A CLASS OF BASIC INTEGRAL EQUATIONS WITH BASIC HYPERGEOMETRIC FUNCTION ${}_1\Phi_1$ IN THE KERNELS

by T. R. PRABHAKAR, *Department of Mathematics, University of Delhi, Delhi 7*
and
M. CHAKRABARTY, *Department of Mathematics, Ramjas College, University of Delhi, Delhi 7*

(Communicated by F. C. Auluck, F.N.A.)

(Received 14 May 1974)

A basic integral equation whose kernel contains the basic analogue of the confluent hypergeometric function is studied by a systematic use of fractional q-integral operators. Properties of a related basic integral operator are drived in the first part of the paper; these are subsequently used to discuss theorems on the basic integral equation and to obtain its explicit analytic solutions.

1. INTRODUCTION

Recently, Prabhakar (1969, 1971) used fractional integration to discuss the integral equation

$$K(a, b, c) f(x) = \int_0^x \frac{(x-t)^{b-1}}{\Gamma(b)} {}_1F_1(a; b; c(x-t)) f(t)\, dt = g(x) \qquad (1\cdot1)$$

and to prove results on the operator $K(a, b, c)$ the function ${}_1F_1(a; b; z)$ being Kummer's confluent hypergeometric function Slater (1966). The object of the present paper is to study certain properties of the basic integral operator $H_q(a, b, c)$ defined by

$$H_q(a, b, c) f(x) = \frac{1}{\Gamma_q(b)} \int_0^x [x - tq]_{b-1}\, {}_1\Phi_1(a; b; c[x - tq^b])\, f(t)\, d(t; q),$$

$$\text{for } x > 0, \quad \ldots \quad \ldots \qquad (1\cdot2)$$

provided $|cx| < 1$ and the series $\sum_{j=0}^{\infty} |q^j f(xq^j)|$ converges. Here ${}_1\Phi_1(a; b; z)$ is the basic confluent hypergeometric function and $F([x-y])$ stands for $\sum_{r=0}^{\infty} A_r [x-y]_r$

if $F(x) = \sum_{r=0}^{\infty} A_r x^r$ (Hahn 1949); the basic integral in (1·2) is also defined in Section 2.

The main tool employed is the fractional q-integral operator I_q^a defined recently by Agarwal (1969). The results proved for $H_q(a, b, c)$ are subsequently used to discuss solutions of the basic integral equation

$$\int_0^x \frac{[x - tq]_{b-1}}{\Gamma_q(b)} \, {}_1\Phi_1(a; b; c\,[x - tq^b])\, f(t)\, d(t; q) = g(x). \quad \ldots \quad (1·3)$$

Explicit analytic solutions are obtained in two different forms and indeed, under different assumptions.

2. Definitions and Notation

For $|q| < 1$, let

$[q^a]_n = (1 - q^a)\,(1 - q^{a+1}) \ldots (1 - q^{a+n-1})\ ;\ [q^a]_0 = 1.$

The following basic hypergeometric functions are defined in the usual manner (Hahn 1949, Slater 1966) :

$${}_r\Phi_s\left[\begin{matrix} a_1, a_2, \ldots, a_r\ ; z \\ b_1, b_2, \ldots, b_s \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{[q^{a_1}]_n\ [q^{a_2}]_n \ldots [q^{a_r}]_n\ z^n}{[q]_n\ [q^{b_1}]_n \ldots [q^{b_s}]_n}\ ;\ |z| < 1,$$

$$e_q(x) = \prod_{n=0}^{\infty} [1 - xq^n]^{-1} = \sum_{r=0}^{\infty} \frac{x^r}{(q)_r}\ ;\ |x| < 1,$$

$$\mathrm{E}_q(x) = \prod_{n=0}^{\infty} [1 - xq^n] = \sum_{r=0}^{\infty} \frac{(-x)^r\, q^{r(r-1)/2}}{[q]_r}$$

$$[x - y]_\nu = x^\nu \prod_{n=0}^{\infty} \left[\frac{1 - yq^n/x}{1 - yq^{n+\nu}/x}\right] = x^\nu \sum_{r=0}^{\infty} \frac{[q^{-\nu}]_r\ (yq^\nu/x)^r}{[q]_r}\ ;$$

$$|yq^\nu/x| < 1$$

$$\Gamma_q(a) = \frac{(1 - q)_{a-1}}{(1 - q)^{a-1}}\ ;\ (a \neq 0, -1, -2, \ldots).$$

