SOME APPLICATIONS OF (f, d_n) SUMMABILITY

by RICHARD A. SHOOP, *Department of Mathematics, Kent State University, Kent, OH 44242 U.S.A.*

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In this paper it is shown that a class of (f, d_n) summability matrices corresponding to sequences (d_n) for which $d_n \geqslant 0$ for all in preserve the Gibbs Phenomenon for Furier Series. Also, (f, d_n) summability of series of Legendre Polynomials is investigated in the case where f is a bilinear transformation with real coefficients.

1. INTRODUCTION

The (f, d_n) summability method was introduced by Smith (1965) as a generalization of the $[F, d_n]$ method of Jakimovski. Let f be a non-constant function, analytic on $\{z: |z| < R\}$ $(1 < R)$, and let $\{d_n\}$ be a sequence of complex numbers with $-f(0) \neq d_n \neq -f(1)$ $(n = 1, 2..)$. Let $a_{00} = 1$, $a_{0k} = 0$ $(k \geqslant 1)$, and

$$\prod_{j=1}^{n} \frac{f(z) + d_j}{f(1) + d_j} = \sum_{k=0}^{\infty} a_{nk} z^k \quad .. \quad .. \quad .. \quad (n \geqslant 1).$$

The matrix $A = (a_{nk})$ is the (f, d_n) method generated by the function f and the sequence $\{d_n\}$. The class of (f, d_n) methods contains several well-known special cases: If $f(z) = z$, the corresponding matrix is the $[F, d_n]$ method of Jakimovski (1959). If $f(1) = 1$, and $d_n = 0$ for all n, the resulting (f, d_n) method is the Sonnenschein matrix generated by f:

$$a_{00} = 1, \; a_{0k} = 0 \quad .. \quad .. \quad (k \geqslant 1)$$

and

$$[f(z)]^n = \sum_{k=0}^{\infty} a_{nk} z^k \quad .. \quad .. \quad (n \geqslant 1).$$

It is known that the Euler method, the Taylor method, the Karamata method, and the Borel method are all Sonnenschein matrices for the appropriate choices of the function f. Smith showed that if $\{d_n\}$ is a sequence of nonnegative real numbers and if the coefficients in the Maclaurin expansion of f are real and nonnegative, then the resulting (f, d_n) method is regular (i.e., transforms convergent sequences into convergent sequences with the same limit) if and only if

$$\sum_{n=1}^{\infty} \frac{1}{f(1) + d_n} = \infty .$$

In this paper we shall investigate the Gibbs phenomenon for regular (f, d_n) summability methods, and determine domains into which (f, d_n) methods generated by a linear fractional transformation with real coefficients analytically continue a series of Legendre polynomials.

2. The Gibbs Phenomenon for (f, d_n) Summability

Let $\xi(t)$ be a real-valued function of bounded variation on some open interval I, and having only a finite number of discontinuities on I. Let $S_n(t)$ be the n-th partial sum of the Fourier series of ξ at t, and suppose that $t_0 \in I$ is a discontinuity of ξ. It is known that for each extended real number τ, there exists a sequence $\{t_n\}$ in I, converging to t_0, for which

$$\lim_{n \to \infty} S_n(t_n) = \frac{\xi(t_0+0) + \xi(t_0-0)}{2} + \frac{\xi(t_0+0) - \xi(t_0-0)}{\pi} \times \int_0^\tau \frac{\sin u}{u}\, du.$$

This behavior of the sequence $\{S_n(t)\}$ is known as the Gibbs phenomenon. If $A = (a_{nk})$ is a regular summability matrix and if the sequence $\{\sigma_n(t)\}$, where

$$\sigma_n(t) = \sum_{k=0}^{\infty} a_{nk} S_k(t)$$

also has the property described, then A is said to preserve the Gibbs phenomenon. Miracle (1960) showed that in order to prove that a regular matrix A preserves the Gibbs phenomenon, it is sufficient to show that for each $\tau \in [-\pi, \pi,]$ there exists a real null sequence $\{t_n\}$, such that

$$\lim_{n \to \infty} \sigma_n(t_n) = \int_0^\tau \frac{\sin u}{u}\, du$$

where $\{\sigma_n(t_n)\}$ is the A-transform of the sequence of partial sums of the Fourier series of the particular 2π-periodic function

$$\phi(t) = \begin{cases} \frac{\pi}{2} \text{ if } 0 < t < \pi \\ -\frac{\pi}{2} \text{ if } -\pi < t \leqslant 0 \\ 0 \text{ if } t \equiv 0 \pmod{\pi}. \end{cases} \qquad (2{\cdot}1)$$

Assume now that $\{d_n\}$ is a sequence of non-negative real numbers, that the coefficients in the Maclaurin expansion of $f(z)$ are real and non-negative, and that $|f(z)| \leqslant 1$ for $|z| \leqslant 1$, with $f(1) = f'(1) = 1$. Then the resulting (f, d_n) matrix is regular (Smith 1965) and has real entries.

Theorem 2·1—Let $S_n(t)$ be the n-th partial sum of the Fourier series of the function $\phi(t)$ defined by (2·1), let $\{\sigma_n(t)\}$ be the (f, d_n)-transform of the sequence $\{S_n(t)\}$, and let $H_n = \sum_{j=1}^{n} (1 + d_j)^{-1}$. If $\{t_n\}$ is a positive (resp. negative) null sequence, such that $\lim_{n\to\infty} H_n t_n = \frac{\tau}{2}$ $(0 \leqslant \tau \leqslant \pi)$, (resp. $-\pi \leqslant \tau \leqslant 0$), then

$$\lim_{n\to\infty} \sigma_n(t_n) = \int_0^{\tau} \frac{\sin u}{u}\, du$$

i.e., the (f, d_n) method preserves the Gibbs phenomenon.

