ON THE $|E, q|$ SUMMATION OF DERIVED FOURIER SERIES AND ITS CONJUGATE SERIES

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Mohanty and Mahapatra (1968) have proved the $|E, q|$ summability of Fourier series and its conjugate series. In this paper we prove $|E, q|$ summability of derived Fourier series and its conjugate series.

§1. *Definition*—A series $\sum_{n=1}^{\infty} a_n$ is said to be summable (E, q) for $q < 0$ to S, if $\sum_{n=1}^{\infty} (q+1)^{-n} b_n \to S$,

where

$$b_n = \sum_{k=1}^{\infty} \binom{n}{k} q^{n-k} a_k.$$

If $\sum_{1}^{\infty} (q+1)^{-n} b_n$ is absolutely convergent, then the series $\sum_{n=1}^{\infty} a_n$ is said to be summable $|E, q|$ (Hardy 1949).

§2. Let $f(t)$ be a periodic function with period 2π and integrable (L) over $(-\pi, \pi)$. Let its Fourier series be

$$\tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=0}^{\infty} A_n(t) \qquad \text{...(2.1)}$$

then the series conjugate to it is

$$\sum_{n=1}^{\infty} (b_n \cos nt - a_n \sin nt) = \sum_{n=1}^{\infty} B_n(t). \qquad \text{...(2.2)}$$

The series obtained by formally differentiating the Fourier series (2.1) at $t = x$ is

$$\sum_{n=1}^{\infty} n(b_n \cos nx - a_n \sin nx) = \sum_{n=1}^{\infty} nB_n(x) \qquad \text{...(2.3)}$$

and the series obtained by formally differentiating the conjugate series (2.2) at $t = x$ is

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$$\sum_{n=1}^{\infty} n (a_n \cos nx + b_n \sin nx) = \sum_{n=1}^{\infty} nA_n (x). \quad \ldots(2.4)$$

We write

$$\phi (t) = \tfrac{1}{2} \{f(x + t) + f(x - t)\},$$

$$\psi (t) = \tfrac{1}{2} \{f(x + t) - f(x - t)\},$$

$$g_1 (n, t) = \int_t^{\delta} (1 + q^2 + 2q \cos u)^{(n-1)/2} \sin\left\{(n-1) \tan^{-1} \frac{\sin u}{q + \cos u} + u\right\} du$$

and

$$g_2 (n, t) = \int_t^{\delta} (1 + q^2 + 2q \cos u)^{(n-1)/2} \cos \left\{(n-1) \tan^{-1} \frac{\sin u}{q + \cos u} + u\right\} du.$$

§3. Recently, Mohanty and Mohapatra (1968) have proved the following theorems for Fourier series and its conjugate series.

Theorem A—If $\phi (t) \log t^{-1}$ is of bounded variation in $(0, \delta)$, where $0 < \delta < 1$, then the series (2.1) is summable $| E, q |$ $(0 < q < 1)$ at $t = x$.

Theorem B—If (i) $\psi (t) \log t^{-1}$ is of bounded variation in $(0, \delta)$, where $0 < \delta < 1$, (ii) $| \psi (t) | t^{-1}$ is integrable in $(0, \delta)$, then the series (2.2) is summable $| E, q |$ $(0 < q < 1)$.

The object of this paper is to prove the following theorems for derived Fourier series and its conjugate series.

Theorem 1—If

$$\psi (+ 0) = 0 \quad \ldots(3.1)$$

and

$$\int_0^{\delta} \frac{| d\psi (t) |}{t^2} < \infty, \quad \ldots(3.2)$$

then the series (2.3) is summable $| E, q |$, $(0 < q < 1)$, where

$$0 < \delta < 1.$$

Theorem 2—If

$$\phi (+ 0) = 0 \quad \ldots(3.3)$$

and

$$\int_0^{\delta} \frac{| d\phi (t) |}{t^2} < \infty, \quad \ldots(3.4)$$

then the series (2.4) is summable $|E, q|$, $(0 < q < 1)$, where

$0 < \delta < 1$.

§4. In order to prove the theorems, we use the following estimates of the functions $g_1(n, t)$ and $g_2(n, t)$, which can be obtained easily by applying the second mean value theorem.

$$g_1(n,t) = 0\left\{\frac{1}{n}(1+q^2+2q\cos t)^{(n+1)/2}\right\}, \quad \ldots(4.1)$$

$$g_2(n,t) = 0\left\{\frac{1}{n}(1+q^2+2q\cos t)^{(n+1)/2}\right\}. \quad \ldots(4.2)$$

for $0 < n^{-1} < \delta < 1$ and $0 < q < 1$.

