ON A CLASS OF INTEGRAL INEQUALITIES OF BELLMAN-BIHARI TYPE

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The present paper deals with some new integral inequalities of the Bellman-Bihari type that have a wide range of applications in the theory of differential and integral equations of the more general type.

§ 1. In the study of ordinary differential equations and integral equations one has to deal with certain integral inequalities. Gronwall's inequality, also known as Bellman's lemma (Bellman and Cooke 1963, p. 36) embodies an inequality which is often referred to, and justifiably so, as the fundamental inequality of differential equations. On the basis of various motivations, it has been extended and used considerably in various contexts. The most commonly used generalization of the Gronwall-Bellman inequality is due to Bihari (1956) which provides explicit bounds on unknown function. Recently, in a series of papers the author (Pachpatte 1973; 1975*a*, *b*, *c*) has established some new integral inequalities of the Bellman-Bihari type which can be used as a tool in applications. The purpose of this paper is arisen by a strong influence of a slight variant of Gronwall-Bellman inequality given in the book by Bellman and Cooke (1963, p. 56) in the following form.

Lemma 1 (Bellman and Cooke 1963, p. 56)—Let $n(t)$ be positive, monotonic, nondecreasing function and $x(t) \geqslant 0$, $f(t) \geqslant 0$. If all these functions are continuous and if

$$x(t) \leq n(t) + \int_a^t f(s)\, x(s)\, ds, \quad a \leq t \leq b$$

then

$$x(t) \leq n(t) \exp \left(\int_a^t f(s)\, ds \right), \quad a \leq t \leq b.$$

Pachpatte (1975*b*, *c*) has profitably employed the spirit of this inequality to establish some new integral inequalities. Our objective here is to present a number

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of theorems concerning the integral inequalities which claim their origin to the integral inequality given in Lemma 1. Although the integral inequalities of Bellman-Bihari type are widely known and used, there appear to be no results of this kind for integral inequalities.

§ 2. In this section, we wish to establish some nonlinear generalizations of Lemma 1 which can be used as a tool in applications. For this purpose, we require the class of functions S as defined by the author in his earlier papers (Pachpatte 1975*a*, *b*).

A function $H : [0, \infty) \to [0, \infty)$ is said to belong to the class S if

(i) $H(u)$ is positive, nondecreasing and continuous on $[0, \infty)$,

(ii) $\frac{1}{v} H(u) \leqslant H\left(\frac{u}{v}\right)$, for $u > 0$, $v \geqslant 1$.

Before giving the main results in this section, we first establish the following interesting and useful nonlinear generalization of Lemma 1 which is useful in our further discussion.

Theorem 1—Let $x(t)$, $f(t)$ and $g(t)$ be real valued nonnegative continuous functions defined on $I = [0, \infty)$, $n(t)$ be a positive, monotonic, nondecreasing continuous function defined on I, and $H \in S$, for which the inequality

$$x(t) \leqslant n(t) + \int_0^t f(s) H\left(x(s) + \int_0^s g(\tau) H(x(\tau))\, d\tau\right) ds, \quad t \in I \qquad \ldots(1)$$

holds. Then

$$x(t) \leqslant n(t)\left[1 + \int_0^t f(s) H\left(G^{-1}\left[G(1) + \int_0^s (f(\tau) + g(\tau))\, d\tau\right]\right) ds\right],$$
$$0 \leqslant t \leqslant b \qquad \ldots(2)$$

for $0 \leqslant t \leqslant b$, where

$$G(r) = \int_{r_0}^{r} \frac{ds}{H(s)}, \quad r \geqslant r_0 > 0 \qquad \ldots(3)$$

and G^{-1} is the inverse of G, and t is in the subinterval $[o, b]$ of I so that

$$G(1) + \int_0^t (f(s) + g(s))\, ds \in \mathrm{Dom}\ (G^{-1}).$$

PROOF: Since $n(t)$ is positive, monotonic, nondecreasing and $H \in S$, we have

$$\frac{x(t)}{n(t)} \leqslant 1 + \int_0^t f(s) H\left(\frac{x(s)}{n(t)} + \int_0^s g(\tau) H\left(\frac{x(\tau)}{n(t)}\right) d\tau\right) ds$$

$$\leqslant 1 + \int_0^t f(s) H\left(\frac{x(s)}{n(s)} + \int_0^s g(\tau) H\left(\frac{x(\tau)}{n(\tau)}\right) d\tau\right) ds. \qquad \ldots(4)$$

