively denote the components of velocity and external force in the x, y and z directions, $\Omega=(0,0,\Omega)$, T the temperature, ρ the density, p the pressure, $r=(x,y,z)$ and α, μ and k_1 (assumed constants) are respectively the coefficients of volume expansion, viscosity and thermometric conductivity. Further, c_v (assumed constant) stands for the specific heat of the fluid at constant volume and

$$\Phi=2\mu\,e_{ij}^2-\tfrac{2}{3}\,\mu e_{jj}^2\,, \qquad \ldots(6)$$

with

$$e_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right). \qquad \ldots(7)$$

Clearly the initial stationary state whose stability we wish to examine is then characterized by the following solutions for the velocity, temperature, density and pressure fields with $X_i = (0, 0, -g)$:

$$\left.\begin{aligned} & u_i \equiv 0, \\ & T = T_0 - \beta z, \\ & \rho = \rho_0 [e^{-\delta z} + \alpha\beta z], \\ & P = p - \frac{\rho_0}{2} |\Omega \times r|^2 = P_0 - g\rho_0 \left[\frac{1}{\delta}(1 - e^{-\delta z}) + \frac{\alpha\beta z^2}{2}\right] \end{aligned}\right\} \qquad ...(8)$$

where P_0 is the value of P at the lower boundary $z = 0$ and $\beta = T_0 - T_1/d > 0$ is the maintained uniform adverse temperature gradient.

Let the initial stationary state described by equations (8) be slightly perturbed so that the perturbed state (denoted by primed symbols) is given by (Banerjee 1973)

$$\left.\begin{aligned} & u_i' = (u, v, w), \\ & T' = T_0 - \beta z + \theta, \\ & \rho' = \rho_0 \left[e^{-\delta z} + \frac{\triangle\rho}{\rho_0} + \alpha (T_0 - T - \theta)\right], \\ & P' = P + \triangle P. \end{aligned}\right\} \qquad ...(9)$$

where (u, v, w), θ, $\triangle\rho$ and $\triangle P$ respectively stand for the perturbations in the velocity, temperature, initial density and pressure fields at the initial stationary state.

Then the linearized perturbation equations (using Boussinesq approximation and the small-δ-approximation (Banerjee 1972, 1973)) of momentum, incompressibility, continuity and heat conduction become

$$\rho_0 \frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}(\triangle P) + \mu\nabla^2 u + 2\rho_0 v\Omega \qquad ...(10)$$

$$\rho_0 \frac{\partial v}{\partial t} = -\frac{\partial}{\partial y}(\triangle P) + \mu\nabla^2 v - 2\rho_0 u\Omega \qquad ...(11)$$

$$\rho_0 \frac{\partial w}{\partial t} = -\frac{\partial}{\partial z}(\triangle P) + g\alpha\rho_0\theta - g\triangle\rho + \mu\nabla^2 w \qquad ...(12)$$

$$\frac{\partial}{\partial t}(\triangle\rho) - \rho_0 w\delta = 0 \qquad ...(13)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad ...(14)$$

$$\frac{\partial \theta}{\partial t} = \beta w + \kappa\nabla^2\theta, \ \kappa = \frac{\kappa_1}{\rho_0 c_v}. \qquad ...(15)$$

Now analysing the perturbations in terms of normal modes by seeking solutions whose dependence on x, y and t is given by

$$\exp\,[i(k_x x + k_y y) + nt] \qquad ...(16)$$

eqns. (10)–(15) become

$$\rho_0 n u = -ik_x \triangle P + \mu\left[\frac{d^2}{dz^2} - k^2\right] u + 2\rho_0 v\Omega \qquad \ldots(17)$$

$$\rho_0 n v = -ik_y \triangle P + \mu\left[\frac{d^2}{dz^2} - k^2\right] v + 2\rho_0 u\Omega \qquad \ldots(18)$$