§5. *Proof of Theorem* 1—Let I denotes the $|E, q|$ summability of the series $\Sigma\, nB_n^{(x)}$, then we have

$$I = \frac{2}{\pi}\sum_{n=1}^{\infty} n(q+1)^{-n}\left|\int_0^{\pi} \psi(t)(1+q^2+2q\cos t)^{(n-1)/2} \times \sin\left\{(n-1)\tan^{-1}\frac{\sin t}{q+\cos t} + t\right\} dt\right|$$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty} n(q+1)^{-n}\left|\left(\int_{\delta}^{\pi} + \int_0^{\delta}\right)\psi(t)(1+q^2+2q\cos t)^{(n-1)/2} \times \sin\left\{(n-1)\tan^{-1}\frac{\sin t}{q+\cos t} + t\right\} dt\right|$$

$$= I_1 + I_2, \text{ say.}$$

We have

$$I_1 \leqslant \frac{2}{\pi}\sum_{n=1}^{\infty} n(q+1)^{-n}\int_{\delta}^{\pi} |\psi(t)|(1+q^2+2q\cos t)^{(n-1)/2}\,dt$$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty} n(q+1)^{-n}(1+q^2+2q\cos\delta)^{(n-1)/2}\int_{\delta}^{s} |\psi(t)|\,dt \times (\delta < s < \pi)$$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty} n(1+q)^{-n}(1+q)^{n-1}\left(1 - \frac{4q}{(1+q)^2}\sin^2\delta/2\right)^{\frac{n-1}{2}} \times \int_{\delta}^{s} |\psi(t)|\,dt$$

$$= \frac{2}{\pi}(1+q)^{-1} \sum_{n=1}^{\infty} n\left(1 - \frac{4q}{(1+q)^2}\sin^2(\delta/2)\right)^{\frac{n-1}{2}} \int_{\delta}^{\pi} |\psi(t)|\, dt$$

$$< \infty.$$

Using the condition (3.1), we have

$$\int_0^{\delta} \psi(t)(1 + q^2 + 2q\cos t)^{(n-1)/2} \sin\left\{(n-1)\tan^{-1}\frac{\sin t}{q+\cos t} + t\right\} dt$$

$$= -\Big[\psi(t) \int_t^{\delta} (1 + q^2 + 2q\cos u)^{(n-1)/2}$$

$$\times \sin\left\{(n-1)\tan^{-1}\frac{\sin u}{q+\cos u} + u\right\} du\Big]_0^{\delta}$$

$$+ \int_0^{\delta} d\psi(t)\, g_1^{(n,t)}$$

$$= \int_0^{\delta} d\psi(t)\, g_1(n, t).$$

Using (4.1), we have

$$I_2 \leqslant \sum_{n=1}^{\infty} n(q+1)^{-n} \int_0^{\delta} |d\psi(t)|\, |g_1(n,t)|$$

$$= A \int_0^{\delta} |d\psi(t)| \sum_{n=1}^{\infty} n(q+1)^{-n} |g_1(n,t)|$$

$$= A \int_0^{\delta} |d\psi(t)| \sum_{n=1}^{\infty} (q+1)^{-n} \,.\, 0\{(1 + q^2 + 2q\cos t)^{(n+1)/2}\}$$

$$= A \int_0^{\delta} |d\psi(t)| \sum_{n=1}^{\infty} \left(1 - \frac{4q}{(1+q)^2}\sin^2(t/2)\right)^{\frac{n+1}{2}}$$

$$= A \int_0^{\delta} |d\psi(t)| \sum_{n=1}^{\infty} (\cos t)^{n+1}\left(\text{if } \frac{2\sqrt{q}}{1+q}\sin(t/2) = \sin t\right)$$

$$= A \int_0^{\delta} \frac{|d\psi(t)|}{t^2}$$

$$< \infty,$$

where A is some constant not necessary same at each occurrence. This completes the proof of the theorem.

§6. *Proof of Theorem* 2— $\sum_{n=1}^{\infty} nA_n(x)$ is summable $|E, q|$

if

$$J = \sum_{n=1}^{\infty} (q+1)^{-n} \left| \sum_{k=1}^{n} \binom{n}{k} q^{n-k} kA_k(x) \right| < \infty.$$

Now,

$$J = \frac{2}{\pi} \sum_{n=1}^{\infty} n(q+1)^{-n} \left| \int_0^{\pi} \phi(t)(1+q^2+2q\cos t)^{(n-1)/2} \right.$$

$$\left. \times \cos\left\{(n-1)\tan^{-1}\frac{\sin t}{q+\cos t} + t\right\} dt \right|$$

$$= \frac{2}{\pi} \sum_{n=1}^{\infty} n(q+1)^{-n} \left| \left(\int_{\delta}^{\pi} + \int_0^{\delta} \right) \phi(t)(1+q^2+2q\cos t)^{(n-1)/2} \right.$$

$$\left. \times \cos\left\{(n-1)\tan^{-1}\frac{\sin t}{q+\cos t} + t\right\} dt \right|$$

$$= J_1 + J_2, \text{ say}.$$

Proceeding on the similar way as in the proof of Theorem 1, it is easy to see that

$$J_1 = o(1)$$

and

$$J_2 = o(1).$$

References

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