Define $v(t)$ by the right member of (4). Then

$$v'(t) = f(t)\, H\left(\frac{x(t)}{n(t)} + \int_0^t g(\tau)\, H\left(\frac{x(\tau)}{n(\tau)}\right) d\tau\right), \quad v(0) = 1,$$

which in view of (4) implies

$$v'(t) \leqslant f(t)\, H\left(v(t) + \int_0^t g(\tau)\, H(v(\tau))\, d\tau\right). \qquad \ldots(5)$$

If we put

$$m(t) = v(t) + \int_0^t g(\tau)\, H(v(\tau))\, d\tau, \quad m(0) = v(0) = 1, \qquad \ldots(6)$$

it follows from (6) and (5) and the fact that $v(t) \leqslant m(t)$, we have

$$m'(t) \leqslant (f(t) + g(t))\, H(m(t)). \qquad \ldots(7)$$

Dividing both sides of (7) by $H(m(t))$, using (3) and integrating from 0 to t, we obtain

$$G(m(t)) - G(m(0)) \leqslant \int_0^t (f(\tau) + g(\tau))\, d\tau. \qquad \ldots(8)$$

Then from (5) and (8) we have

$$v'(t) \leqslant f(t)\, H\left(G^{-1}\left[G(1) + \int_0^t (f(\tau) + g(\tau))\, d\tau\right].\right) \qquad \ldots(9)$$

Now, integrating both sides of (9) from 0 to t and substituting the value of $v(t)$ in (4) we obtain the desired bound in (2). The subinterval $[0, b]$ is obvious.

We note that the estimate for a slightly more general form of (1) when $n(t)$ is not monotonic nondecreasing and H is a positive, continuous, nondecreasing, submultiplicative and subadditive function for $u \geqslant 0$, 0 is already obtained by Pachpatte (1975*a*).

We now apply Theorem 1 to establish the following useful integral inequalities.

Theorem 2—Let $x(t)$, $f(t)$, $g(t)$ and $h(t)$ be real valued nonnegative continuous functions defined on I; $H \in S$; $W(u)$ be a positive, continuous, monotonic, nondecreasing and submultiplicative function for $u \geqslant 0$, and suppose further that the inequality

$$x(t) \leqslant x_0 + \int_0^t f(s)\, H\left(x(s) + \int_0^s g(\tau)\, H(x(\tau))\, d\tau\right) ds$$
$$+ \int_0^t h(s)\, W(x(s))\, ds \qquad \ldots(10)$$

is satisfied for all $t \in I$, where x_0 is a positive constant. Then

$$x(t) \leqslant \Omega^{-1}\Big[\Omega(x_0) + \int_0^t h(s)\, W\Big(1 + \int_0^s f(\tau)\, H\big(G^{-1}[G(1) + \int_0^\tau (f(k) + g(k))\, dk]\big)\, d\tau\Big)\, ds\Big]\Big[1 + \int_0^t f(s)\, H\Big(G^{-1}[G(1) + \int_0^s (f(\tau) + g(\tau))\, d\tau]\Big)\, ds\Big], \quad 0 \leqslant t \leqslant b \tag{11}$$

where G, G^{-1} are as defined in Theorem 1, Ω is defined by

$$\Omega(r) = \int_{r_0}^{r} \frac{ds}{W(s)}, \; r \geqslant r_0 > 0, \qquad \ldots(12)$$

and Ω^{-1} is the inverse function of Ω, and t is in the subinterval $[0, b]$ of I such that

$$G(1) + \int_0^t (f(s) + g(s))\, ds \in \mathrm{Dom}\,(G^{-1})$$

and

$$\Omega(x_0) + \int_0^t h(s)\, W\Big(1 + \int_0^s f(\tau)\, H\big(G^{-1}[G(1) + \int_0^T (f(k) + g(k))\, dk]\big)\, d\tau\Big)\, ds \in \mathrm{Dom}\,(\Omega^{-1}).$$