$$\rho_0 n w = -\frac{d}{dz}(\triangle P) + \mu\left[\frac{d^2}{dz^2} - k^2\right] w + g\alpha\rho_0\theta - g\triangle\rho \qquad \ldots(19)$$

$$n\triangle\rho - \rho_0\delta w = 0 \qquad \ldots(20)$$

$$i(k_x u + k_y v) = -\frac{dw}{dz} \qquad \ldots(21)$$

$$n\theta = \beta w + \kappa\left[\frac{d^2}{dz^2} - k^2\right]\theta \qquad \ldots(22)$$

where $k = \sqrt{k_x^2 + k_y^2}$ is the wave number of the perturbation, n is a constant which can be complex and u, v, w, $\triangle P$, θ, $\triangle\rho$ are now functions of z only.

Using the non-dimensional quantities defined by

$$\left.\begin{aligned} z_* &= z/d;\quad \sigma_* = \frac{nd^2}{\nu};\quad P_{1*} = \nu/\kappa;\quad a_* = kd;\\ w_* &= w;\quad \theta_* = \theta;\quad M_* = d\delta;\quad D_* = dD. \end{aligned}\right\} \qquad \ldots(23)$$

where

$$\nu = \mu/\rho_0 \quad \text{and} \quad D = \frac{d}{dz},$$

and dropping the asterisk for convenience in writing, we derive the following equations from the above system of equations :

$$(D^2 - a^2 - P_1\sigma)\,\theta = -\left(\frac{\beta d^2}{\kappa}\right) w \qquad \ldots(24)$$

$$(D^2 - a^2 - \sigma)\,\zeta = -\left(\frac{2\Omega d}{\nu}\right) Dw \qquad \ldots(25)$$

$$(D^2 - a^2)(D^2 - a^2 - \sigma)\,w - \left(\frac{2\Omega d^3}{\nu}\right) D\zeta = \left(\frac{g\alpha d^2}{\nu}\right) a^2\theta - \frac{ga^2\delta d^2}{n\nu} w \qquad \ldots(26)$$

where

$$\zeta(z) = i\,[k_x v(z) - k_y u(z)].$$

It is easily seen that eqns. (24), (25) and (26) coincide exactly with eqns. (93), (94) and (95) respectively of Chandrasekhar's treatise (1961, p. 89) when δ is zero.

Solution of eqns. (24), (25) and (26) must now be sought which satisfy the following appropriate boundary conditions :

$$w = 0 \text{ and } \theta = 0 \text{ for } z = 0 \text{ and } z = 1, \qquad \ldots(27)$$

and

either $\zeta = 0$ and $Dw = 0$ (on a rigid boundary)

or $D\zeta = 0$ and $D^2 w = 0$ (on a free boundary).

3. Theorems

We prove the following theorems;

Theorem 3.1—The principle of exchange of stabilities is not valid for the problem under discussion.

Proof : We have from eqn. (26) that

$$\sigma\left[(D^2 - a^2)(D^2 - a^2 - \sigma) w - \left(\frac{2\Omega d^3}{\nu}\right) D\zeta - \left(\frac{g\alpha d^2}{\nu}\right) a^2 \theta \right]$$

$$= - \frac{g a^2 \delta d^4}{\nu^2} w. \qquad \ldots(29)$$

If possible let the principle of exchange of stabilities be satisfied so that $\sigma_r = 0 \Rightarrow \sigma_i = 0$ where $\sigma = \sigma_r + i\sigma_i$. Then $\sigma = 0$ should be allowed by the governing equations and boundary conditions.