Proof: It is known that the partial sum $S_n(t)$ is given by

$$S_n(t) = \int_0^{t} \frac{\sin 2n u}{\sin u}\, du.$$

Thus,

$$\sigma_n(t) = \sum_{k=0}^{\infty} a_{nk} \int_0^{t} \frac{\sin 2k u}{\sin u}\, du.$$

For $|u| \leqslant \frac{\pi}{2}$, it is shown inductively that

$$|\sin 2k u| \leqslant k\pi |\sin u|.$$

Thus,

$$\left| \sum_{k=0}^{\infty} a_{nk} \frac{\sin 2k u}{\sin u} \right| \leqslant \pi \sum_{k=0}^{\infty} |k\, a_{nk}|$$

and this latter series converges due to the analyticity of f on the closed unit disc.

Hence, for $|u| \leqslant \frac{\pi}{2}$, the series

$$\sum_{k=0}^{\infty} a_{nk} \frac{\sin 2k u}{\sin u}$$

is uniformly convergent, and we may write

$$\sigma_n (t) = \int_0^t \frac{1}{\sin u} \sum_{k=0}^{\infty} a_{nk} \sin 2k\, u\, du.$$

Using the definition of the matrix A and the fact that a_{nk} is real for all n and k, we obtain

$$\sigma_n (t) = \int_0^t \frac{1}{\sin u} Im\left\{ \sum_{k=0}^{\infty} a_{nk} (e^{2iu})^k \right\} du$$

$$= \int_0^t \frac{1}{\sin u} Im \left\{ \prod_{j=1}^{n} \frac{f(e^{2iu}) + d_j}{1 + d_j} \right\} du.$$

Define numbers R_j and θ_j by the relation

$$R_j e^{i\theta_j} = f(e^{2iu}) + d_j.$$

Then

$$\sigma_n (t) = \int_0^t \frac{1}{\sin u} \prod_{j=1}^{n} \frac{R_j}{1 + d_j} \sin \left(\sum_{j=1}^{n} \theta_j \right) du. \quad .. \quad .. \quad (2{\cdot}2)$$

Since $f(1) = f'(1) = 1$, the Taylor expansion of $f(z) - z$ about the point $z_0 = 1$ takes the form

$$f(z) - z = \sum_{=2}^{\infty} a_k (z - 1)^k$$

$$= 0 (| z - 1 |)^2. \quad .. \quad .. \quad (z \to 1).$$

If $z = e^{2iu}$, this becomes

$$f(e^{2iu}) = e^{2iu} + 0 (| e^{2iu} - 1 |^2) .. \quad .. \quad (u \to 0)$$

$$= e^{2iu} + 0(u^2). \quad .. \quad .. \quad .. \quad (u \to 0).$$

Hence,

$$R_j e^{i\theta_j} \; e^{2iu} + d_j + 0(u^2) \quad .. \quad .. \quad (u \to 0),$$

and the following relations are readily verified :

$$R_j \cos \theta_j = \cos 2u + d_j + 0\,(u^2) \quad \ldots \quad (u \to 0), \qquad (2.3a)$$

$$R_j \sin \theta_j = \sin 2u + 0\,(u^2) \quad \ldots \quad \ldots \quad (u \to 0), \qquad (2{\cdot}3b)$$

$$R_j^2 = 1 + 2d_j \cos 2u + d_j^2 + (1 + d_j).\, 0\,(u^2) \quad \ldots \quad (u \to 0), \qquad (2.3c)$$

$$R_j \leqslant 1 + d_j. \quad \ldots \quad \ldots \quad \ldots \qquad (2{\cdot}3d)$$

If $t \geqslant 0$, then $\theta_j \geqslant 0$. If $t \leqslant 0$, then $\theta_j \leqslant 0$. In either case,

$$\theta_j = 0\,(|u|) \quad \ldots \quad \ldots \quad (u \to 0). \qquad (2{\cdot}3e)$$

Temporarily assume that $\tau \geqslant 0$. Then the sequence $\{t_n\}$, the existence of which is asserted in the theorem, may be chosen to be nonnegative. Thus, assume that $t \geqslant 0$, so that $\theta_j \geqslant 0$ by (2·3e). Since $d_j \geqslant 0$ and $f(1) = 1$, from the equation $R_j e^{i\theta} j = f(e^{2iu}) + d_j$, it follows that $\lim\limits_{u \to 0} R_j \geqslant 1$. Choose $t_0 > 0$, so that whenever $0 \leqslant u \leqslant t_0$, $R_j > \frac{1}{2}$, and assume in what follows that $t \leqslant t_0$.

From (2·3c) and (2·3d), it follows that

$$1 - \frac{R_j}{1+d_j} \leqslant 1 - \frac{R_j^2}{(1 + d_j)^2}$$

$$= \frac{2d_j\,(1 - \cos 2u) - (1 + d_j).\, 0(u^2)}{(1 + d_j)^2} \quad \ldots \quad \ldots \quad (\mathrm{u} \to 0)$$

$$= \frac{4d_j \sin^2 u}{(1 + d_j)^2} + \frac{0\,(u^2)}{1 + d_j} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (\mathrm{u} \to 0)$$

$$= \frac{0\,(u^2)}{1 + d_j} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (\mathrm{u} \to 0)$$

Thus,

$$0 \leqslant 1 - \prod_{j=1}^{n} \frac{R_j}{1 + d_j}$$

$$= \left(1 - \frac{R_1}{1 + d_1}\right) + \sum_{k=2}^{n} \left(\prod_{j=1}^{k-1} \frac{R_j}{1 + d_j}\right)\left(1 - \frac{R_k}{1 + d_k}\right)$$

$$\leqslant \sum_{k=1}^{n} \left(1 - \frac{R_k}{1 + d_k}\right)$$

$$= \sum_{k=1}^{n} \frac{0\,(u^2)}{1 + d_k} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (\mathrm{u} \to 0)$$

$$= (0\, H_n\, u^2) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (H_n u^2 \to 0).$$

If $\prod_{j=1}^{n} \frac{R_l}{1+d_j}$ is replaced by $1 + 0(H_n u^2)$ in (2·2), we obtain

$$\sigma_n(t) = \int_0^t \frac{\sin\left(\sum_{j=1}^{n} \theta_j\right)}{\sin u} du$$

$$+ 0\left\{H_n \int_0^t \frac{u^2}{\sin u} \sin\left(\sum_{j=1}^{n} \theta_j\right) du\right\},$$

or

$$\sigma_n(t) = \int_0^t \frac{\sin\left(\sum_{j=1}^{n} \theta_j\right)}{\sin u} du + 0(H_n t^2) \qquad .. \quad (H_n t^2 \to 0). \ (2{\cdot}4)$$