PROOF : Define

$$n(t) = x_0 + \int_0^t h(s)\, W(x(s))\, ds, \qquad n(0) = x_0.$$

Then (10) can be restated as

$$x(t) \leqslant n(t) + \int_0^t f(s)\, H\Big(x(s) + \int_0^s g(\tau)\, H(x(\tau))\, d\tau\Big)\, ds.$$

Since $n(t)$ is positive, monotonic, nondecreasing, and $H \in S$, we have from Theorem 1

$$x(t) \leqslant n(t)\Big[1 + \int_0^t f(s)\, H\Big(G^{-1}[G(1) + \int_0^s (f(\tau) + g(\tau))\, d\tau]\Big)\, ds\Big], \qquad 0 \leqslant t \leqslant b. \qquad \ldots(13)$$

Further

$$W(x(t)) \leqslant W(n(t))\, W\Big(1 + \int_0^t f(s)\, H\big(G^{-1}[G(1) + \int_0^s (f(\tau) + g(\tau))\, d\tau]\big)\, ds\Big),$$

since W is submultiplicative. Hence

$$\frac{h(t)\,W(x(t))}{W(n(t))} \leqslant h(t)\,W\left(1 + \int_0^t f(s)\,H\left(G^{-1}\left[G(1) + \int_0^s (f(\tau) + g(\tau))\,d\tau\right]\right)ds\right)$$

Because of (12) this reduces to

$$\frac{d}{dt}\,\Omega(n(t)) \leqslant h(t)\,W\left(1 + \int_0^t f(s)\,H\left(G^{-1}\left[G(1) + \int_0^s (f(\tau) + g(\tau))\,d\tau\right]\right)ds\right).$$

Now, integrating from 0 to t, we obtain

$$\Omega(n(t)) - \Omega(n(0)) \leqslant \int_0^t h(s)\,W\left(1 + \int_0^s f(\tau)\,H(G^{-1}[G(1) + \int_0^\tau (f(k) + g(k))\,dk])\,d\tau\right)ds. \quad \ldots(14)$$

The desired bound in (11) follows from (13) and (14) on the subinterval $[0, b]$ of I.

We next state and prove a still more general form of Theorem 2 which basically involves the comparison principle.

Theorem 3—Let $x(t)$, $f(t)$, $g(t)$, $h(t)$ and $q(t)$ be real valued nonnegative continuous functions defined on I; $H \in S$; $W(t, u)$ be a nonnegative, continuous, monotonic, nondecreasing in u, $u \geqslant 0$, for each fixed $t \in I$; the functions $p(t) > 0$, $\Phi(t) \geqslant 0$ be nondecreasing in t and continuous on I, $\Phi(0) = 0$, and suppose further that the inequality

$$x(t) \leqslant p(t) + \int_0^t f(s)\,H\left(x(s) + \int_0^s g(\tau)\,H(x(\tau))\,d\tau\right)ds + h(t)\,\Phi\left(\int_0^t q(s)\,W(s, x(s))\,ds\right) \quad \ldots(15)$$

is satisfied for all $t \in I$. Then

$$x(t) \leqslant k(t)\,[p(t) + h(t)\,\Phi(r(t))], \quad t \in I_0. \quad \ldots(16)$$

where

$$k(t) = 1 + \int_0^t f(s)\, H\Big(G^{-1}[G(1) + \int_0^s (f(\tau) + g(\tau))\, d\tau]\Big)\, ds, \qquad ...(17)$$

G and G^{-1} are as in Theorem 2 and I_0 is the largest subinterval of I on which the right-hand side of (17) exists, and $r(t)$ is the maximal solution of

$$r'(t) = q(t)\, W(t, k(t)[p(t) + h(t)\, \Phi(r(t))]), \qquad r(0) = 0 \qquad ...(18)$$

existing on I.