Equation (29), when $\sigma = 0$, gives $w \equiv 0$. $\qquad \ldots(30)$

Equation (24) then gives

$$(D^2 - a^2)\,\theta = 0, \qquad \ldots(31)$$

which together with the boundary conditions on θ imply $\theta \equiv 0$. Further from equation (25) we have

$$(D^2 - a^2)\,\zeta = 0. \qquad \ldots(32)$$

Equation (32) when solved with the boundary conditions on ζ yield $\zeta \equiv 0$. Further $\zeta \equiv 0$ together with eqn. (21) (with $w \equiv 0$) imply that $u \equiv v \equiv 0$. Thus $\sigma = 0$ corresponds to the trivial solution which is contrary to the starting assumption, namely that the initial stationary state solution is non-trivially perturbed. Hence the theorem.

Theorem 3.2—For the case of free boundaries with $\delta > 0$, the Rayleigh number and the frequency of oscillations at the marginal state are respectively given by

$$R_1 a^2 = (\pi^2 + a^2)^2 + T\pi^2 + R_2 a^2\left(1 + \frac{1}{P_1}\right) - (1 + 2P_1)(\pi^2 + a^2)\,\sigma_i^2 \qquad ...(33)$$

$$\sigma_i^2 = \frac{-B + \sqrt{B^2 - 4AC}}{2A} \qquad ...(34)$$

where

$$\left.\begin{aligned} R_1 &= \frac{g\alpha\beta d^4}{\kappa\nu}\,; \quad A = (1 + P_1)(\pi^2 + a^2) > 0; \\ T &= \frac{4\Omega^2 d^4}{\nu^2}\,; \quad B = (1 + P_1)(\pi^2 + a^2)^3 - T\pi^2(1 - P_1) - \frac{R_2 a^2}{P_1}\,; \\ R_2 &= \frac{g\delta d^4}{\kappa\nu}\,; \quad C = -\frac{R_2 a^2}{P_1}(\pi^2 + a^2)^2. \end{aligned}\right\} \qquad ...(35)$$

and R_1, T and R_2 respectively stand for the Rayleigh number, the Taylor number and the initial non-homogeneity number.

PROOF: The marginal state according to Theorem 3.1 is characterized by $\sigma = i\sigma_i$, σ_i being real and non-zero.

For the case of free boundaries, eqns. (24) and (26) give

$$D^2\theta = 0 = D^4 w \text{ for } z = 0 \text{ and } z = 1. \qquad ...(36)$$

Equation (25) then gives

$$D^3\zeta = 0 \text{ for } z = 0 \text{ and } z = 1. \qquad ...(37)$$

Now differentiating eqn. (26) twice with respect to z and using the boundary conditions (36) and (37), we derive

$$D^6 w = 0 \text{ at } z = 0 \text{ and } z = 1. \qquad ...(38)$$

Likewise it is easy to see that w satisfies the boundary conditions

$$w = 0 \text{ and } D^{2m}(w) = 0 \text{ for } z = 0 \text{ and } z = 1 \ (m = 1, 2, 3, \ldots). \qquad ...(39)$$

The proper solution for w belonging to the lowest mode is (Chandrasekhar 1961)

$$w = C_1 \sin \pi z, \qquad ...(40)$$

C_1 being a constant.

Now by applying the operator $(D^2 - a^2 - \sigma)(D^2 - a^2 - P_1\sigma)$ to eqn. (26), we can eliminate ζ and θ with the help of eqns. (24) and (25) and obtain

$$\sigma(D^2 - a^2)(D^2 - a^2 - P_1\sigma)(D^2 - a^2 - \sigma)^2 w + \sigma T D^2 (D^2 - a^2 - P_1\sigma) w$$
$$= -\sigma R_1 a^2 (D^2 - a^2 - \sigma) w - \frac{R_2 a^2}{P_1}(D^2 - a^2 - \sigma)(D^2 - a^2 - P_1\sigma) w. \quad \text{...(41)}$$

Substituting the above solution for w in eqn. (41), we obtain the characteristic eqation

$$\sigma(\pi^2 + a^2)(\pi^2 + a^2 + P_1\sigma)(\pi^2 + a^2 + \sigma)^2 + \sigma T\pi^2 (\pi^2 + a^2 + P_1\sigma)$$
$$= \sigma R_1 a^2(\pi^2 + a^2 + \sigma) - \frac{R_2 a^2}{P_1}(\pi^2 + a^2 + \sigma)(\pi^2 + a^2 + P_1\sigma). \quad \text{...(42)}$$