We now estimate $\sum_{j=1}^{n} \theta_j$ by a simpler expression, noting the error of approximation. According to (2·3*b*),

$$\left| \theta_j - \frac{2u}{1+d_j} \right| \leqslant \frac{R_j}{1+d_j}(\theta_j - \sin\theta_j) + \frac{1}{1+d_j}(2u - \sin 2u + 0(u^2))$$

$$+ \left(1 - \frac{R_j}{1+d_j}\right)\theta_j$$

$$\leqslant \frac{R_j \theta_j^3}{1+d_j} + \frac{8u^3 + 0(u^2)}{1+d_j} + \frac{0(u^2)}{1+d_j}$$

$$= \frac{0(u^2)}{1+d_j} \qquad .. \quad .. \quad .. \quad (u \to 0)$$

Hence,

$$\theta_j = \frac{2u}{1+d_j} + \frac{0(u^2)}{1+d_j} \qquad .. \quad .. \quad .. \quad (u \to 0)$$

so,

$$\sum_{j=1}^{n} \theta_j = 2H_n u + 0(H_u u^2) \qquad (H_n u^2 \to 0)$$

and

$$\sigma_n(t) = \int_0^t \frac{\sin [2H_n u + 0 (H_n u^2)]}{\sin u} du + 0 (H_n t^2)$$

$$= \int_0^t \frac{\sin (2H_n u) \cos 0(H_n u^2)}{\sin u} du$$

$$+ \int_0^t \frac{\cos (2H_n u) \sin 0(H_n u^2)}{\sin u} du + 0 (H_n t^2)$$

$$= \int_0^t \frac{\sin (2H_n u) \cos 0(H_n u^2)}{\sin u} du + 0(H_n t^2)$$

$$= \int_0^t \frac{\sin 2H_n u}{\sin u} du + \int_0^t \frac{0 (H_n u^2)}{\sin u} du + 0(H_n t^2)$$

$$= \int_0^t \frac{\sin 2H_n u}{\sin u} du + 0(H_n t^2) \quad \ldots \quad \ldots \quad (H_n t^2 \to 0).$$

Now let $[t_n]$ be a positive null sequence such that $H_n t_n \to \frac{\tau}{2}$. Then $H_n t_n^2 \to 0$, and

$$\sigma_n(t_n) = \int_0^{t_n} \frac{\sin 2H_n u}{\sin u} du + o(1) \quad \ldots \quad \ldots \quad \ldots \quad (n \to \infty)$$

$$= \int_0^{t_n} \frac{\sin 2H_n u}{u} du + o(1) \quad \ldots \quad \ldots \quad (n \to \infty)$$

$$= \int_0^{2H_n t_n} \frac{\sin u}{u} du + o(1) \quad \ldots \quad (n \to \infty)$$

$$= \int_0^{\tau} \frac{\sin u}{u} du + o(1) \quad \ldots \quad \ldots \quad \ldots \quad (n \to \infty)$$

for $0 \leqslant \tau \leqslant \pi$. If $-\pi \leqslant \tau \leqslant 0$, then $0 \leqslant -t \leqslant \pi$, and since σ_n is an odd function of t, we still obtain

$$\sigma_n(t_n) = \int_0^\tau \frac{\sin u}{u}\,du + o(1) \qquad \ldots \qquad \ldots \qquad (n \to \infty).$$

As corollaries of Theorem 2·1, we obtain that the Gibbs phenomenon is preserved both by regular $[F, d_n]$ matrices (Miracle 1960), and by regular Sonnenschein matrices (Sledd 1962).

Corollary 2·1—If A is a regular $[F, d_n]$ matrix for which d_n is real and non negative for each n, then A preserves the Gibbs phenomenon.

Corollary 2·2—Let f be analytic on $\{z : |z| < R\}$ $(R > 1)$, with $|f(z)| \leqslant 1$ for $|z| \leqslant 1$, and $f(1) = f'(1) = 1$. Suppose that a_n is real for all n, where $f(z) = \sum_{n=0}^{\infty} a_n z^n$, and that the Sonnenschein matrix $F = (f_{nk})$ generated by F is regular. If $\{t_n\}$ is a positive null sequence, with $\lim_{n \to \infty} n\,t_n = \frac{\tau}{2}$ $(0 \leqslant \tau \leqslant \pi)$, and if $[\sigma_n(t)]$ is the F-transform of the sequence of partial sums of the Fourier series of the function ϕ defined by (2·1), then

$$\lim_{n \to \infty} \sigma_n(t_n) = \int_0^\tau \frac{\sin u}{u}\,du.$$

3. The (f, d_n) Summability of Series of Legendre Polynomials

Let $P_n(z)$ and $Q_n(t)$ denote the Legendre polynomials of degree n of the first and second kind, respectively. Let t be a fixed complex number not in the interval $[-1, 1]$. It is known (Whittaker and Watson 1952, p. 321) that

$$\frac{1}{t - z} = \sum_{n=0}^{\infty} (2n + 1)\, P_n(z)\, Q_n(t)$$

for all z interior to the ellipse E with foci ± 1, which passes through t.