PROOF : Define

$$n(t) = p(t) + h(t)\, \Phi\big(\int_0^t q(s)\, W(s, x(s))\, ds\big). \qquad ...(19)$$

Then (15) can be restated as

$$x(t) \leqslant n(t) + \int_0^t f(s)\, H\Big(x(s) + \int_0^s g(\tau)\, H(x(\tau))\, d\tau\Big)\, ds.$$

Since $n(t)$ is positive, monotonic, nondecreasing on I, we have from Theorem 1

$$x(t) \leqslant k(t)\, n(t) \qquad ...(20)$$

where $k(t)$ is as given in (17). Now from (19) and (20) we have

$$x(t) \leqslant k(t)[p(t) + h(t)\, \Phi(v(t))] \qquad ...(21)$$

where

$$v(t) = \int_0^t q(s)\, W(s, x(s))\, ds, \qquad v(0) = 0.$$

Consequently it follows that

$$v'(t) \leqslant q(t)\, W(t, k(t)[p(t) + h(t)\, \Phi(v(t))]). \qquad ...(22)$$

A suitable application of Theorem 1.4.1 given in Lakshmikantham and Leela (1959) to (22) and (18) yields

$$v(t) \leqslant r(t) \qquad ...(23)$$

where $r(t)$ is the maximal solution of (18) such that $r(0) = v(0) = 0$. Now from (21) and (23) the desired bound in (16) follows.

It is important to note that, by adding the monotonicity condition on the function $n(t)$ in Theorem 1, the estimate in (2) takes a simple form. In Theorems 2 and 3 this property is profitably employed to establish some new integral inequalities.

§ 3. The next step in this development is to establish a slightly different class of nonlinear integral inequalities which generalizes Lemma 1 consider bly.

Before giving the main results in this direction, we first etablish the following useful nonlinear generalization of Lemma 1. However, this generalization is only preparatory for establishing our main results in this section.

Theorem 4—Let $x(t)$, $f(t)$ and $g(t)$ be real valued nonnegative continuous functions defined on I; $n(t)$ be a positive, monotonic, nondecreasing continuous function defined on I; $H(u)$ be a positive, continuous, monotonic nondecreasing, subadditive and submultiplicative function for $u > 0$ $H(0) = 0$, and H^{-1} denote the inverse function of H, for which the inequality

$$x(t) \leqslant n(t) + H^{-1}\Big[\int_0^t f(s)\, H\big(x(s)\big)\, ds + \int_0^t f(s)\Big(\int_0^s g(\tau)\, H\big(x(\tau)\big)\, d\tau\Big)\, ds\Big] \quad \ldots(24)$$

holds for all $t \in I$. Then

$$x(t) \leqslant n(t)\, H^{-1}\Big[1 + \int_0^t f(s) \exp\Big(\int_0^s \big(f(\tau) + g(\tau)\big)\, d\tau\Big)\, ds\Big] \quad \ldots(25)$$

for all $t \in I$.

PROOF : Since H is subadditive and submultiplicative we have from (24)

$$H\big(x(t)\big) \leqslant H\big(n(t)\big) + \int_0^t f(s)\, H\big(x(s)\big)\, ds + \int_0^t f(s)\Big(\int_0^s g(\tau)\, H\big(x(\tau)\big)\, d\tau\Big)\, ds. \quad \ldots(26)$$

Since $H\big(n(t)\big)$ is positive, monotonic, nondecreasing, we have

$$\frac{H\big(x(t)\big)}{H\big(n(t)\big)} \leqslant 1 + \int_0^t f(s) \frac{H\big(x(s)\big)}{H\big(n(t)\big)}\, ds + \int_0^t f(s)\Big(\int_0^s g(\tau)\frac{H\big(x(\tau)\big)}{H\big(n(t)\big)}\, d\tau\Big)\, ds$$

$$\leqslant 1 + \int_0^t f(s) \frac{H\big(x(s)\big)}{H\big(n(s)\big)}\, ds + \int_0^t f(s)\Big(\int_0^s g(\tau)\frac{H\big(x(\tau)\big)}{H\big(n(\tau)\big)}\, d\tau\Big)\, ds \quad \ldots(27)$$

