SOME NEW GENERATING RELATIONS FOR GENERALIZED HORN FUNCTIONS

by B. M. SHRIVASTAVA, *Department of Mathematics, Government Science College, Rewa (M.P.)*

and

F. SINGH, *Department of Mathematics, Government Engineering College, Rewa (M.P.)*

(Communicated by P. L. Bhatnagar, F.N.A.)

(Received 18 September 1974)

In this note the authors have established the generating relations of hypergeometric functions of three variables defined by Pandey (1963), Dhawan (1970), and Srivastava (1972a). Some known particular cases have also been derived.

1. The subject of generating relations plays an important role in the development and study of special functions. In our present paper we establish some new generating relations for certain hypergeometric functions of three variables by using the technique of Srivastava (1972*b*). In section 3, we give some interesting relations for Appell's functions F_1, F_3, and Humbert's confluent hypergeometric functions Φ_1, Ξ_1 of two variables.

If we use the notation $(a, n) = a\,(a+1)\,(a+2)\,..\,(a+n-1)$; $(a, 0) = 1$ where a is arbitrary and n a positive integer, then the generalized Horn functions of three variables, defined by Pandey (1963), Dhawan (1970), and Srivastava (1972*a*), are as follows :

$$G_A(\alpha, \beta, \beta^1; \gamma; x, y, z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha, n+p-m)\,(\beta, m+p)\,(\beta^1, n)}{(1, m)\,(1, n)\,(1, p)\,(\gamma, n+p-m)}\, x^m y^n z^p \quad .. \quad .. \quad .. \quad (1{\cdot}1)$$

$$G_B(\alpha, \beta_1, \beta_2, \beta_3; \gamma; x, y, z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha, n+p-m)\,(\beta_1, m)\,(\beta_2, n)\,(\beta_3, p)}{(1, m)\,(1, n)\,(1, p)\,(\gamma, n+p-m)}\, x^m y^n z^p, \quad .. \quad .. \quad (1{\cdot}2)$$

$$G_C(\alpha, \alpha_1, \beta, \beta_1; \gamma; x, y, z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha, p-m)\,(\alpha_1, n)\,(\beta, m+p)\,(\beta_1, n)}{(1, m)\,(1, n)\,(1, p)\,(\gamma, n+p-m)}\, x^m y^n z^p \quad .. \quad .. \quad (1{\cdot}3)$$

$$G_D\ (\alpha, \alpha_1, \beta_1, \beta_2, \beta_3; \gamma; x, y, z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha, p-m)(\alpha_1, n)(\beta_1, m)(\beta_2, n)\ (\beta_3, p)}{(1, m)\ (1, n)\ (1, p)\ (\gamma, n+p-m)} x^m y^n z^p, \quad \ldots \quad \ldots \quad \ldots \quad (1{\cdot}4)$$

$$G_C^*(\alpha, \beta, \beta_1; \gamma; x, y, z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha, m+p)\ (\beta, m+n)\ (\beta_1, n-p)}{(1, m)\ (1, n)\ (1, p)\ (\gamma, m+n-p)} x^m y^n z^p \quad \ldots \quad \ldots \quad \ldots \quad (1{\cdot}5)$$

where the notations for G_A to G_D are considerably abbreviated on the lines used by Srivastava (1972*a*), and, for convergence, $|x| < 1$, $|y| < 1$, $|z| < 1$.

2. In this section we obtain the following generating relations :

$$\sum_{r=0}^{\infty} \frac{(\lambda, r)}{(1, r)} G_A\ (\lambda + r, \beta_1, \beta_2; \gamma; x, y, z)\ t^r$$

$$= (1-t)^{-\lambda}\, G_A\left(\lambda, \beta_1, \beta_2; \gamma; x(1-t), \frac{y}{1-t}, \frac{z}{1-t}\right) \quad \ldots \quad \ldots \quad (2{\cdot}1)$$

$$\sum_{r=0}^{\infty} \frac{(\lambda, r)}{(1, r)} G_A\ (\alpha, \lambda + r, \beta_2; \gamma; x, y, z)\ t^r$$

$$= (1-t)^{-\lambda}\, G_A\left(\alpha, \lambda, \beta_2; \gamma; \frac{x}{1-t}, y, \frac{z}{1-t}\right). \quad \ldots \quad \ldots \quad \ldots \quad (2{\cdot}2)$$

$$\sum_{r=0}^{\infty} \frac{(\lambda, r)}{(1, r)} G_A\ (\alpha, \beta_1, \lambda + r; \gamma; x, y, z) t^r$$

$$= (1-t)^{-\lambda}\, G_A\left(\alpha, \beta_1, \lambda; \gamma; x, \frac{y}{1-t}, z\right). \quad \ldots \quad \ldots \quad \ldots \quad (2{\cdot}3)$$

$$\sum_{r=0}^{\infty} \frac{(\lambda, r)}{(1, r)} G_B\ (\lambda + r, \beta_1, \beta_2, \beta_3; \gamma; x, y, z) t^r$$

