AN EXTENSION OF SEHGAL'S RESULT ON FIXED POINTS

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The Principle of Banach contraction has been extended in various ways by several authors. Sehgal extended this principle for mapping which were not necessarily contractions. In this paper this result of Seghal, is extended for a mapping different from contractions.

1. The well known Banach contraction principle states that for a complete metric space (X, ρ), each contraction mapping has a unique fixed point. Several authors viz. Belluce and Kirk (1969), Subrahmanyam (1972), Sehgal (1969), Edelstein (1961) and others have studied its various extensions. Sehgal considered a contractive iterate at each point of the space for a continuous self mapping and proved the following theorem:

Theorem A—Let (X, ρ) be a complete metric space and f a continuous self mapping defined on X satisfying: there exists a $k < 1$ such that for each $x \in X$ there is a positive integer $n(x)$ such that for all $y \in X$,

$$\rho\,(f^{n(x)}(x), f^{n(x)}(y)) \leqslant k\,\rho\,(x, y). \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

Then f has a unique fixed point u and $f^n(x_0) \rightarrow u$ for each $x_0 \in X$.

In a paper, Sharma (1973) generalized the following theorem due to Subrahmanyam (1972):

Theorem B—If f is any continuous self mapping of a complete metric space (X, ρ) such that

$$\rho\,(f(x), f(y)) \leqslant a\,\rho\,(x, f(x)) + b\,\rho\,(y, f(y)) + c\,\rho\,(x, f(y)) + d\,\rho\,(y, f(x)) + e\,\rho\,(x, y) \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

where a, b, c, d and e real constants satisfying either of the following conditions:

$$1 - b - c \neq 0,\ \frac{a + c + e}{1 - b - c} < 1 \quad \ldots \quad \ldots \quad \ldots \quad \text{(i)}$$

$$1 - a - d \neq 0\ \frac{b + d + e}{1 - a - d} < 1 \quad \ldots \quad \ldots \quad \ldots \quad \text{(ii)}$$

then f has a fixed point in X. If in addition $c + d + e < 1$, then the fixed point is unique.

In the present paper, we prove an analogue of Theorem B on the same line of argument as in Theorem A.

2. We prove the following :

Theorem—Let (X, ρ) be a complete metric space and f a continuous self mapping of X satisfying: for each $x \in X$, there exists a positive integer $n(x)$ such that for all $y \in X$

$$\rho(f^{n(x)}(x), f^{n(x)}(y)) \leqslant a\,\rho(x, f^{n(x)}(x)) + b\,\rho(y, f^{n(x)}(y)) + c\,\rho(x, f^{n(x)}(y)) + d\,\rho(y, f^{n(x)}(x)) + e\,\rho(x, y) \quad \ldots \quad \ldots \quad (3)$$

where the positive real constants a, b, c, d and e satisfy

$$a + b + 2c + 2d + e < 1 \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

and

$$\frac{a + c + e}{1 - a - d} < 1, \frac{d + b + e}{1 - b - c} < 1. \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

or

$$1 - a - d \neq 0 \qquad 1 - b - c \neq 0$$

Then f has a unique fixed point u and $f^n(x_0) \to u$ for each $x_0 \in X$,

Firstly, we prove the following lemma.

Lemma—If $f: X \to X$ is any mapping satisfying the conditions of the above theorem, then for each $x \in X$, $\gamma(x) = \sup_n \rho(f^n(x), x)$ is finite.

PROOF: Let $x \in X$ and let

$$l(x) = \max. \{\rho(f^k(x), x) : k = 1, 2, .., n(x)\}$$

If n is a positive integer, there exists an integer $t \geqslant 0$ such that $t.\, n(x) < n \leqslant (t + 1).\, n(x)$, and

$$\begin{aligned} \rho(f^n(x), x) &\leqslant \rho(f^{n(x)} f^{n-n(x)}(x), f^{n(x)}(x)) + \rho(f^{n(x)}(x), x) \\ &\leqslant a\,\rho(f^{n-n(x)}(x), f^n(x)) + b\,\rho(x, f^{n(x)}(x)) + c\,\rho(f^{n-n(x)}(x), f^{n(x)}(x) \\ &\quad + d\,\rho(x, f^n(x)) + e\,\rho(f^{n-n(x)}(x), x) + \rho(f^{n(x)}(x), x) \\ &\leqslant (a + c + e)\,\rho(f^{n-n(x)}(x), x) + (1 + b + c)\,\rho(f^{n(x)}(x), x) + \\ &\qquad (a + d)\,\rho(f^n(x), x) \end{aligned}$$