Let

$$S_n = \sum_{k=0}^{n} (2k + 1)\, P_k(z)\, Q_k(t), \qquad \ldots \qquad \ldots \qquad \ldots \qquad (3·1)$$

and

$$\delta_n = P_{n+1}(z)\, Q_n(t) - P_n(z)\, Q_{n+1}(t). \quad \ldots \qquad \ldots \qquad \ldots \qquad (3·2)$$

By the Christoffel formula (Whittaker and Watson 1952, p. 321)

$$\frac{1}{t-z} = S_n + \frac{1}{t-z}(n+1)\,\delta_n.$$

Choose the branch of $(z^2-1)^{1/2}$ for which $z+(z^2-1)^{1/2}$ lies outside the unit circle. Let

$$\mu = \mu(\phi) = z + (z^2-1)^{1/2}\cos\phi \qquad (0 \leqslant \phi \leqslant \pi)$$

and

$$\nu = \nu(u) = t + (t^2-1)^{1/2}\cosh u \qquad .. \qquad (0 \leqslant u < \infty)$$

The Laplace integral representations (Whittaker and Watson 1952, p. 319) for $P_n(z)$ and $Q_n(t)$ are

$$P_n(z) = \frac{1}{\pi}\int_0^{\pi} \mu^n\, d\phi \qquad .. \qquad .. \qquad .. \qquad (3{\cdot}3)$$

and

$$Q_n(t) = \int_0^{\infty} \nu^{-n-1}\, du. \qquad .. \qquad .. \qquad .. \qquad (3{\cdot}4)$$

From (3·2), (3·3), and (3·4) it follows that

$$\delta_n = \frac{1}{\pi}\int_0^{\pi}\int_0^{\infty} \left(\frac{\mu}{\nu}\right)^n \left[\frac{\mu}{\nu} - \frac{1}{\nu^2}\right] du\, d\phi. \qquad .. \qquad .. \qquad (3{\cdot}5)$$

If $A = (a_{nk})$ is a regular summability matrix, to determine the domain in which z must lie for the series (3·1) to be A-summable to $\frac{1}{t-z}$, it is sufficient to determine the values of z for which the A-transform of the sequence $\{(n+1)\,\delta_n\}$ tends to zero. This problem has been considered for Euler means by Prachar (1948—49); for Taylor and $[F, d_n]$ means by Cowling and King (1962—63); for the $\mathcal{J}(r_n)$ method by Powell (1967); and for $\mathcal{J}(g_m, z)$ summability by Wood (1968). We shall determine the domain of summability of the series (3·1) for a family of regular (f, d_n) methods.

Let f be analytic on the disc $\{w: |w| \leqslant R\}$ $(R > 1)$, and let $[d_j]$ be a sequence of nonnegative real numbers. Suppose that $A = (a_{nk})$, the resulting (f, d_n) matrix, is regular. For $0 < r \leqslant R$, let

$$\Delta_r = \{w: |w| < r\},$$

and

$$\bar{\Delta}_r = \{w: |w| \leqslant r\}.$$

Lemma 3·1—Let t be a fixed complex number not in the interval [—1, 1]. Then the series (3·1) is A-summable to $(t - z)^{-1}$ for all z for which

$$\frac{\mu}{v} \in \triangle_R \text{ for all } \phi \text{ in } [0, \pi] \text{ and } u \text{ in } [0, \infty),$$

and

$$\lim_{n \to \infty} \left[\prod_{j=1}^{n} \frac{f(\frac{\mu}{\nu}) + d_j}{f(1) + d_j} \right] \left[1 + \sum_{j=1}^{n} \frac{\frac{\mu}{\nu} f'(\frac{\mu}{\nu})}{f(\frac{\mu}{\nu}) + d_j} \right] = 0, \quad .. \quad (3·6)$$

uniformly for ϕ in $[0, \pi]$ and u in $[0, \infty]$.

PROOF: By (3·5), the A-transform of the sequence $\{(k+1)\delta_k\}$ is given by

$$\sigma_n = \frac{1}{\pi} \sum_{k=0}^{\infty} a_{nk} (k+1) \int_0 \int_0^{\infty} \left(\frac{\mu}{\nu}\right)^k \left[\frac{\mu}{\nu} - \frac{1}{\nu^2}\right] du \, d\phi. \quad .. \quad (3·7)$$

Let $w = \dfrac{\mu}{\nu}$. Then according to the definition of the matrix A.

$$\sum_{k=0}^{\infty} a_{nk} w^{k+1} = w \prod_{j=1}^{n} \frac{f(w) + d_j}{f(1) + d_j},$$

so that

$$\sum_{k=0}^{\infty} a_{nk} (k+1) w^k = \frac{d}{dw} \left[w \prod_{j=1}^{n} \frac{f(w) + d_j}{f(1) + d_j} \right]$$

$$= w \sum_{i=1} \frac{f'(w)}{f(1) + d_i} \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{f(w) + d_j}{f(1) + d_j} + \prod_{=1}^{n} \frac{f(w) + d_j}{f(1) + d_j}$$

$$= wf'(w) \left[\prod_{j=1}^{n} \frac{1}{f(1) + d_j} \right] \sum_{i=1}^{n} \prod_{\substack{j=1 \\ j \neq i}}^{n} [f(w) + d_j] + \prod_{j=1}^{n} \frac{f(w) + d_j}{f(1) + d_i}$$

$$= wf'(w) \prod_{j=1}^{n} \frac{f(w) + d_j}{f(1) + d_j} \sum_{j=1}^{n} \frac{1}{f(w) + d_j} + \prod_{j=1}^{n} \frac{f(w) + d_j}{f(1) + d_j}$$

$$= \left[\prod_{j=1}^{n} \frac{f(w) + d_j}{f(1) + d_j} \right] \left[1 + wf'(w) \sum_{j=1}^{n} \frac{1}{f(w) + d_j} \right].$$

Hence,

$$\sum_{k=0}^{\infty} a_{nk}(k+1)\left(\frac{\mu}{\nu}\right)^k$$

$$= \left[\prod_{j=1}^{n} \frac{f(\frac{\mu}{\nu}) + d_j}{f(1) + d_j}\right]\left[1 + \frac{\mu}{\nu} f'(\tfrac{\mu}{\nu}) \sum_{j=1}^{n} \frac{1}{f(\frac{\mu}{\nu}) + d_j}\right]. \qquad (3{\cdot}8)$$