$$= (1-t)^{-\lambda}\, G_B\left(\lambda, \beta_1, \beta_2, \beta_3; \gamma; x(1-t), \frac{y}{1-t}, \frac{z}{1-t}\right). \quad \ldots \quad (2{\cdot}4)$$

$$\sum_{r=0}^{\infty} \frac{(\lambda, r)}{(1, r)} G_B\ (\alpha, \lambda + r, \beta_2, \beta_3; \gamma; x, y, z)\ t^r$$

$$= (1-t)^{-\lambda}\, G_B\left(\alpha, \lambda, \beta_2, \beta_3; \gamma; \frac{x}{1-t}, y, z\right). \quad \ldots \quad \ldots \quad \ldots \quad (2{\cdot}5)$$

$$\sum_{r=0}^{\infty} \frac{(\lambda, r)}{(1, r)} G_B\,(\alpha, \beta_1, \lambda + r, \beta_3; \gamma; x, y, z)\, t^r$$

$$= (1 - t)^{-\lambda}\, G_B\left(\alpha, \beta_1, \lambda, \beta_3; \gamma; x, \frac{y}{1 - t}, z\right). \quad .. \quad .. \quad .. \quad (2{\cdot}6)$$

$$\sum_{r=0}^{\infty} \frac{(\lambda, r)}{(1, r)} G_C\,(\lambda + r, \alpha_1, \beta, \beta_1; \gamma; x, y, z)\, t^r$$

$$= (1 - t)^{-\lambda}\, G_C\left(\lambda, \alpha_1, \beta, \beta_1; \gamma; x(1 - t), y, \frac{z}{1 - t}\right). \quad .. \quad .. \quad (2{\cdot}7)$$

$$\sum_{r=0}^{\infty} \frac{(\lambda, r)}{(1, r)} G_C\,(\alpha, \alpha_1, \lambda + r, \beta_1; \gamma; x, y, z,)\, t^r$$

$$= (1 - t)^{-\lambda}\, G_C\left(\alpha, \alpha_1, \lambda, \beta_1; \gamma; \frac{x}{1 - t}, y, \frac{z}{1 - t}\right). \quad .. \quad .. \quad .. \quad (2{\cdot}8)$$

$$\sum_{r=0}^{\infty} \frac{(\lambda, r)}{(1, r)} G_D\,(\lambda + r, \alpha_1, \beta_1, \beta_2, \beta_3; \gamma; x, y, z)\, t^r$$

$$= (1 - t)^{-\lambda}\, G_D\left(\lambda, \alpha_1, \beta_1, \beta_2, \beta_3; \gamma; x(1 - t), y, \frac{z}{1 - t}\right). \quad .. \quad .. \quad (2{\cdot}9)$$

$$\sum_{r=0}^{\infty} \frac{(\lambda, r)}{(1, r)} G_D\,(\alpha, \alpha_1, \lambda + r, \beta_2, \beta_3; \gamma; x, y, z)\, t^r$$

$$= (1 - t)^{-\lambda}\, G_D\left(\alpha, \alpha_1, \lambda, \beta_2, \beta_3; \gamma; \frac{x}{1 - t}, y, z\right). \quad .. \quad .. \quad .. \quad (2{\cdot}10)$$

$$\sum_{r=0}^{\infty} \frac{(\lambda, r)}{(1, r)}\, G_D\,(\alpha, \alpha_1, \beta_1, \beta_2, \lambda + r; \gamma; x, y, z)\, t^r$$

$$= (1 - t)^{-\lambda}\, G_D\left(\alpha, \alpha_1, \beta_1, \beta_2, \lambda; \gamma; x, y, \frac{z}{1 - t}\right). \quad .. \quad .. \quad .. \quad (2{\cdot}11)$$

$$\sum_{r=0}^{\infty} \frac{(\lambda, r)}{(1, r)}\; G^*_C\,(\lambda + r, \beta, \beta^1; \gamma; x, y, z)\, t^r$$

$$= (1 - t)^{-\lambda}\, G^*_C\left(\lambda, \beta, \beta^1; \gamma; \frac{x}{1 - t}, y, \frac{z}{1 - t}\right). \quad .. \quad .. \quad .. \quad (2{\cdot}12)$$

$$\sum_{r=0}^{\infty} \frac{(\lambda, r)}{(1, r)} G^*_C (\alpha, \lambda + r, \beta^1; \gamma; x, y, z) t^r$$

$$= (1-t)^{-\lambda} G^*_C \left(\alpha, \lambda, \beta^1; \gamma; \frac{x}{1-t}, \frac{y}{1-t}, z \right) \quad \ldots \quad (2\cdot13)$$

$$\sum_{r=0}^{\infty} \frac{(\lambda, r)}{(1, r)} G^*_C (\alpha, \beta, \lambda + r; \gamma; x, y, z) t^r$$

$$= (1-t)^{-\lambda} G^*_C \left(\alpha, \beta, \lambda; \gamma; x, \frac{y}{1-t}, z(1-t) \right) \quad \ldots \quad (2\cdot14)$$

The convergence conditions for all of these results are $|x| < 1$, $|y| < 1$, $|z| < 1$, and $|t| < 1$.

