ON THE THEORY OF WEAK SPHERICAL SHOCKS

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The shock wave theory of Kirkwood and Bethe is modified to study the propagation of weak spherical shocks arising from arbitrary piston motions. The proposed theory is shown to be in good agreement with the existing weak shock theories of Lighthill (1948), Witham (1956) and Varley and Cumberbatch (1966).

INTRODUCTION

The spherical 'piston problem' poses serious difficulties when the piston path is arbitrary. Only in special cases such as the uniformly expanding sphere, Taylor (1946), or a similar power law type of motion, Kochina and Melnikova (1958), it has been found possible to find an exact 'similarity' solution. Lighthill (1948) has analytically treated the uniformly expanding sphere problem involving weak shocks and has given a general method for arbitrary weak piston motions to correct the linear solution which is itself adequate except that it gives inaccurate value of pressure near the piston. Whitham (1950) treated the spherical piston motion, simulating a weak explosion, and derived a uniformly valid (first order) approximate solution which led to his well known theory of weak shocks. Whitham (1956) again considered this problem in the context of his general theory.

In the present paper, we develop and modify the shock wave theory of Kirkwood and Bethe, given briefly in Cole (1948), to study weak spherical shocks, resulting from an arbitrary piston motion. The main assumption in Kirkwood and Bethe theory, besides that of isentropy which is common to all theories of weak shocks and whose validity has been thoroughly discussed by Whitham (1950), is that total enthalpy $r \dfrac{\partial \phi}{\partial t}$ remains constant along the forward characteristic. Otherwise the full equations of motion are used. This assumption is, in fact, an exact relation in the linear theory and also corresponds to the first term in the asymptotic expansion of Whitham (1950) giving the solution at large distance from the source. Cole (1948) has discussed this approximation in the appendix but has not come to any firm conclusion.

In the present paper we briefly show how this theory may be used to study arbitrary weak piston motions and give a comparison of this theory with the existing theories of Whitham (1956), Lighthill (1948) and Varley and Cumberbatch (1966). We conclude that this theory is as accurate as the theory of 'relatively undistorted waves', given by Varley and Cumberbatch but is not a second order theory as, for example, is Friedrich's (1948) for plane shocks.

2. Basic Equation

The equations of motion and continuity for spherical symmetry are

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} = -\frac{1}{\rho} \frac{\partial p}{\partial r} \qquad (2{\cdot}1)$$

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \rho u) = 0 \qquad (2{\cdot}2)$$

where p, ρ and u are pressure, density and velocity at any point with spatial coordinate r and time t. It is assumed that the entropy is constant in the entire disturbed flow so that $p = \bar{k} \rho^\gamma$ where $\bar{k}$ is a constant and $\gamma = c_p / c_v$, the ratio of specific heats. If we introduce the velocity potential (Φ), defined by $u = -\partial\phi/\partial r$, in eqns. (2·1) and (2·2) and combine them suitably, we get a single equation in $\Phi = r\phi$:

$$\frac{\partial^2 \Phi}{\partial r^2} - \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} = \frac{1}{c^2}\left[\frac{u}{2} \frac{\partial u^2}{\partial r} - \frac{du^2}{dt^2}\right] \qquad (2{\cdot}3)$$

Here c is the speed of sound. This equation shows that in the acoustic approximation when the non-linear term on the right may be neglected, Φ and hence the kinetic enthalpy $\partial\Phi/\partial t \equiv r\,(\partial\phi/\partial t)$ remains constant during propagation, Kirkwood and Bethe assume that for waves of finite amplitude, $G \equiv r\,(\partial\phi/\partial t)$ is propagated with the exact speed of forward characteristics :

$$\frac{\partial G}{\partial t} + (u + c) \frac{\partial G}{\partial r} = 0. \qquad (2{\cdot}4)$$

3. Equation for Sound Speed at the Piston

We express the kinetic enthalpy G in the form

$$G(r, t) = r \frac{\partial \phi}{\partial t} = r\left(h + \frac{u^2}{2}\right) \qquad (3{\cdot}1)$$

where

$$h = \int_{p_0}^{p} \frac{dp}{\rho} . \qquad (3{\cdot}2)$$

This is, in fact, Bernoulli's equation. To find out the speed of sound at the piston as it moves and hence other physical variables, we first substitute eqn. (3·1) into (2·4). We then express the derivatives of p and ρ in eqns. (2·1) and (2·2) in terms of those of h according to eqn. (3·2) and make use of the resulting equations to write eqn. (2·4) along the piston path $\frac{dr}{dt} = u$. We thus obtain

$$\frac{r}{c}\left(1-\frac{u}{c}\right)\left(\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial r}\right)+\left(\frac{u}{c}+1\right)h$$
$$= r\left(1-\frac{u}{c}\right)\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}\right)+\frac{3}{2}\left(1-\frac{u}{3c}\right)u^2. \quad \ldots \quad \ldots \quad (3{\cdot}3)$$