Define $v(t)$ by the right member of (27). Then

$$v'(t) = f(t)\left[\frac{H(x(t))}{H(n(t))} + \int_0^t g(\tau)\frac{H(x(\tau))}{H(n(\tau))}\,d\tau\right], \quad v(0) = 1,$$

which in view of (27) implies

$$v'(t) \leqslant f(t)\left[v(t) + \int_0^t g(\tau)\, v(\tau)\, d\tau\right]. \qquad \ldots(28)$$

If we put

$$m(t) = v(t) + \int_0^t g(\tau)\, v(\tau)\, d\tau, \qquad m(0) = v(0) = 1 \qquad \ldots(29)$$

it follows from (28), (29) and the fact that $v(t) \leqslant m(t)$, we have

$$m'(t) \leqslant (f(t) + g(t))\, m(t)$$

which implies the estimate for $m(t)$ such that

$$m(t) \leqslant \exp\left(\int_0^t (f(s) + g(s))\, ds\right) \qquad \ldots(30)$$

since $m(0) = 1$, Then from (28) and (30) we have

$$v'(t) \leqslant f(t) \exp\left(\int_0^t (f(s) + g(s))\, ds\right). \qquad \ldots(31)$$

Now, integrating both sides of (31) from 0 to t and substituting the value of $v(t)$ in (27) and then applying H^{-1} to both sides, we obtain the desired bound in (25).

An estimate for a slightly more general form of (24) when $n(t)$ is not monotonic nondecreasing is already obtained by Pachpatte (1975*a*). We observe that, by adding the monotonicity condition on the function $n(t)$ in the above theorem, the estimate in (25) takes a simple form.

We now apply Theorem 4 to establish the following more general integral inequalities.

Theorem 5—Let $x(t)$, $f(t)$, $g(t)$ and $h(t)$ be real valued positive continuous functions defined on I; H, H^{-1} are as defined in Theorem 4; and W is the same function as defined in Theorem 2, for which the inequality

$$x(t) \leqslant x_0 + H^{-1}\left[\int_0^t f(s)\, H(x(s))\, ds + \int_0^t f(s)\left(\int_0^s g(\tau)\, H(x(\tau))\, d\tau\right) ds\right] + \int_0^t h(s)\, W(x(s))\, ds \qquad \ldots(32)$$

is satisfied for all $t \in I$, where x_0 is a positive constant. Then

$$x(t) \leqslant \Omega^{-1}\Big[\Omega(x_0) + \int_0^t h(s)\, W\Big(H^{-1}\,[1 + \int_0^s f(\tau)\exp\big(\int_0^\tau (f(k) + g(k))\,dk\big)\,d\tau]\Big)\,ds\Big] \,.\, H^{-1}\Big[1 + \int_0^t f(s)\exp\big(\int_0^s (f(\tau) + g(\tau))\,d\tau\big)\,ds\Big], \qquad 0 \leqslant t \leqslant b \qquad \text{...(33)}$$

where Ω is as defined in (12), and Ω^{-1} is the inverse function Ω, and t is in the sub-interval $[0, b]$ of I such that

$$\Omega(x_0) + \int_0^t h(s)\, W\Big(H^{-1}\,[1 + \int_0^s f(\tau)\exp\big(\int_0^\tau (f(k) + g(k))\,dk\big)\,d\tau]\Big)\,ds \in \mathrm{Dom}\,(\Omega^{-1}).$$

We next establish a still more general form of Theorem 5 which may be used in certain situations.