Since f is analytic on Δ_R, the series $\sum_{k=0}^{\infty} a_{nk}\ (k+1)\ w^k$ is uniformly convergent on Δ_r $(r < R)$. Thus, the order of summation and integration in (3·7) may be changed. From (3·7) and (3·8), we obtain

$$\sigma_n = \frac{1}{\pi}\int_0^{\pi}\int_0^{\infty}\left[\frac{\mu}{\nu} - \frac{1}{\nu^2}\right]\left[\sum_{k=0}^{\infty} a_{nk}(k+1)\left(\frac{\mu}{\nu}\right)^k\right] du\, d\phi$$

$$= \frac{1}{\pi}\int_0^{\pi}\int_0^{\infty}\left[\frac{\mu}{\nu} - \frac{1}{\nu^2}\right]\left[\prod_{j=1}^{n}\frac{f(\frac{\mu}{\nu}) + d_j}{f(1) + d_j}\right]\left[1 + \frac{\mu}{\nu} f'\left(\frac{\mu}{\nu}\right)\right.$$

$$\left.\sum_{j=1}^{n} \frac{1}{f(\frac{\mu}{\nu}) + d_j}\right] du\, d\phi.$$

Thus,

$$|\sigma| \leqslant \left[\frac{1}{\pi}\int_0^{\pi}\int_0^{\infty}\left|\frac{\mu}{\nu} - \frac{1}{\nu^2}\right| du\, d\phi\right]\left[\operatorname*{Max}_{u;\phi}\left|\prod_{j=1}^{n}\frac{f(\frac{\mu}{\nu})+d_j}{f(1)+d_j}\left(1 + \sum_{j=1}^{n}\frac{\frac{\mu}{\nu} f'(\frac{\mu}{\nu})}{f(\frac{\mu}{\nu}) + d_j}\right)\right|\right].$$

Since the integral

$$\frac{1}{\pi}\int_0^{\pi}\int_0^{\infty}\left|\frac{\mu}{\nu} - \frac{1}{\nu^2}\right| du\, d\phi$$

is bounded, it follows that $\sigma_n \to 0$ whenever

$$\lim_{n\to\infty}\left[\prod_{j=1}^{n}\frac{f(\frac{\mu}{\nu}) + d_j}{f(1) + d_j}\right]\left[1 + \sum_{j=1}^{n}\frac{\frac{\mu}{\nu} f'(\frac{\mu}{\nu})}{f(\frac{\mu}{\nu}) + d_j}\right] = 0$$

uniformly for $\phi \in (0, \pi)$, and $u \in (0, \infty)$.

The following two lemmas are modifications of results of Smith (1965, p. 514). The proofs are omitted since they are similar to those in the aforementioned paper.

Lemma 3·2—If $d_j > 0$ for all j, $\lim_{j\to\infty} d_j = \infty$ and $\sum_{j=1}^{\infty} \frac{1}{d_j} = \infty$, then (3·6) is satisfied for all z such that

$$\operatorname{Re}\left\{f\left(\frac{\mu}{\nu}\right)\right\} < \operatorname{Re}\{f(1)\},$$

for all ϕ in $[0, \pi]$, and u in $(0, \infty)$.

Lemma 3·3—If $f(1) \neq 0$, $d_j \geqslant 0$ for all j, and $\lim_{j\to\infty} d_j = 0$, then (3·6) is satisfied for all z such that

$$|f(\tfrac{\mu}{\nu})| < |f(1)|$$

for all ϕ in $[0, \pi]$ and u in $(0, \infty)$.

Suppose now that $f(w) = \dfrac{aw + b,}{cw + d}$ where a, b, c, and d are real, with $ad - bc > 0$, and $\left|\dfrac{d}{c}\right| > 1$. We shall obtain three results (Theorems 3·1, 3·2, and 3·3) which provide the domain of (f, d_n)—summability of the series (3·1) under the condition that $\lim_{j\to\infty} d_j = 0$. Since both the Euler means and the Taylor means are (f, d_n) methods generated by functions f of the type considered here, our results will generalize both those of Prachar (1948-49) and of Cowling and King (1962-63).

According to Lemma 3·3, condition (3·6) will be satisfied whenever $|f(\frac{\mu}{\nu})| < |f(1)|$. This occurs if and only if

$$|c + d|^2 \, |a\mu + b\nu|^2 < |a + b|^2 \, |c\mu + d\nu|^2. \quad .. \qquad (3{\cdot}9)$$

By writting μ and ν in terms of their real and imaginary parts, and simplifying the resulting expression, inequality (3·9) may be expressed as

$$(ad + bc + 2ac)\,|\mu|^2 - 2(ac - bd)\operatorname{Re}\{\mu\bar{\nu}\} < (ad + bc + 2bd)\,|\nu|^2. \qquad (3{\cdot}10)$$

This inequality takes one of three forms, depending upon whether $ad + bc + 2ac$ is positive, negative, or zero. We have

$$-2(ac - bd)\operatorname{Re}\{\mu\bar{\nu}\} < (ad + bc + 2bd)\,|\nu|^2 \text{ if } ad + bc + 2ac = 0, \quad .. \qquad (3{\cdot}11a)$$

$$|\mu|^2 - \frac{2(ac - bd)}{ad + bc + 2ac}\operatorname{Re}\{\mu\bar{\nu}\} < \frac{ad + bc + 2bd}{ad + bc + 2ac}\,|\nu|^2 \text{ if } ad + bc + 2ac > 0,$$

(3·11*b*)

$$|\mu|^2 - \frac{2(ac - bd)}{ad + bc + 2ac} \operatorname{Re}\{\mu \bar{\nu}\} > \frac{ad + bc + 2bd}{ad + bc + 2ac} |\nu|^2 \text{ if } ad + bc + 2ac < 0. \tag{3·11c}$$

The following reformulation of the equation

$$|\mu|^2 - \frac{2(ac - bd)}{ad + bc + 2ac} \operatorname{Re}\{\mu \bar{\nu}\} = \frac{ad + bc + 2bd}{ad + bc + 2ac} |\nu|^2$$

is obtained by writing μ and ν in terms of their real and imaginary parts, and completing the square.

$$\left|\mu - \frac{ac - bd}{ad + bc + 2ac} \nu\right| = \left|\frac{(a + b)(c + d)}{ad + bc + 2ac}\right| |\nu|. \quad \ldots \quad \ldots \tag{3·12}$$

For fixed ν, equation (3·12) defines a circle K_ν with center $\frac{ac - bd}{ad + bc + 2ac}\nu$, which passes through ν.