The basic integrals are defined through the relations (Hahn 1949) :

$$\int_0^x f(t)\, d(t\ ; q) = x(1 - q) \sum_{k=0}^{\infty} q^k f(xq^k)$$

$$\int_x^\infty f(t)\, d(t\,;q) = x(1-q)\sum_{k=1}^{\infty} q^{-k} f(xq^{-k})$$

$$\int_0^\infty f(t)\, d(t\,;q) = (1-q)\sum_{k=-\infty}^{\infty} q^{k} f(q^{k}).$$

The fractional q—integral operator I_q^α is defined by (Agarwal 1969) :

$$I_q^\alpha f(x) = \frac{1}{\Gamma_q(\alpha)} \int_0^x (x-tq)_{\alpha-1}\, f(t)\, d(t\,;q).$$

3. Properties of $H_\alpha\ (a, b, c)$

Theorem 1—If Re $b > 0$, $|cx| < 1$, and $\sum_{i=0}^{\infty} |q^i f(xq^i)|$ converges then

$$I_q^\alpha H_q(a, b, c) f(x) = H_q(a, b+\alpha, c) f(x). \quad \ldots \quad \ldots \quad \ldots \quad (3{\cdot}1)$$

Proof : $I_q^\alpha\ H_q(a, b, c) f(x) = \dfrac{1}{\Gamma_q(\alpha)\ \Gamma_q(b)} \displaystyle\int_0^x (x-tq)_{\alpha-1}$

$$\times \int_0^t [\,t-uq\,]_{b-1}\ {}_1\Phi_1(a; b; c\,[\,t-uq^b]) f(u)\, d(u; q)\, d(t; q)$$

$$= \frac{1-q}{\Gamma_q(b)\ \Gamma_q(\alpha)} \int_0^x (x-tq)_{\alpha-1}\, t^b \sum_{j=0}^{\infty} q^j (1-q^{j+1})_{b-1}$$

$$\times\ {}_1\Phi_1\ (a; b; ct\,[\,1-q^{b+j}])\ f(tq^j)\, d(t\,;q)$$

$$= \frac{(1-q)^2 x^{b+\alpha}}{\Gamma_q(b)\ \Gamma_q(\alpha)} \sum_{n=0}^{\infty} \frac{(q^q)_n\,(cx)^n}{[q]_n\ [q^b]_n} \sum_{k=0}^{\infty} (1-q^{k+1})_{\alpha-1}\, q^{k(1+b+n)}$$

$$\times \sum_{j=0}^{\infty} q^j\,[\,1-q^{j+1}\,]_{b+n-1}\ f(xq^{k+j})$$

$$= (1-q)^{b+\alpha}\, x^{b+\alpha} \sum_{n=0}^{\infty} \frac{[q^q]_n\ (cx)^n}{[q]_n} \sum_{k=0}^{\infty} \frac{[\,q^\alpha\,]_k}{[\,q\,]_k}\, q^{k(1+b+n)}$$

$$\times \sum_{j=0}^{\infty} \frac{[q^{b+n}]_j\, q^j}{[q]_j} f\,(xq^{k+j})$$

$$= (1-q)^{b+a}\, x^{b+a} \sum_{n=0}^{\infty} \frac{[q^a]_n\,(cx)^n}{[q]_n} \sum_{i=0}^{\infty} \frac{[q^{b+n}]_i}{[q]_i}\, q^i f\,(xq^i)$$

$$\times \sum_{k=0}^{i} \frac{[q^a]_k\,[q^{-i}]_k\, q^k}{[q]_k\,[q^{1-b-n-i}]_k}.$$

