AN EXTREMAL PROBLEM FOR STARLIKE FUNCTIONS†

by SURINDER PAUL, ‡Department of Mathematics, Punjabi University, Patiala

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Let $S^*_{-2a_1}$ denote the class of functions $f(z) = z - 2a_1z^2 + b_3z^3 + \cdots$ which are regular, univalent and starlike with respect to origin in $E\{z : |z| < 1\}$ and have fixed second coefficient $-2a_1$, which is assumed to be real. In this paper bounds on $|f(z)|, f(z) \in S^*_{-2a_1}$ which depend on a_1, $|z|$ and argument of z have been obtained.

§ 1. Let S^* denote the class of functions $f(z) = z + b_2z^2 + b_3z^3 + \ldots$ which are regular, univalent and starlike with respect to origin in $E\{z: |z| < 1\}$ It is known (Nehari 1952) that a necessary and sufficient condition for $f(z) \in S^*$ is that

Re $zf'(z)/f(z) > 0$, $z \in E$. Let us denote by $S^*_{-2a_1}$ the sub-class of S^* whose functions $f(z) = z - 2a_1z^2 + b_3z^3 + \ldots$

have fixed second coefficient '$-2a_1$'. Without loss of generality we may assume that a_1 is real, so that $-1 \leqslant a_1 \leqslant 1$. The class $S^*_{-2a_1}$ has been investigated by various authors (Finkelstein 1967, Tepper 1970). Finkelstein (1967) established bounds on $|f(z)|$, $|f'(z)|$ for $f(z) \in S^*_{-2a_1}$. His results depend only on a_1 and $|z|$ but not on the argument of z.

In this paper we propose to determine distortion theorems for the class $S^*_{-2a_1}$ which depend upon the argument of z as well.

Let $\mathcal{P}$ denote the class of function $P(z) = 1 + p_1z + p_2z^2 + \ldots$ regular in E and with Re $P(z) > 0$ there. It is well known (Nehari 1952) that $|p_n| \leqslant 2$, $n = 1, 2, \ldots$ Further we denote by $\mathcal{P}_{-2a_1}$ the sub-class of $\mathcal{P}$ whose functions $P(z) = 1 - 2a_1z + p_2z^2 + \ldots\ldots$ have fixed second coefficient '$-2a_1$'. Without loss of generality we may assume that a_1 is real, $-1 \leqslant a_1 \leqslant 1$.

It is easy to see that $f(z) \in S^*_{-2a_1}$ iff

$$z\frac{f'(z)}{f(z)} \in \mathcal{P}_{-2a_1}.$$

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‡ *Present Address:* Department of Mathematics, P.M.N. College, Rajpura 140401, Punjab.

Hence to determine the bounds on $|f(z)|$, $f(z) \in S^*_{-2a_1}$ one has to establish the corresponding bounds on

$$\text{Re } P(z), \; P(z) \in \mathcal{P}_{-2a_1}.$$

Finally, we shall denote by B the class of regular functions

$\phi(z) = a_1 z + a_2 z^2 + \ldots$ in E which satisfy the condition $|\phi(z)| < 1$ for $|z| < 1$ and by B_{a_1} the sub-class of B whose functions have pre-assigned first coefficient 'a_1', which is assumed to be real.

§ 2. *Lemma*:—If $P(z) \in \mathcal{P}_{-2a_1}$ then

$$\frac{1 - r^2}{1 + r^2 + 2a_1 r \cos\theta} \leqslant \text{Re } P(re^{i\theta}) \leqslant \frac{(1 - r^2)^2 + 4r^2(1 - a_1^2)}{(1 - r^2)(1 + r^2 + 2a_1 r \cos\theta)} \quad \ldots(1)$$

Equality in the left-hand side holds for

$$P_0(z) = \frac{(1 - a_1 z) + (a_1 - z)\epsilon z}{(1 + a_1 z) + (a_1 + z)\epsilon z} \text{ for } \epsilon = e^{-2i\theta}$$

and in the right-hand side for

$$\epsilon = \frac{(1 + r^2)\sin\theta + i\{(1 + r^2)\cos\theta + 2a_1 r\}}{(1 + r^2)\text{Sin }\theta - i\{(1 + r^2)\cos\theta + 2a_1 r\}}$$

where $z = re^{i\theta}$.