On the other hand, the equation

$$-2(ac - bd) \operatorname{Re}\{\mu \bar{\nu}\} = (ad + bc + 2bd) |\nu|^2$$

which corresponds to inequality (3·11a), may be rewritten as

$$\alpha x + \beta y = \frac{ad + bc + 2bd}{2(bd - ac)} |\nu|^2.$$

This equation describes a line L_ν passing through ν, which is perpendicular to the segment from the origin to ν. Assume now that $bd - ac > 0$. Then (3·11a) becomes

$$\operatorname{Re}\{\mu \bar{\nu}\} < \frac{ad + bc + 2bd}{2(bd - ac)} |\nu|^2. \quad \ldots \tag{3·13}$$

Since $bd > ac$, we have

$$ad + bc + 2bd > ad + bc + 2ac,$$

so that

$$ad + bc + 2bd > 0.$$

Thus, inequality (3·13) is of the form

$$\operatorname{Re}\{\mu \bar{\nu}\} < K|\nu|^2.$$

where $K > 0$. The half-plane H_ν determined by (3·13) contains the origin. In order for (3·11a) to be satisfied, it is necessary to require that $\mu \in H_\nu$ for all ϕ in $[0, \pi]$. Since

$$\nu = t + (t^2 - 1)^{1/2} \cosh u,$$

as u takes values in the interval $[0, \infty]$, v describes a ray h_t, with initial point $t + (t^2 - 1)^{1/2}$, which, if extended, would pass through $t - (t^2 - 1)^{1/2}$. Since $t + (t^2 - 1)^{1/2}$ lies outside the unit circle, the half-plane H_v containes the unit disc for each value of u. Since $\mu = z + (z^2 - 1)^{1/2} \cos \phi$, it follows that, as ϕ takes values in the interval $[0, \pi]$, μ describes a segment with endpoints $z - (z^2 - 1)^{1/2}$ and $z + (z^2 - 1)^{1/2}$. Since $\mu(\pi) = z - (z^2 - 1)^{1/2}$ lies interior to the unit circle, $\mu(\pi) \in \bigcap_u H_v$. As an intersection of half-planes, $\bigcap_u H_v$ is a convex set. Thus, in order that $\mu \in \bigcap_u H_v$ for all ϕ in $[0, \pi]$, it is sufficient to require that $z + (z^2 - 1)^{1/2} \in \bigcap_u H_v$. This is the same as requiring that $z \in B_t$, where B_t is the image of $\bigcap_u H_v$ under the mapping

$$z = \tfrac{1}{2}(w + w^{-1}). \qquad (3{\cdot}14)$$

Additionally, μ/ν must lie within the domain of analyticity of f. Thus, μ/v must satisfy

$$|\mu/\nu| \leqslant \rho < |d/c|, \qquad (\rho > 1).$$

This occurs for all $u \in [0, \infty)$ and $\phi \in [0, \pi]$ provided that

$$|z + (z^2 - 1)^{1/2}| \leqslant \rho\, |t + (t^2 - 1)^{1/2}|.$$

Hence, $z + (z^2 - 1)^{1/2}$ must lie within the circle centered at the origin, of radius $|d/c|\, |t + (t^2 - 1)^{1/2}|$. Now the mapping (3·14) carries circles centered at the origin, of radius greater than one, onto ellipses with foci ± 1. Thus, we must require that z lie interior to the ellipse E_t with foci ± 1, which passes through

$$\tfrac{1}{2}\left[\, |d/c|\ (t + (t^2 - 1)^{1/2}) + \frac{1}{|d/c|\ (t + (t^2 - 1)^{1/2})}\right].$$

Note that E_t properly contains the ellipse E, within which the series (3·1) is known to converge. In terms of this notation, we have established the following theorem.

Theorem 3·1—If $f(w) = \dfrac{aw + b}{cw + d}$, with $ad + bc + 2ac = 0$, and $bd - ac > 0$ and if $\{d_n\}$ is a sequence of nonnegative real numbers such that $\lim_{n \to \infty} d_n = 0$, then the series (3·1) is (f, d_n)-summable to $(t - z)^{-1}$ for all z satisfying $z \in B_t \cap E_t$.

Suppose now that $ad + bc + 2ac > 0$, and that $bd - ac > 0$. In order for the (f, d_n) method to sum the series (3·1) in this case, it is sufficient that

$$\left|\mu - \frac{ac - bd}{ad + bc + 2ac}\, \nu\right| < \frac{(a + b)\,(c + d)}{ad + bc + 2ac}\, |\nu|.$$

Thus, μ must lie in the interior I_ν of the circle K_ν defined by equation (3·12). Using the fact that $|\nu| > 1$ for all u and the assumption that $bd - ac > 0$, we have

$$1 + \frac{|ac - bd|}{ad + bc + 2ac} |\nu| \leqslant \frac{ad + bc + 2ac + (bd - ac)}{ad + bc + 2ac} |\nu|$$
$$= \frac{(a+b)(c+d)}{ad + bc + 2ac} |\nu| .$$

Hence, the radius of K_ν exceeds the modulus of the center of K_ν by at least one, so I_ν contains the unit disc for each value of u. Since $\bigcap_u I_\nu$ is a convex set which contains the point $z-(z^2-1)^{1/2}$, in order that μ lie in $\bigcap_u I_\nu$ for all ϕ in $[0, \pi]$, it is sufficient to require that $z + (z^2-1)^{1/2} \in \bigcap_u I_\nu$. Let B_t denote the image of $\bigcap_u I_\nu$ under the mapping (3·14), and let E_t have the same meaning as in Theorem 3·1.

Theorem 3.2—If $f(w) = \dfrac{aw + b}{cw + d}$, with $ad+bc+2ac > 0$ and $bd-ac > 0$, and if $\{d_n\}$ is a sequence of nonnegative real numbers such that $\lim_{n\to\infty} d_n = 0$, then the series (3·1) is (f, d_n)—summable to $(t-z)^{-1}$ for all z satisfying $z \in B_t \cap E_t$.