i.e. $$(f^n(x), x) \leqslant \frac{a + c + e}{1 - a - d}\,\rho(f^{n-n(x)}(x), x) + \frac{1 + b + c}{1 - a - d}\,l(x)$$

$$\leqslant \ldots\ldots$$

$$\leqslant \frac{1 + b + c}{1 - a - d}\left[1 + \frac{a + c + e}{1 - a - d} + .. + \left(\frac{a + c + e}{1 - a - d}\right)^t\right] l(x)$$

$$< \frac{1+b+c}{1-a-d} \; \frac{1}{1-\frac{a+c+e}{1-a-d}} \; l\,(x)$$

$$= \frac{1+b+c}{1-(2a+c+d+e)} \; l\,(x) \text{ for all } n \geqslant 0$$

Hence $\gamma\,(x) = \sup\limits_{n}\,(f^{n}\,(x), x)$ is finite.

PROOF OF THE THEOREM : Let x_0 be arbitrary. Let $m_0 = n\,(x_0)$, $x_1 = f^{m_0}\,(x_0)$ and successively $m_i = n\,(x_i)$, $x_{i+1} = f^{m_i}\,(x_i)$. We show that $\{x_n\}$ is a convergent sequence. We have

$$\begin{aligned}
\rho\,(x_2, x_1) &= \rho\,(f^{m_0}\,f^{m_1}\,(x_0), f^{m_0}\,(x_0)\\
&\leqslant a\;\rho\,(f^{m_1}\,(x_1), f^{m_1}\,(x_0)) + b\,\rho\,(x_0, x_1) + c\,\rho\,(f^{m_1}\,(x_0), x_1)\\
&\quad + d\,\rho\,(x_0, x_2) + e\,\rho\,(f^{m_1}\,(x_0), x_0)\\
&\leqslant a^2\rho\,(x_2, x_1) + ab\,\rho\,(x_0, f^{m_1}\,(x_0)) + a\,c\rho\,(x_1, x_0) + ac\,\rho\,(x_0, f^{m_1}\,(x_0))\\
&\quad + ad\,\rho\,(x_1, x_0) + ad\,\rho\,(x_2, x_1) + ae\,\rho\,(x_1, x_0) + b\,\rho\,(x_1, x_0)\\
&\quad + c\,\rho\,(f^{m_1}\,(x_0), x_0) + c\,\rho\,(x_1, x_0) + d\,\rho\,(x_1, x_0) + d\,\rho\,(x_2, x_1)\\
&\quad + e\,\rho\,(f^{m_1}\,(x_0)\,x_0)
\end{aligned}$$

i.e. $\rho\,(x_2, x_1) \leqslant \alpha\,\rho\,(x_1, x_0) + \beta\,\rho\,(f^{m_1}\,(x_0), x_0)$

wher $\alpha = \dfrac{ac + a + ae + b + c + d}{1 - a^2 - ad - d}$, $\beta = \dfrac{ab + ac + c + e}{1 - a^2 - ad - d}$.

Clearly $\alpha + \beta < 1$ and $\beta < 1$ in view of (4).

Again $\rho\,(x_3, x_2) = \rho\,(f^{m_1}\,f^{m_2}\,(x_1), f^{m_1}\,(x_1))$

$$\begin{aligned}
&\leqslant a\,\rho\,(f^{m_2}\,(x_2), f^{m_2}\,(x_1)) + b\,\rho\,(x_1, x_2) + c\,\rho\,(f^{m_2}\,(x_1), x_2)\\
&\quad + d\,\rho\,(x_1, x_3) + e\,\rho\,(f^{m_2}\,(x_1), x_1),\\
&\leqslant a^2\,\rho\,(x_2, x_3) + ab\,\rho\,(x_1, f^{m_2}\,)\,(x_1)\,) + ac\,\rho\,(x_2, x_1\\
&\quad + ac\,\rho\,(x_1, f^{m_2}\,(x_1)) + ad\,\rho\,(x_1, x_2) + ad\,\rho\,(x_2, x_3)\\
&\quad ae\,\rho\,(x_2, x_1) + b\rho\;x_1, x_2\,) + c\rho\,(f^{m_2}\,(\,x_1\,), x_1\,)\\
&\quad + c\,\rho\,(x_1, x_2) + d\,\rho\,(x_1, x_2) + d\,\rho\,(x_2, x_3) + e\,\rho\,(f^{m_2}\,(x_1), x_1)
\end{aligned}$$

i.e. $\rho\,(x_3, x_2) \leqslant \alpha\,\rho\,(x_2, x_1) + \beta\,\rho\,(f^{m_2}\,(x_1), x_1)$.