Theorem 6—Let $x(t), f(t), g(t), h(t)$ and $q(t)$ be real valued positive continuous functions defined on I; H, H^{-1} are as defined in Theorem 4; $W(t, u), p(t), \Phi(t)$ are as defined in Theorem 3, and suppose further that the inequality

$$x(t) \leqslant p(t) + H^{-1}\Big[\int_0^t f(s)\, H(x(s))\,ds + \int_0^t f(s)\big(\int_0^s g(\tau)\, H(x(\tau))\,d\tau\big)\,ds\Big] + h(t)\,\Phi\big(\int_0^t q(s)\, W(s, x(s))\,ds\big) \qquad \text{...(34)}$$

is satisfied for all $t \in I$. Then

$$x(t) \leqslant k_1(t)\,[p(t) + h(t)\,\Phi(r(t))], \quad t \in I \qquad \text{...(35)}$$

where

$$k_1(t) = H^{-1}\Big[1 + \int_0^t f(s)\exp\big(\int_0^s (f(\tau) + g(\tau))\,d\tau\big)\,ds\Big],$$

and $r(t)$ is the maximal solution of

$$r'(t) = q(t)\, W\big(t, k_1(t)\,[p(t) + h(t)\,\Phi(r(t))]\big), \qquad r(0) = 0 \qquad \text{...(36)}$$

existing on I.

The details of the proofs of Theorems 5 and 6 follows by the similar arguments as in the proofs of Theorems 2 and 3 given in Section 2, by making use of Theorem 4. We omit the details.

§4. In this section we wish to establish some new integral inequalities which can be used as a tool in applications. We first establish the following generalization of Lemma 1 which is useful in our further discussion.

Theorem 7—Let $x(t)$ and $f(t)$ be real-valued positive continuous functions defined on I, $n(t)$ be a positive, monotonic, nondecreasing continuous function defined on I, and $H \in S$, for which the inequality

$$x(t) \leqslant n(t) + \int_0^t f(s)\Big(x(s) + \int_0^s f(\tau)\Big(\int_0^\tau f(k)\,H(x(k))\,dk\Big)\,d\tau\Big)\,ds \quad \ldots(37)$$

holds for all $t \in I$. Then

$$x(t) \leqslant n(t)\Big[1 + \int_0^t f(s)\Big(1 + \int_0^s f(\tau)\,G^{-1}\Big[G(1) + \int_0^\tau f(k)\,dk\Big]\,d\tau\Big)\,ds\Big], \quad 0 \leqslant t \leqslant b, \quad \ldots(38)$$

where

$$G(r) = \int_{r_0}^{r} \frac{ds}{s + H(s)}, \quad r \geqslant r_0 > 0 \quad \ldots(39)$$

and G^{-1} is the inverse of G, and t is in the subinterval $[0, b]$ of I so that

$$G(1) + \int_0^t f(s)\,ds \in \mathrm{Dom}\,(G^{-1}).$$

PROOF : Since $n(t)$ is positive, monotonic, nondecreasing we observe from (37) that

$$\frac{x(t)}{n(t)} \leqslant 1 + \int_0^t f(s)\Big(\frac{x(s)}{n(t)} + \int_0^s f(\tau)\Big(\int_0^\tau f(k)\,H\Big(\frac{x(k)}{n(t)}\Big)\,dk\Big)\,d\tau\Big)\,ds$$

$$\leqslant 1 + \int_0^t f(s)\Big(\frac{x(s)}{n(s)} + \int_0^s f(\tau)\Big(\int_0^\tau f(k)\,H\Big(\frac{x(k)}{n(k)}\Big)\,dk\Big)\,d\tau\Big)\,ds. \quad (40)$$

Now, by setting $v(t)$ is equal to the right member of (40) and following the similar argument as in the proof of Theorem 1 (see, Theorem 4 of Pachpatte 1975*b*) with suitable modifications we obtain the desired bound in (38).

Finally, we apply Theorem 7 to establish the following interesting and useful integral inequalities.