PROOF: For $P(z) \in \mathcal{P}_{-2a_1}$ we may write

$$P(z) = \frac{1 - \phi(z)}{1 + \phi(z)}$$

for some $\phi(z) \in B_{a_1}$

Clearly we have

$$\phi(z) = z\frac{a_1 + \phi_1(z)}{1 + a_1\phi_1(z)}, \quad \phi_1(z) \in B.$$

Therefore we may write

$$P(z) = \frac{(1 - a_1 z) + (a_1 - z)\phi_1(z)}{(1 + a_1 z) + (a_1 + z)\phi_1(z)}.$$

or equivalently

$$\phi_1(z) = \frac{(1 - a_1 z) - (1 + a_1 z)P(z)}{(z - a_1) + (z + a_1)P(z)}$$

Since by Schwarz lemma $|\phi_1(z)| \leqslant |z|$, $z \in E$

we obtain

$$\left| \frac{(1 - a_1 z) - (1 + a_1 z) P(z)}{(z - a_1) + (z + a_1) \mathrm{P}(z)} \right| \leqslant |z|$$

which on simplifying gives

$$|P(z) - A| \leqslant \rho_1 \qquad \ldots(2)$$

where

$$A = \frac{1 + |z|^4 - 2a_1^2 |z|^2 - i2a_1 (1 - |z|^2 \operatorname{Im} z}{(1 - |z|^2)(1 + |z|^2 + 2a_1 \operatorname{Re} z)} \qquad \ldots(3)$$

$$\rho_1 = \frac{2|z|^2 (1 - a_1^2)}{(1 - |z|^2)(1 + |z|^2 + 2a_1 \operatorname{Re} z)} \qquad \ldots(4)$$

from (2) we obtain

$$\operatorname{Re} A - \rho_1 \leqslant \operatorname{Re} P(z) \leqslant \operatorname{Re} A + \rho_1$$

which in view of (3) and (4) yields (1)

To prove sharpness of inequalities in (1) we notice that equalities will be attained for function $P_0(z)$ defined as

$$P_0(z) = \frac{(1 - a_1 z) + (a_1 - z)\epsilon z}{(1 + a_1 z) + (a_1 + z)\epsilon z}, \ |\epsilon| = 1 \qquad \ldots(5)$$

which certainly belongs to the class $\mathscr{P}_{-2a_1}$

It is obvious that in the left hand side equality is attained when $\epsilon = e^{-2i\theta}$, $z = re^{i\theta}$ and for the right-hand side equality is attained by (5) when

$$\tan \phi/2 = \frac{(1 + r^2)\cos\theta + 2a_1 r}{(1 + r^2)\sin\theta}$$

where $\epsilon = c^{i\theta}$ and $z = re^{i\theta}$,

which completes the proof of lemma.

§ 3. *Theorem*:—If $f(z) \in S^*_{-2a_1}$ then

$$\frac{r}{(1 + r^2 + 2a_1 r\cos\theta)} \leq |f(re^{i\theta})|$$

$$\leq \frac{r}{(1 + r^2 + 2a_1 r\cos\theta)^{\frac{a_1^2 \sin^2\theta}{1 - a_1^2\cos^2\theta}} (1 + r)^{\frac{1 - a_1^2}{1 - a_1\cos\theta}} (1 - r)^{\frac{(1 - a_1^2)}{1 + a_1\cos\theta}}} \qquad \ldots(6)$$

PROOF: Since $f(z) \in S^*_{-2a_1}$ we have $\operatorname{Re}\left(zf'(z)/f(z)\right) > o,\ z \in E.$

Now if $z = re^{i\theta}$ then by Cauchy-Riemann equations

$$\frac{\partial}{\partial r}\left[\log\left|\frac{f(z)}{z}\right|\right] = \frac{1}{r}\ \operatorname{Re}\ (P(z) - 1),\ P(z) \in \mathcal{P}_{-2a_1}$$

Hence in view of (1) we have

$$\frac{1}{r}\left\{\frac{1 - r^2}{1 + r^2 + 2a_1 r\cos\theta} - 1\right\} \leqslant \frac{\partial}{\partial r}\left\{\log\left|\frac{f(z)}{z}\right|\right\}$$

$$\leqslant \frac{1}{r}\left\{\frac{(1 - r^2)^2 + 4r^2(1 - a_1^2)}{(1 - r^2)\ (1 + r^2 + 2a_1 r\cos\theta)} - 1\right\}$$

which on integration yields (6).

It is not difficult to verify that the left-hand side equality in (6) is attained for the function $f(z)$ defined as

$$f(z) = \frac{z}{1 + a_1(1 + \epsilon)\ z + \epsilon z^2}$$

where

$$\epsilon = e^{-2i\theta},\ z = re^{i\theta}.$$

Mnimizing the left-hand side and maximizing the right-hand side of (6) with respect to 'θ' we obtain

Corollary: If $f(z) \in S^*_{-2a_1}$ then we have

$$\frac{r}{1 + 2\,|a_1|\,r + r^2} \leqslant |f(re^{i\theta})| \leqslant \frac{r}{(1 - r^2)}\left(\frac{1 + r}{1 - r}\right)^{(a_1)}$$

These inequalities were obtained by Finkelstein (1967).

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