Summing the innermost series with the help of q—analogue of Vandermonde's theorem (Slater 1966, 3.3.2.7.), we get

$$I_q^a \; H_q\,(a, b, c)\, f\,(x) = (1-q)^{b+} \;\; x^{b+a}$$

$$\times \sum_{n=0}^{\infty} \frac{[q^a]_n\,(cx)^n}{[q]_n} \sum_{i=0}^{\infty} \frac{[q^{b+n+a}]_i}{[q]_i}\, q^i \;\; f\,(xq^i)$$

$$= \frac{(1-q)x^{b+a}}{\Gamma_q \;\; (b+a)} \sum_{i=0}^{\infty} q^i f\,(xq^i)\;\; [1-q^{\;+i}]_{b+a-1}$$

$$\times \sum_{n=0}^{\infty} \frac{[q^a]_n\;\; [1-q^{b+a+i}]_n\,(cx)^n}{[q]_n\,[q^{b+a}]_n}$$

$$= \int_0^x \frac{(x-tq)_{b+a-1}}{\Gamma_q\,(b+a)} \;\; {}_1\Phi_1\,(a;\, b+a;\, c\,[x - tq^{b+a}])\, f(t)\; d(t;\, q)$$

$$= H_q(a,\, b+a,\, c)\, f(x),$$

and the theorem is proved.

In a similar manner, one can easily prove that

$$H_q\,(a, b, c)\; I_q^a f(x) = H_q\,(a, b+a, c)\, f(x),$$

which, on using Theorem 1 gives the following theorem :

Theor m 2—If Re $b > 0$, Re $a > 0$, $|cx| < 1$ and $\sum_{i=0}^{\infty} |q^i f(xq^i)|$ converges, then

$$H_q\,(a, b, c)\; I_q^a f(x) = I_q^a \; H_q\,(a, b, c)\, f(x). \qquad .. \qquad .. \qquad .. \qquad (3{\cdot}2)$$

Theorem 3—If Re $b' > 0$, $|cx| < 1$, $|cxq^a| < 1$ and $\sum_{i=0}^{\infty} |q^i f(xq^i)|$ converges, then

$$H \;\; (a, b, c)\; H_q\,(a', b', cq^a)\, f(x) = H_q\,(a+a', b+b', c) f(x). \qquad .. \qquad (3{\cdot}3)$$

PROOF : $H_q(a, b, c)\, H_q(a', b', cq^a) f(x)$

$$= \frac{1}{\Gamma_q(b)\, \Gamma_q(b')} \int_0^x [x - tq]_{b-1}\ {}_1\Phi_1(a; b; c[x - tq^b])$$

$$\times \int_0^t f(u)[t - uq]_{b'-1}\ {}_1\Phi_1(a'; b'; cq^a[t - uq^{e'}])\, d(u; q)\, d(t; q)$$

$$= \frac{(1-q)^2 x^{b+b'}}{\Gamma_q(b)\, \Gamma_q(b')} \sum_{j=0}^{\infty} q^j (1 - q\quad)_{b-1}\ {}_1\Phi_1(a; b; cx[1 - q^{b+j}])$$

$$\times \sum_{k=0}^{\infty} q^k [1 - q^{k+1}]_{b'-1}\, q^{b'j}\ {}_1\Phi_1(a'; b'; cq^{a+j} x[1 - q^{b'+k}]) f(xq^{k+j})$$

$$= (1-q)^{b+b'}\, x^{b+b'} \sum_{i=0}^{\infty} q^i f(xq^i) \sum_{m,n=0}^{\infty} \frac{[q^a]_m\, [q^{a'}]_n\, [q^{b+m}]_i}{[q]_m\, [q]_n\, [q]_i}$$

$$\times\ q^{i(b'+n)+an}\, (cx)^{m+n} \sum_{k=0}^{i} \frac{[q^{b'+n}]_k (q^{-i})_k\, q^{k(1-b-b'-m-n)}}{[q]_k\, [q^{1-b-m-i}]_k}.$$