Putting $\sigma = i\sigma_i$ in eqn. (42) and separating the real and imaginary parts we get

$$P_1(\pi^2 + a^2)\sigma_i^4 + [R_1 a^2 - R_2 a^2 - 2(\pi^2 + a^2)^3 - P_1(\pi^2 + a^2)^3 - P_1 T\pi^2]\sigma_i^2$$
$$+ \frac{R_2 a^2}{P_1}(\pi^2 + a^2)^2 = 0 \quad \text{...(43)}$$

$$(\pi^2 + a^2)^3 + T\pi^2 + R_2 a^2\left(1 + \frac{1}{P_1}\right) - R_1 a^2 = (1 + 2P_1)(\pi^2 + a^2)\sigma_i^2. \quad \text{...(44)}$$

Substituting for $R_1 a^2$ from eqn. (44) in eqn. (43) we get

$$(1 + P_1)(\pi^2 + a^2)\sigma_i^4 + \left[(1 + P_1)(\pi^2 + a^2)^3 - T\pi^2(1 - P_1) - \frac{R_2 a^2}{P_1}\right]\sigma_i^2$$
$$- \frac{R_2 a^2}{P_1}(\pi^2 + a^2)^2 = 0. \quad \text{...(45)}$$

The solutions for σ_i^2 satisfying eqn. (45) are

$$\sigma_i^2 = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}. \quad \text{...(46)}$$

Further, $\delta > 0 \Rightarrow R_2 > 0$ and hence $C < 0$. Thus, since $\sigma_i^2 > 0$, the only admissible value of σ_i^2 is given by

$$\sigma_i^2 = \frac{-B + \sqrt{B^2 - 4AC}}{2A}. \quad \text{...(47)}$$

Substituting the above value of σ_i^2 in eqn. (44), we have

$$R_1 a^2 = (\pi^2 + a^2)^3 + T\pi^2 + R_2 a^2\left(1 + \frac{1}{P_1}\right) - (1 + 2P_1)(\pi^2 + a^2)\sigma_i^2. \quad \text{...(48)}$$

This establishes the theorem.

Theorem 3.3—For $R_2 > 0$, $\partial R_1/\partial T > 0$ for all $T > 0$ provided $P_1 > 1$.

PROOF: From eqn. (38), we derive

$$a^2 \frac{\partial R_1}{\partial T} = \pi^2 - (1 + 2P_1)(\pi^2 + a^2) \frac{\partial \sigma_i^2}{\partial T}. \quad ...(49)$$

Further, since $R_2 > 0$, we have from eqn. (47)

$$2A \frac{\partial \sigma_i^2}{\partial T} = \pi^2 (1 - P_1) \left[1 - \frac{B}{\sqrt{B^2 - 4AC}} \right] > 0, \quad ...(50)$$

since

$$\frac{B}{\sqrt{B^2 - 4AC}} < 1,$$

A being positive while C negative.

Substituting the above value of $\partial \sigma_i^2/\partial T$ in eqn. (49) and simplifying a little, we derive

$$a^2 \frac{\partial R_1}{\partial T} = \pi^2 \left[1 - \frac{(1 + 2P_1)(1 - P_1)}{2(1 + P_1)} \left\{ 1 - \frac{B}{\sqrt{B^2 - 4AC}} \right\} \right]. \quad ...(51)$$

Now since

$$P_1 > 1 \quad \text{and} \quad \frac{B}{\sqrt{B^2 - 4AC}} < 1,$$

it follows that

$$\partial R_1/\partial T > 0 \text{ for all } T > 0. \quad ...(52)$$

This proves the theorem.