Proofs—Let us define

$$F(h, x, y, z) = \sum_{m,n,p=0}^{\infty} (h, am + bn + cp) Amnp\, x^m y^n z^p \quad \ldots \quad (2\cdot15)$$

Now consider

$$\triangle = \sum_{r=0}^{\infty} \frac{(\lambda, r)}{(1, r)} t^r F(\lambda + r, x, y, z),$$

Express F in series form as given by (2·15), employ the elementary relation

$$(\lambda, r)(\lambda + r, am + bn + cp) = (\lambda, r + am + bn + cp) = (\lambda, am + bn + cp)$$
$$(\lambda + am + bn + cp, r),$$

we get

$$\triangle = (1-t)^{-\lambda} \sum_{m,n,p=0}^{\infty} (\lambda, am+bn+cp) \; Amnp \left(x(1-t)^{-a} \right)^m \left(y(1-t)^{-b} \right)^n \left(z(1-t)^{-c} \right)^p$$

$$= (1-t)^{-\lambda} \; F \left(\lambda, x(1-t)^{-a}, y(1-t)^{-b}, z(1-t)^{-c} \right), \quad \ldots \quad (2\cdot16)$$

provided that both sides have a meaning.

Proceeding on parallel lines as mentioned above, the relations (2·1) to (2·14) are immediate consequences of the definitions (1·1) to (1·5).

It may be of interest to mention further that (2·16) yields future relations involving hypergeometric functions not yet considered

3. In this section we give the following results as specialized or limiting cases of some of the generating relations in section 2.

$$\sum_{r=0}^{\infty} \frac{(\lambda, r)}{(1, r)} \; F_1 \; (\lambda + r, \beta, \beta^1; \gamma; y, z) \; t^r$$

$$= (1 - t)^{-\lambda} \; F_1 \left(\lambda, \beta, \beta^1; \gamma; \frac{y}{1 - t}, \; \frac{z}{1 - t} \right), \quad .. \quad .. \quad .. \quad (3\cdot1)$$

$$\sum_{r=0}^{\infty} \frac{(\lambda, r)}{(1, r)} \; F_3 \; (\alpha, \lambda + r, \beta, \beta^1; \gamma; y, z) \; t^r$$

$$= (1 - t)^{-\lambda} \; F_3 \left((\alpha, \lambda, \beta, \beta^1; \gamma; y, \; \frac{z}{1 - t} \right), \quad .. \quad .. \quad .. \quad (3\cdot2)$$

$$\sum_{r=0}^{\infty} \frac{(\lambda, r)}{(1, r)} \; \Phi_1 \; (\lambda + r, \beta; \gamma; z, y) t^r$$

$$= (1 - t)^{-\lambda} \; \Phi_1 \left(\lambda, \beta; \gamma; \frac{z}{1 - t}, \; \frac{y}{1 - t} \right), \quad .. \quad .. \quad .. \quad (3\cdot3)$$

$$\sum_{r=0}^{\infty} \frac{(\lambda, r)}{(1, r)} \; \Xi \; (\lambda + r, \alpha, \beta; \gamma; z, y) \; t^r$$

$$= (1 - t)^{-\lambda} \; \Xi \left(\lambda, \alpha, \beta; \gamma; \frac{z}{1 - t}, y \right), \quad .. \quad .. \quad .. \quad .. \quad (3\cdot4)$$

where Φ_1 and Ξ, are Humberts' confluent hypergeometric functions of two variables Bateman project [(1953, p. 225 (20) and p. 226 [25)].

If we replace t by $-t$ in (3·1) and (3·2) we get two recent results of Anandani [(1970, p. 51-52)] involving Appell's F_1 and F_3. All of these results are evidently contained in the general formulae (1·9), (1·10), or (1·11), of Srivastava [1972*b*, p. 75].

It will not be out of place to mention here that a number of results similar to those given in section 3 can easily be derived from the results of section 2, or as special cases of the aforementioned formulae of Srivastava (1972*b*).

Acknowledgement

The authors take this opportunity to express their sincere thanks to Professor H.M. Srivastava of the University of Victoria for his kind encouragement and helpful suggestions.

References

Anandani, P. (1970). On some generating functions. *Glasnik Mat.*, Ser. III, **5** (25), 51-53.

Bateman Project (1953). Higher Transcendental Functions, Vol. I, McGraw-Hill Book Co.. Inc., New york.

Dhawan, G. K. (1970). Hypergeometric functions of three variables. *Proc. Math. Acad. Sci. India*, A, **40**, 43-48).

Pandey, R. C. (1963). On certain hypergeometric transformations. *J. Math. Mech.*, **12**, 113-118.

Srivastava, H. M. (1972*a*). A note on certain hypergeometric differential equations. *Mat. Vesnik*, **9** (24), 101-107.

————(1972*b*). Certain formulae involving Appell functions. *Comment. Math., Univ. St. Paul.*, **21**, 73-99.