Summing the innermost series with the help of basic analogue of Gauss' theorem (Slater 1966, 3.3.2.5), we have

$$H_q(a, b, c)\, H_q(a', b', cq^a) f(x) = (1-q)^{b+b'}\, x^{b+b'} \sum_{i=0}^{\infty} q^i f(xq^i)$$

$$\times \sum_{m,n=0}^{\infty} \frac{[q^a]_m\, [q^{a'}]_n\, (q^{b+b'+m+n})_i\, (cx)^{m+n}\, q^{an}}{(q)_m\, (q)_u\, (q)_i}$$

$$= (1-q)^{b+b'}\, x^{b+b'} \sum_{i=0}^{\infty} \frac{q^i f(xq^i)}{[q]_i} \sum_{r=0}^{\infty} \frac{[q^a]_r\, [q^{b+b'+r}]_i\, (cx)^r}{(q)_r}$$

$$\times \sum_{n=0}^{r} \frac{(q^{a'})_n\, (q^{-r})_n\, q^n}{[q]_n\, [q^{1-a-r}]_n}.$$

Again, summing the innermost series with the help of basic analogue of Vandermonde's Theorem, we find that

$H_q(a, b, c)\, H_q(a', b', cq^a) f(x)$

$$= (1-q)^{b+b'}\, x^{b+b'} \sum_{i=0}^{\infty} \frac{q^i f(xq^i)}{[q]_i} \sum_{r=0}^{\infty} \frac{[q^{a+a'}]_r\, [q^{b+b'+r}]_i\, (cx)^r}{(q)^r}$$

$$= \frac{(1-q]^{b+b'} \quad x^{b+b'}}{(1-q)_{b+b'-1}} \sum_{i=0}^{\infty} q^i [1-q^{i+1}]_{b+b'-1}$$

$$\times \sum_{r=0}^{\infty} \frac{[q^{a+a'}]_r \quad [q^{b+b'+i}]_r}{[q]_r \quad [q^{b-b'}[} (cx)^r f(xq^i)$$

$$= H_q(a+a', b+b', c) f(x)$$

Theorem 4—If Re $b > 0$, Re $a > 0$, $|cx| < 1$ and $\sum_{i=0}^{\infty} |q^i(xq^i)|$ converges, then

$$H_q(a, b, c) f(x) = I_q^{b-a} e_q(cx) I_q^a \{E_q(cxq^a) f(x)\}. \quad \ldots \quad \ldots \quad (3\cdot4)$$

PROOF : By Theorem 1

$$I_q^a H_q(a, b, c) f(x) = H_q(a, a+b, c) f(x)$$
$$= I_q^b H_q(a, a, c) f(x)$$

But $H_q(a, a, c) = \int_0^x \frac{[x-tq]_{a-1}}{\Gamma_q(a)} e_q(c[x-tq^a]) f(t)\, d(t; q).$

Using the relation

$$\frac{(e_q)(x)}{(e_q)(y)} = e_q([x-y])$$

given by Hahn (1949) we can write

$$H_q(a, a, c) = e_q(cx) \int_0^x \frac{(x-tq)_{a-1}}{\Gamma_q(a)} E_q(ctq^a) f(t)\, d(t; q)$$

$$= e_q(cx) I_q^a \{E_q(cxq^a) f(x)\}.$$

Hence, $I_q^a H_q(a, b, c) f(x) = I_q^b e_q(cx) I_q^a \{E_q(cxq^a) f(x)\}$, which immediately leads to (3·4).

4. Basic Integral Equation (1·3)

In this section we study the basic integral equation (1·3) by using the theorems proved in the previous section. In Theorem 5 the solution of (1·3) is obtained in terms of fractional q-integral operators and basic exponential functions $e_q(x)$ and $E_q(x)$. In Theorem 6, the solution is given as a basic integral involving, ${}_1\Phi_1$.

Theorem 5—If Re $a >$ Re $b > 0$ and $\sum_{i=0}^{\infty} | \; q^i \; g(xq^i) \; |$ converges, then the basic integral equation

$$\int_0^x \frac{(x - tq)_{b-1}}{\Gamma_q(b)} \; {}_1\Phi_1(a; b; c[x - tq^b]) f(t) \, d(t; q) = g(x) \qquad \ldots \qquad (4{\cdot}1)$$

has solution f given by

$$f(x) = e_q(cxq^a) \, I_q^{-a} \, E_q(cx) \, I_q^{a-b} \, g(x) \qquad \ldots \qquad (4{\cdot}2)$$

where

$$\sum_{i=0}^{\infty} | \, q^i f(xq^i) \, | \text{ converges.}$$

Proof: Suppose Re $a>0$, Re $b>0$, $|cx| < 1$. With the help of Theorem 4, eqn. (4·1) can be written

for $\sum_{=0}^{\infty} | \, q^i f(xq^i) \, |$ convergent, as

$$I_q^{b-a} \, e_q(cx) \, I_q^a \{E_q(cxq^a) f(x)\} = g(x).$$

First operating on both sides by I_q^{a-b}, we have

$$I_q^a \{E_q(cxq^a) f(x)\} = E_q(cx) \, I_q^{a-b} \, g(x).$$

Again operating by I_q^{-a}, we get our required result (4·2).