Theorem 8—Let $x(t)$, $f(t)$ and $g(t)$ be real valued nonnegative continuous functions defined on I; $H \in S$; and W is the same function as defined in Theorem 2, and suppose further that the inequality

$$x(t) \leqslant x_0 + \int_0^t f(s)\Big(x(s) + \int_0^s f(\tau)\Big(\int_0^\tau f(k) H(x(k))\, dk\Big)\, d\tau\Big)\, ds$$

$$+ \int_0^t g(s) W(x(s))\, ds \qquad \ldots(41)$$

is satisfied for all $t \in I$, where x_0 is a positive constant. Then

$$x(t) \leqslant \Omega^{-1}\Big[\Omega(x_0) + \int_0^t g(s) W\Big(1 + \int_0^s f(\tau)\Big(1 + \int_0^\tau f(k) G^{-1}[G(1)$$

$$+ \int_0^k f(n)\, dn]\, dk\Big)\, d\tau\Big)\, ds\Big] \cdot \Big[1 + \int_0^t f(s)\Big(1 + \int_0^s f(\tau) G^{-1}[G(1)$$

$$+ \int_0^\tau f(k)\, dk]\, d\tau\Big)\, ds\Big], \qquad 0 \leqslant t \leqslant b, \qquad \ldots(42)$$

where G, G^{-1} are as defined in Theorem 7, Ω is as defined in (12), and Ω^{-1} is the inverse function of Ω, and t is in the subinterval $[0, b]$ of I such that

$$G(1) + \int_0^t f(s)\, ds \in \mathrm{Dom}\,(G^{-1}),$$

and

$$\Omega(x_0) + \int_0^t g(s) W\Big(1 + \int_0^s f(\tau)\Big(1 + \int_0^\tau f(k) G^{-1}[G(1)$$

$$+ \int_0^k f(n)\, dn]\, dk\Big)\, d\tau\Big)\, ds \in \mathrm{Dom}\,(\Omega^{-1}).$$

We next formulate a still more general form of Theorem 8 which may be convenient in some applications.

Theorem 9—Let $x(t)$, $f(t)$, $h(t)$ and $q(t)$ be real valued nonnegative continuous functions defined on I; $H \in S$; $W(t, u)$, $p(t)$, $\Phi(t)$ are as defined in Theorem 3, and suppose further that the inequality

$$x(t) \leqslant p(t) + \int_0^t f(s)\Big(x(s) + \int_0^s f(\tau)\Big(\int_0^\tau f(k) H(x(k))\, dk\Big)\, d\tau\Big)\, ds$$

$$+ h(t)\, \Phi\Big(\int_0^t q(s) W(s, x(s))\, ds\Big) \qquad \ldots(43)$$

is satisfied for all $t \in I$. Then

$$x(t) \leqslant k_2(t)\,[p(t) + h(t)\, \Phi(r(t))], \quad t \in I_0, \qquad \ldots(44)$$

where

$$k_2(t) = 1 + \int_0^t f(s)\Big(1 + \int_0^s f(\tau)\, G^{-1}[G(1) + \int_0^T f(k)\, dk]\, d\tau\Big) ds, \quad \ldots(45)$$

G and G^{-1} are as in Theorem 7 and I_0 is the largest subinterval of I on which the right-hand side of (45) exists, and $r(t)$ is the maximal solution of

$$r'(t) = q(t)\, W\big(t, k_2(t)[p(t) + h(t)\,\Phi(r(t))]\big), \qquad r(0) = 0,$$

existing on I.

The proofs of Theorems 8 and 9 proceed much as in Theorems 2 and 3 given in Section 2, by making use of Theorem 7, and we leave the details to the reader.

Using the present approach, one can also obtain bounds for inequalities of the form

$$x(t) \leqslant n(t) + \int_0^t f(s)\Big(x(s) + \int_0^s g(\tau)\, x(\tau)\, d\tau + \int_0^s \big(f(\tau) + g(\tau)\big)\, H\big(x(\tau)\big)\, d\tau\Big) ds$$

where x, n, f, g and H are as in Theorem 1. In view of this remark, one can use this inequality to establish the inequalities similar to those obtained in Theorems 2 and 3. Since this translation is quite straightforward in view of the results of this paper, and we leave it for the reader to fill in where needed.

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