However

$\rho\,(f^{m_2}(x_1), x_1) = \rho\,(f^{m0} f^{m_2}(x_0), f^{m0}(x_0)$ and proceding on the same ine as above we get

$$\rho\,(f^{m_2}(x_1), x_1) \leqslant \alpha\,\rho\,(x_1, x_0) + \beta\,\rho\,(f^{m_2}(x_0), x_0).$$

Thus,

$$\rho\,(x_3, x_2) \leqslant \alpha\,(\alpha+\beta)\,\rho\,(x_1, x_0) + \alpha\beta\,\rho\,(f^{m_1}(x_0), x_0) + \beta^2\,\rho\,(f^{m_2}(x_0), x_0)$$

Similarly,

$$\rho\,(x_4, x_3) \leqslant \alpha\,(\alpha+\beta)^2\,\rho(x_1, x_0) + \alpha\beta\,(\alpha+\beta)\,\rho\,(f^{m_1}(x_0), x_0) + \alpha\,\beta^2\,\rho\,(f^{m2}(x_0), x_0) + \beta^3\,\rho\,(f^{m3}(x_0, x_0)$$

and in gene al,

$$\rho\,(x_n, x_{n+1}) \leqslant \alpha\,(\alpha+\beta)^{n-1}\,\rho\,(x_1, x_0) + \alpha\beta\,(\alpha+\beta)^{n-2}\rho\,(f^{m_1}(x_0, x_0) +$$
$$\alpha\beta^2\,(\alpha+\beta)^{n-3}\,\rho\,(f^{m_2}(x_0), x_0 + \ldots + \alpha\beta^{n-1}\,(f^{m_{n-1}}(x_0), x_0) + \beta^n\rho\,(f^{m_n}(x_0), x_0)$$
$$\leqslant [\alpha\,(\alpha+\beta)^{n-1} + \alpha\beta\,(\alpha+\beta)^{n-2} + \alpha\beta^2\,(\alpha+\beta)^{n-3} + \ldots + \alpha\beta^{n-1} + \beta^n]\,\gamma(x_0)$$

Now for $k > n$

$$\rho\,(x_n, x_k) \leqslant \rho\,(x_n, x_{n+1}) + \rho\,(x_{n+1}, x_{n+2}) + \ldots + (\rho\,(x_{k-1}, x_k)$$
$$\leqslant [\,\alpha\,(\alpha+\beta)^{n-1} + \alpha\beta\,(\alpha+\beta)^{n-2} + \ldots + \alpha\beta^{n-2}\,(\alpha+\beta) + \alpha\beta^{n-1}\,]$$
$$[\,1 + (\alpha+\beta) + (\alpha+\beta)^2 + \ldots + (\alpha+\beta)^{k-n-1}\,]\;\gamma\,(x_0)$$
$$+ \alpha\beta^n\,[1 + (\alpha+\beta) + (\alpha+\beta)^2 + \ldots + (\alpha+\beta)^{k-n-2}]\;\gamma\,(x_0)$$
$$+ \alpha\beta^{n+1}\,[\,1 + (\alpha+\beta) + \ldots + (\alpha+\beta)^{k-n-3}\,]\;\gamma\,(x_0)$$
$$+ \ldots + \alpha\beta^{k-3}\,[\,1 + (\alpha+\beta)]\;\gamma\,(x_0) + \alpha\beta^{k-2}\,\gamma\,(x_0)$$
$$+ \beta^n\,[1 + \beta + \ldots + \beta^{k-n-1}]\;\gamma\,(x_0)$$

$$< [\,\alpha\sum_{i=0}^{n-1}\beta^i\,(\alpha+\beta)^{n-i-1}]\left[\frac{1}{1-(\alpha+\beta)}\right]\gamma(x_0) + \frac{\alpha\beta^n}{1-(\alpha+\beta)}\,\gamma(x_0)$$
$$+ \frac{\alpha\beta^{n+1}}{1-(\alpha+\beta)}\,\gamma\,(x_0) + \ldots + \frac{\alpha\beta^{k-3}}{1-(\alpha+\beta)}\,\gamma(x_0) + \alpha\beta^{k-2}\,\gamma(x_0) + \frac{\beta^n}{1-\beta}\;\gamma\,(x_0)$$
$$\to 0 \text{ as } n, k \to \infty,$$

which implies that $\{x_n\}$ is Cauchy sequence in complete metric space X.