Theorem 6—If Re $b >$ Re l, Re $b>0$, Re $l<0$, $|cx| < 1$, $|cxq^a| < 1$ and $\sum_{i=0}^{\infty} | \, q^i g(xq^i) \, |$ converges, then the basic integral equation

$$\int_0^x \frac{(x - tq)_{b-1}}{\Gamma_q(b)} \; {}_1\Phi_1(a; b; c[x - tq^b]) f(t) \, d(t; q) = g(x) \qquad \ldots \qquad (4{\cdot}3)$$

has solution f given by

$$f(x) = \int_0^x \frac{[x - tq]_{l-b-1}}{\Gamma_q(l - b)} \; {}_1\Phi_1(-a; l-b; cq^a(x - tq^{l-b}) \times I_q^{-l} g(t) \, d(t; q) \qquad \ldots \qquad (4{\cdot}4)$$

where $\sum_{i=0}^{\infty} | \, q^i f(xq^i) \, |$ converges.

PROOF : In our notation of operators we have to show that

$$H_q\,(a, b, c)\; f = g \qquad \text{(4·5)}$$

has the solution f given by

$$H_q\,(-a, l-b, cq^a)\; I_q^{-l}\; g = f. \qquad \text{(4·6)}$$

Assume Re $b>$ Re l, Re $b>0$, Re $l<0$, $|cx|<1$, $|cxq^a|<1$ and $\sum\limits_{=0}^{\infty} |\,q^i f(xq^i)\,|$ to be convergent. Operate on both sides of (4·5) by $H_q\,(-a, l-b, cq^a)\; I_q^{-l}$, to get

$$H_q\,(-a, l-b, cq^a)\; I_q^{-l}\; H_q\,(a, b, c) f = H_q\,(-a, l-b, cq^a)\; I_q^{-l}\; g.$$

On using Theorem 2, we are led to

$$H_q\,(-a, l-b, cq^a)\; H_q\,(a, b, c)\; I_q^{-l} f = H_q\,(-a, l-b, cq^a)\; I_q^{-l}\; g.$$

An application of Theorem 3, then gives

$$H_q\,(0, l, cq^a)\; I_q^{-l} f = H_q\,(-a, l-b, cq^a)\; I_q^{-l}\; g. \qquad \text{(4·7)}$$

But $H_q\,(0, l, cq^a)\; I_q^{-l}\; f = \int\limits_0^x \frac{[x-tq]_{l-1}}{\Gamma_q\,(l)}$

$$\times\; {}_1\Phi_1\,(0; l; cq^a\,[x - tq^l])\; I_q^{-l} f\,(t)\; d\,(t; q).$$

$$= I_q^l\; I_q^{-l} f.$$

Hence (4·7) gives

$$f = H_q\,(-a, l-b, cq^a)\, I_q^{-l}\; g$$

which is (4·6).

RFFERENCES

Agarwal R. P. (1969). Certain fractional g-integrals and g-derivatives. *Proc. Camb. Phil. Soc.*, **66**, 365-70.

Hahn W. (1949). Beitrage Zur theorie der Heineschen Reihen. Die 24 integrale der hypergeometrischen q-Differenzengleichung Das q-Analogen der Laplace transformation. *Math. Nachr.*, **2**, 340-79.

Prabhakar T. R. (1969). Two singular integral equations invoking confluent hypergeometric functions. *Proc. Camb. Phil. Soc.*, **66**, 71-89.

————(1971). Some integral equations with Kumer's functions in the Kernels. *Canad, Math. Bull.*, **14** (3) 391-404.

Slater, L. J. (1966). Generalized Hypergeometric Functions. Cambridge University Press, London.