Theorem 3.4—For $R_2 > 0$, $\partial R_1/\partial T > 0$ for all $T > 0$ provided $P_1 < 1$.

PROOF: For $R_2 > 0$, we have $A > 0$ and $C < 0$ while B could either be positive or negative. We first consider the case when B is positive. It then follows from eqn. (51) that $\partial R_1/\partial T$ is positive because the conditions

$$\frac{B}{\sqrt{B^2 - 4AC}} < 1 \quad ...(53)$$

$$0 < \frac{(1 + 2P_1)(1 - P_1)}{2(1 + P_1)} < 1 \quad ...(54)$$

are true.

For the case when B is negative, let, if possible, $\partial R_1/\partial T < 0$. Then, eqn. (51) gives

$$\frac{1 + P_1 + 2P_1^2}{(1 + 2P_1)(1 - P_1)} < \frac{|B|}{\sqrt{B^2 + 4A|C|}} \quad ...(55)$$

which is not possible as

$$\frac{1 + P_1 + 2P_1^2}{(1 + 2P_1)(1 - P_1)} > 1, \qquad \ldots(56)$$

while

$$\frac{|B|}{\sqrt{B^2 + 4A|C|}} < 1. \qquad \ldots(57)$$

This proves the theorem.

Theorem 3.5—For $R_2 < 0$, no marginal state exists provided $P_1 > 1$.

PROOF : Let the marginal state exist under the given conditions so that $\sigma_r = 0$ is allowed by the governing equations and boundary conditions. Clearly, σ_i cannot be zero for otherwise principle of exchange of stabilities will be valid which contradicts Theorem 3.1 which is true irrespective of whether R_2 is positive or negative and P_1 is greater than or less than one. We then have from eqn. (46) (which still holds good)

$$\sigma_i^2 = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}, \qquad \ldots(58)$$

where we must now take $R_2 < 0$ and $P_1 > 1$ in the expressions for A, B and C as given by eqns. (35). Thus, under the conditions of the theorem it is clear that

$$\left.\begin{array}{l} A > 0 \\ B > 0 \\ C > 0 \end{array}\right\}. \qquad \ldots(59)$$

Clearly, eqn. (58) does not allow any positive root for σ_i^2 when inequalities (59) are satisfied. However, σ_i^2 is essentially positive since σ_i is real. Therefore, our starting assumption namely $\sigma_r = 0$ is possible is not correct and this establishes the theorem. It implies that under the above conditions an arbitrary mode is either damped or amplified. We discuss this point in the subsequent theorems.

Theorem 3.6—For $R_2 < 0$ and $P_1 > 1$, non-oscillatory modes exist and are unstable.

PROOF : Consider the non-oscillatory modes, *i.e.*, modes for which $\sigma_i = 0$ and obviously $\sigma_r \neq 0$ (Theorem 3.5). By putting $\sigma = \sigma_r$ in the characteristic eqn. (42) we derive the characteristic equation for such modes as

$$\sigma_r(\pi^2 + a^2)(\pi^2 + a^2 + P_1\sigma_r)(\pi^2 + a^2 + \sigma_r)^2 + \sigma_r T\pi^2(\pi^2 + a^2 + P_1\sigma_r)$$
$$= \sigma_r R_1 a^2(\pi^2 + a^2 + \sigma_r) - \frac{R_2 a^2}{P_1}(\pi^2 + a^2 + \sigma_r)(\pi^2 + a^2 + P_1\sigma_r). \qquad \ldots(60)$$

where now R_2 is negative and $P_1 > 1$.

Equation (60) is an equation of degree four in σ_r with real coefficients and hence allows four roots for σ_r which may be real or complex. If we denote the roots by σ_{r1}, σ_{r2}, σ_{r3}, and σ_{r4}, we have from eqn. (60)

$$\sigma_{r1},\ \sigma_{r2},\ \sigma_{r3},\ \sigma_{r4} < 0, \qquad \ldots(61)$$

under the conditions of the theorem.