Now suppose that $ad + bc + 2ac < 0$. In order for the (f, d_n) method to sum the series (3·1) in this case, μ must lie exterior to the circle k_ν defined by (3·12). Assume that $(a+b)(c+d) > 0$. Then from the inequalities

$$ad + bc + ac + bd > 0 \text{ and } -ad - bc - 2ac > 0$$

it follows that $bd - ac > 0$. Thus,

$$1 + \left| \frac{(a+b)(c+d)}{ad + bc + 2ac} \right| |\nu| \leqslant \frac{bd - ac}{-(ad + bc + 2ac)} |\nu|$$
$$= \left| \frac{ac - bd}{ad + bc + 2ac} \nu \right| .$$

Therefore, the modulus of the center of K_ν exceeds the radius of K_ν by at least one, so that the unit disc is contained in the exterior of K_ν. For each value of u in $(0, \infty)$, let h_ν and l_ν denote the internal common tangents to K_ν and the unit circle. Let H_ν and L_ν be the half-planes containing the origin determined by h_ν and l_ν, respectively. Let J_ν be the finite region exterior to K_ν, bounded by K_ν, and h_ν and l_ν, and let $C_\nu = H_\nu \cup L_\nu \cup J_\nu$. Then C_ν contains the unit disc for each value of u. Although $\bigcap_u C_\nu$ is not convex, by its construction, if $z + (z^2-1)^{1/2} \in \bigcap_u C_\nu$, then $\mu \in \bigcup_u C_\nu$ for all ϕ in $[0, \pi]$.

Let B_t denote the image of $\bigcap_u C_\nu$ under the mapping (3·14), and let E_t have the same meaning as in Theorem 3.1.

Theorem 3·3—If $f(w) = \dfrac{aw + b}{cw + d}$, with $ad + bc + 2ac < 0$ and $(a + b)(c + d) > 0$, and if $\{d_n\}$ is a sequence of nonnegative real numbers such that $\lim_{n\to\infty} d_n = 0$, then the series (3·1) is (f, d_n)—summable to $(t - z)^{-1}$ for all $z \in B_t \cap E_t$.

Theorems 3·1, 3·2, and 3·3 were obtained under the assumption that the sequence $\{d_n\}$ satisfied $\lim_{n\to} d_n = 0$. We now assume that d_n is positive for each n, that $\sum_{n=1}^{\infty} \dfrac{1}{d_n} = \infty$, and that $\lim_{n\to\infty} d_n = \infty$, According to Lemma 3·2, condition (3·6) will be satisfied if

$$\operatorname{Re}\left\{f\left(\frac{\mu}{\bar{\nu}}\right)\right\} < \operatorname{Re}\{f(1)\}.$$

A lengthy, though direct, calculation shows that

$$\operatorname{Re}\left\{f\left(\frac{u}{\nu}\right)\right\} = \frac{ac|\mu|^2 + bd|\nu|^2 + (ad + bc)\operatorname{Re}\{\mu\bar{\nu}\}}{c^2|\mu|^2 + d^2|\nu|^2 + 2cd\operatorname{Re}\{\mu\bar{\nu}\}_1}. \quad .. \quad (3\cdot 15)$$

Without loss of generality, assume that $c \geqslant 0$, and consider separately the cases corresponding to $c > 0$ and $c = 0$. If $c > 0$ and $c + d > 0$, then by (3·15),

$$\operatorname{Re}\left\{f\left(\frac{\mu}{\nu}\right)\right\} < \operatorname{Re}\{f(1)\}$$

if and only if

$$[ac|\mu|^2 + bd|\nu|^2 + (ad + bc)\operatorname{Re}\{\mu\bar{\nu}\}](c + d)$$
$$< [c^2|\mu|^2 + d^2|\nu|^2 + 2cd\operatorname{Re}\{\mu\bar{\nu}\}](a + b).$$

When simplified, this inequality takes the form

$$|\mu|^2 - (1 - d/c)\operatorname{Re}\{\mu\bar{\nu}\} < d/c\,|\nu|^{2\prime} \quad .. \quad .. \quad (3\cdot 16)$$

If $c > 0$ and $c + d < 0$, then

$$\operatorname{Re}\left\{f\left(\frac{\mu}{\nu}\right)\right\} < \operatorname{Re}\{f(1)\}$$

if and only if

$$|\mu|^2 - (1 - d/c)\operatorname{Re}\{\mu\bar{\nu}\} > d/c\,|\nu|^2. \quad .. \quad .. \quad (3\cdot 17)$$

If $c = 0$, then the function f takes the form

$$f(w) = aw + b,$$

so that

$$\text{Re}\left\{f\left(\frac{\mu}{\nu}\right)\right\} = \frac{b|\nu|^2 + a\,\text{Re}\{\mu\bar{\nu}\}}{|\nu|^2}.$$

Hence,

$$\text{Re}\left\{f\left(\frac{\mu}{\nu}\right)\right\} < \text{Re}\{f(1)\}$$

if and only if

$$b|\nu|^2 + a\,\text{Re}\{\mu\bar{\nu}\} < (a+b)|\nu|^2:$$

i.e.,

$$\left.\begin{array}{ll}\text{Re}\{\mu\bar{\nu}\} < |\nu|^2 & \text{if } a > 0\\ \text{Re}\{\mu\bar{\nu}\} > |\nu|^2 & \text{if a} < 0.\end{array}\right\} \quad \ldots \quad \ldots \quad (3{\cdot}18)$$

Consider the equation

$$|\mu|^2 - (1 - d/c)\,\text{Re}\{\mu\bar{\nu}\} = d/c\;|\nu|^2 \quad \ldots \quad \ldots \quad (3{\cdot}19)$$

which corresponds to the inequalities (3·16) and (3·17). If μ and ν are written in terms of their real and imaginary parts, completion of the square allows (3·19) to to be expressed as

$$\left|\mu - \frac{c-d}{2c}\;\nu\right| = \left|\frac{c+d}{2c}\right|\;|\nu|. \quad \ldots \quad \ldots \quad (3{\cdot}20)$$

For fixed ν, equation (3·20) describes a circle K_ν with center $\frac{c-d}{2c}\,\nu$, which passes through ν.