Let $x_n \to u \in X$. We show that $f(u) = u$. If $f(u) \neq u$,

then there exists a pair of dispoint closed neighbourhood P and Q such that $u \in P$, $f(u \in Q$ and

$$\sigma = \inf\{\,\rho\,(x, y) \;:\; x \in P, y \in Q\} > 0. \qquad \ldots \qquad \ldots \qquad \ldots \qquad (6)$$

Since f is continuous, $x_n \in P$ and $f(x_n) \in Q$ for all n sufficiently large. Now $\rho(f(x_n), x_n) = \rho(f^{m_{n-1}} f(x_{n-1});\ f^{m_{n-1}}(x_{n-1}))$ and using (3) we get

$$\rho(f(x_n), x_n) \leqslant \frac{a+c+e}{1-a-d}\,\rho(f(x_{n-1}), x_{n-1}) + \frac{a+b+c+d}{1-a-d}\,\rho(x_{n-1}, x_n)$$

$$\leqslant \ldots\ldots$$

$$\leqslant \left(\frac{a+c+e}{1-a-d}\right)^n \rho(f(x_0), x_0) + \frac{a+b+c+d}{1-a-d}\left[\left(\frac{a+c+e}{1-a-d}\right)^{n-1} \rho(x_0, x_1)\right.$$

$$\left. + \left(\frac{a+c+e}{1-a-d}\right)^{n-2} \rho(x_1, x_2) + \ldots + \frac{a+c+e}{1-a-d}\,\rho(x_{n-2}, x_{n-1})\right]$$

$$\to 0 \text{ as } n \to \infty, \text{ contradicting (6).}$$

Hence $f(u) = u$.

We now show the uniqueness of the fixed point u.

Let there exist $v\ (\neq u) \in X$ such that $f(v) = v$. Then

$$\rho(u, v) = \rho(f^{n(u)}, f^{n(u)}(v)) \leqslant a\,\rho(u, f^{n(u)}(u)) + b\,\rho(v, f^{n(u)}(v))$$
$$+ c\,\rho(u, f^{n(u)}(v)) + d\,\rho(v, f^{n(u)}(u)) + e\,\rho(u, v)$$
$$= (c + d + e)\,\rho(u, v)$$

Hence $u = v$

We now show that $f^n(x_0) \to u$. Let

$$\delta^* = \max.\{\rho(f^k(x_0), u) : k = 0, 1, 2, .., n(u) - 1\}$$

If n is sufficiently large integer, then $n = r\,n(u) + q$,

$0 \leqslant q < n(u), r > 0$ and

$\rho(f^n(x_0), u) = \rho(f^{rn(u)+q}(x_0), u)$ and using (3) we have

$$\rho(f^n(x_0), u) \leqslant \frac{a+c+e}{1-a-d}\,\rho(f^{(r-1)n(u)+q}(x_0), u)$$

$$\leqslant \ldots\ldots\ldots\ldots$$

$$\leqslant \left\{\frac{a+c+e}{1-a-d}\right\}^r \rho(f^q(x_0), u)$$

$$\leqslant \left\{\frac{a+c+e}{1-a-d}\right\}^r \delta^*$$

As $n \to \infty$ implies $r \to \infty$, we have

$$\rho(f^n(x_0), u) \to 0 \text{ as } n \to \infty$$

This proves the theorem.

We now reduce some particular cases of the theorem proved above.

Case I—If $c = 0$ and $d = 0$ then the conclusion of the theorem for mapping f satisfying

$$\rho\left(f^{n(x)}(x), f^{n(x)}(y)\right) \leqslant a\,\rho\left(x, f^{n(x)}(x)\right) + b\,\rho\left(y, f^{n(x)}(y)\right) + e\,\rho(x,y) \quad .. \quad (7)$$

hold where $a + b + e < 1$ and $2a + e < 1$. (refer 2).

Case II—In addition to $c = 0$, $d = 0$, we take $e = 0$, then the conclusion of the theorem for the mapping f satisfying

$$\rho\left(f^{n(x)}(x), f^{n(x)}(y)\right) \leqslant a\,\rho\left(x, f^{n(x)}(x)\right) + b\,\rho\left(y, f^{n(x)}(y)\right) \quad .. \quad (8)$$

hold where $a < \frac{1}{2}$, $b < \frac{1}{2}$ (refer Khezanchi and Dass 1974).

Case III—If $b = c = d = a = 0$, then theorem reduces to that of Sehgal viz Theorem A.

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