Inequality (61) shows that one of the roots must be negative and one must be positive. Thus, under the conditions of the theorem, for given values of the other parameters, we do have non-oscillatory modes either two or four such that at least one is amplified in time and thus makes the system unstable.

This establishes the theorem.

Theorem 3.7—For $R_2 < 0$ and $P_1 > 1$, the system is unstable.

PROOF: This follows easily from Theorem 3.6 since non-oscillatory modes exist and are amplified. Thus, irrespective of the existence of oscillatory modes and their character, there are always certain modes (the non-oscillatory ones) which are unstable. The system itself is then unstable according to the linear stability theory.

4. CONCLUDING REMARKS

The investigations presented in this paper are based on the rather plausible hypothesis that in reality a fluid is necessarily initially non-homogeneous and one may not be justified in neglecting this initial non-homogeneity, however small, everywhere in the equations of motion. On the contrary, even the slightest amount of this initial non-homogeneity may turn out to be quite significant for the problem under consideration and hence the behaviour of a homogeneous fluid should be deduced from that of a non-homogeneous one in the limit of very small non-homogeneity. The importance of this initial non-homogeneity of the fluid has been clearly demonstrated by Banerjee (1971, 1972, 1973) in the context of the Bénard problem and the Taylor problem of rotating cylinders and therefore it is desirable to analyse the Bénard problem with rotation when the fluid is taken to be initially non-homogeneous. This will also provide a support to the reliability of the results of Chandrasekhar's model. The above point of view of looking at the investigations presented in the paper appears more fundamental than the one given at the outset, namely that of extending the results of the generalized Bénard model in the framework of rotation and investigating its role. The result contained in Theorem 3.1, in a sense, is decisive. Thus, while in the presence of a uniform rotation alone, instability could set in either as stationary convection or as overstability, it definitely sets in as overstability when an initially stable density stratification is also present. Thus, overstability as the mode of the onset of instability appears more likely in Chandrasekhar's model. This result is not surprising at all as it is well known that stable stratification permits internal waves. Theorem 3.2 gives the Rayleigh number and the frequency of oscillations that characterize the marginal state. Putting $R_2 = 0 = T$ (which imply $\sigma_i = 0$) we recover the results of Rayleigh (1916) and Pellew and Southwell (1940), $R_2 = 0$

those of Chandrasekhar (1953, p. 55) and $T = 0$ those of Banerjee (1972). Theorems 3.3 and 3.4 show the stabilizing character of a uniform rotation upon the generalized Bénard problem ($\delta > 0$) for all values of the thermal Prandtl number. Thus, the character of a uniform rotation in the present situation remains what it was in Chandrasekhar's model. The difference lies in the manner in which the instability sets in. The character of the marginal state is dominated by the properties of the initial stratification while the uniform rotation helps in postponing the onset of instability by raising the Rayleigh number at the marginal state. When the initial stratification is monotonically increasing with respect to the vertical coordinate, *i.e.*, $R_2 < 0$, Theorem 3.5 shows that there is no marginal state, the problem thus again being dominated by the initial stratification as the same result holds true for the generalized Bénard model also. This result is a little surprising though could possibly be attributed to the neglect of mass diffusion in the governing equations. The extension of the present model by taking mass diffusion into account will be the subject-matter of a later communication. Theorem 3.6 shows that non-oscillatory modes do exist and are unstable and thus make the system unstable according to the linear stability theory. The investigation of the character of oscillatory modes, if there exists any, under the conditions of the theorem, is thus of secondary importance. The system remains unstable irrespective of their existence and character as concluded in Theorem 3.7. We note that this result is analogous to the corresponding result for the generalized Bénard problem (Banerjee 1972) where the non-oscillatory modes are amplified irrespective of the character of the applied uniform temperature gradient. The character of the non-oscillatory modes appear to be completely guided by the character of the initial density stratification.

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