On the other hand, the equation

$$\text{Re}\{\mu\bar{\nu}\} = |\nu|^2 \quad \ldots \quad \ldots \quad \ldots \quad (3{\cdot}21)$$

corresponding to the inequalities (3·18), may be written as

$$\alpha x + \beta y = |\nu|^2.$$

This equation describes a line L_ν, which passes through ν, and is perpendicular to the segment from the origin to ν.

In summary, we have shown that the inequality

$$\text{Re}\left\{f\left(\frac{\mu}{\nu}\right)\right\} < \text{Re}\{f(1)\}$$

takes one of the following forms :

$$| \mu - \frac{c-d}{2c} \nu | < \frac{c+d}{2c} | \nu | \qquad (c > 0, c + d > 0), \qquad .. \quad (3{\cdot}22a)$$

$$| \mu - \frac{c-d}{2c} \nu | > - \frac{c+d}{2c} | \nu | \qquad (c > 0, c + d < 0), \qquad .. \quad 3{\cdot}22b)$$

$$\text{Re} \{\mu \bar{\nu}\} < | \nu |^2 \qquad (c = 0, a > 0), \quad .. \quad .. \quad (3{\cdot}22c)$$

$$\text{Re} \{\mu \bar{\nu}\} > | \nu |^2 \qquad (c = 0, a < 0). \quad .. \quad .. \quad (3{\cdot}22d)$$

Suppose first that $c > 0$ and $c + d > 0$. We have assumed that $| d/c | > 1$. If d were negative, then we would have $| d/c | = - d/c$, and from $-d/c > 1$, we would conclude that $c + d < 0$. Hence $d > 0$, $| d/c | = d/c$, and $c - d < 0$. Inequality (3·22a) is satisfied for all μ in the interior I_ν of the circle K_ν defined by eqn. (3·20). Now

$$1 + | \frac{c-d}{2c} \nu | \leqslant \left(1 + \frac{d-c}{2c} \nu \right) = \frac{c+d}{2c} | \nu | ,$$

so that each I_ν contains the unit disc. Hence $z - (z^2 - 1)^{1/2} \in \bigcap_u I_\nu$. Since $\bigcap_u I_\nu$ is convex, in order that $\mu \in \bigcap_u I_\nu$ for all ϕ in $[0, \pi]$, it is sufficient to require that $z + (z^2 - 1)^{1/2} \in \bigcap_u I_\nu$. Let B_t denote the image of $\bigcap_u I_\nu$ under the mapping (3·14) and let E_t have the same meanin as in Theorem 3·1.

Theorem 3.4—If $f(w) = \dfrac{aw + b}{cw + d}$, with $c > 0$ and $c + d > 0$, and if $\{d_n\}$ is a sequence of positive numbers such that $\sum_{n=1}^{\infty} \dfrac{1}{d_n} = \infty$ and $\lim_{n \to \infty} d_n = \infty$, then the series (3·1) is (f, d_n)—summable to $(t - z)^{-1}$ for all z satisfying $z \in B_t \cap E_t$.

Now suppose $c > 0$ and $c + d < 0$. Then $c - d > 0$. In order that (3·22b) be satisfied, it is sufficient that μ lie exterior to the circle K_ν. The following sequence of inequalities shows that the unit disc lies exterior to each K_ν,

$$\begin{aligned} 1 + \frac{| c + d |}{2c} | \nu | &= 1 - \frac{c+d}{2c} | \nu | \\ &\leqslant \left(1 - \frac{c+d}{2c} \right) | \nu | \\ &= \frac{c-d}{2c} | \nu | \\ &= | \frac{c-d}{2c} | \quad | \nu | . \end{aligned}$$

Let B_t and E_t have the same meaning as in Theorem 3·3, with the exception that the circle K_ν is now defined by equation (3·19), and not by equation (3·12). Proceeding exactly as in the proof of Theorem 3.3, we obtain the following :

Theorem 3·5—If $f(w) = \dfrac{aw+b}{cw+d}$, with $c > 0$ and $c + d < 0$, and if $\{d_u\}$ is a sequence of positive numbers such that $\sum_{n=1}^{\infty} \dfrac{1}{d_n} = \infty$ and $\lim_{n\to\infty} d_n = \infty$, then the series (3·1) is (f, d_n)—summable to $(t - z)^{-1}$ for all z satisfying $z \in B_t \cap E_t$.

Suppose now that $c = 0$. Then μ must satisfy one of the inequalities (3·22*c*) or (3·22*d*). If a < 0, then the half-plane H_ν defined by the inequality (3·22*d*) fails to contain the unit disc. Thus, it is impossible to have $\mu \in \bigcap_u H_\nu$ for all ϕ in $[0, \pi]$. The (f, d_n) method is therefore ineffective in this case. If $a > 0$, then the half-plane H_ν defined by the inequality (3·22*c*) does contain the unit disc. Thus, in order that $\mu \in \bigcap_u H_\nu$ for all ϕ in $[0, \pi]$, it is sufficient to require that $z + (z^2 - 1)^{1/2} \in \bigcap_u H_\nu$. If B_t denotes the image of $\bigcap_u H_\nu$ under the mapping (3·14), and E_t has the same meaning as in Theorem 3.1, then we have proven the following:

Theorem 3·6—If $f(w) = aw + b$ with $a > 0$, and if $\{d_n\}$ is a sequence of positive numbers such that $\sum_{n=1}^{\infty} \dfrac{1}{d_n} = \infty$ and $\lim_{n\to\infty} d_n = \infty$, then the sequence (3·1) is (f, d_n)-summable to $(t - z)^{-1}$ for all z satisfying $z \in B_t \cap E_t$.

The domains in which the various (f, d_n) methods considered here provide analytic continuation of the general series of Lengendre polynomials

$$\sum_{n=0}^{\infty} a_n P_n(z)$$

to $f(z)$, where

$$a_n = \frac{2n+1}{2\pi i} \int_\gamma f(t)\, Q_n(t)\, dt,$$

may now be determined according to general results obtained by Jakimovski (1964, 1966).

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