Further, since h and the speed of sound c are connected by the relation

$$h = \frac{c^2{}_0}{\gamma-1}\left(\frac{c^2}{c_0^2}-1\right) \quad \ldots \quad \ldots \quad \ldots \quad (3{\cdot}4)$$

we finally obtain the equation for the speed of sound along the piston path as

$$\frac{dc}{dt} = \frac{\gamma-1}{2R\left(1-\frac{u}{c}\right)}\left\{R\frac{d^2R}{dt^2}\left(1-\frac{u}{c}\right)+\frac{3}{2}\left(\frac{dR}{dt}\right)^2\left(1-\frac{u}{3c}\right)\right.$$
$$\left.-\frac{1}{\gamma-1}\left(1+\frac{u}{c}\right)\left(c^2-c_0^2\right)\right\} \quad \ldots \quad \ldots \quad (3{\cdot}5)$$

where $R = R(t)$ is the radius of the piston, $u = dR/dt$ and

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r} = \frac{d^2R}{dt^2}.$$

If we know the piston path $R{=}R\ (t)$, the ordinary differential equation (3·5) gives the speed of sound along this path (cf. Cole 1948, p. 273).

4. Characteristics and the Shock Locus

Eqn. (2·4) shows that if $G\ (r, t)$ is known on the surface of an expanding sphere $R = R\ (\ t_R\ )$, its value at any point r and time t is given by

$$G\ (r, t) = G\ (R, \tau) = G_R\ (\tau) \quad \ldots \quad \ldots \quad \ldots \quad (4{\cdot}1)$$

where

$$t = \tau + \int_R^r \frac{dr}{u + c} . \qquad \text{.. .. ..} \qquad (4\cdot2)$$

τ is the characteristic variable of Whitham (1956) and labels individual characteristics. It is, of course, the time at which a particular wavelet is emitted by the piston. To obtain the equation for the characteristics we write the Rankine-Hugoniot equations at the shock up to second order in its strength. We have

$$G = c_0 \, r \, u \, (1 + \beta \, u) \qquad \text{.. .. .. ..} \qquad (4\cdot3)$$

$$u + c = c_0 \, (1 + 2 \, \beta \, u) \qquad \text{.. .. .. ..} \qquad (4\cdot4)$$

where $\beta = \dfrac{\gamma + 1}{4c_0}$. We substitute eqns. (4·3) and (4·4) into (4·2) and integrate from the piston to the shock. We get

$$t = \tau + \frac{\beta G}{c_0^2} \left[\frac{1 + 2 \, \beta u}{\beta u \, (1 + \beta u)} - \frac{1 + 2 \beta U}{\beta U \, (1 + \beta U)} - 2 \, \text{In} \, \frac{(1 + \beta u) \, \beta U}{(1 + \beta U) \, \beta u} \right] \qquad \text{.. .. ..} \qquad (4\cdot5)$$

where

$$\beta u = \frac{1}{2} \left[\left(1 + \frac{\gamma + 1}{r c_0^2} \, G \right)^{1/2} - 1 \right]$$

$$\beta U = \frac{1}{2} \left[\left(1 + \frac{\gamma + 1}{R c_0^2} \, G \right)^{1/2} - 1 \right] . \qquad \text{.. .. ..} \qquad (4\cdot6)$$

To draw the individual characteristics corresponding to a given piston motion, we have the following procedure. For a given $R = R\,(t)$, Eqns. (3·5) and (3.4) give $h = H$ and hence $G (= R \, (H + \frac{1}{2} \, u^2) \,)$ as functions of τ, which is equal to t at the piston. Choose a value of τ, find $G\,(\tau)$ from (3·4); $R\,(\tau)$ and $U\,(\tau) = \dfrac{dR}{dt}$ are known from the piston motion. Hence eqn. (4·5) gives $t = t(r)$ as the locus of this characteristic. To obtain the shock path, we write the shock conditions up to second order in shock strength:

$$u + c = c_0 \, (1 + 2 \beta u) \; ; \; U_s = c_0 \, (1 + \beta u + \tfrac{1}{2} \, \beta^2 u^2) ;$$

$$G = c_0 \, r \, (1 + \beta u) \, u \qquad \text{.. .. .. ..} \qquad (4\cdot7)$$

where in calculating G we have assumed that the flow behind the shock is a simple wave (cf. Cole 1948 p. 44). Now we follow Whitham (1956) in deriving the shock locus. Let $T = T(r)$ be the shock locus. We substitute this in the characteristic relation (4·6), differentiate it with respect to r and equate dt/dr from the resulting equation to U_s^{-1} as given by (4·7). We finally have

$$\frac{dT}{dr} = \frac{\frac{\gamma+1}{r^2}\bar{G}L_r + (1+\beta u + \frac{1}{2}\beta^2 u^2)^{-1}}{D} \quad \ldots \quad \ldots \quad (4\cdot 8)$$

wh

$$D = 1 + \left[\frac{1+2\beta u}{u(1+\beta u)} - \frac{1+2\beta U}{U(1+\beta U)} - 2\beta \ln \frac{(1+\beta u)U}{(1+\beta U)u}\right]\frac{d\bar{G}}{dT} + \frac{\gamma+1}{r} L_r \frac{d\bar{G}}{dT} - \left(\frac{\gamma+1}{R}\frac{d\bar{G}}{dT} - \frac{\gamma+1}{R^2}\bar{G}\frac{dR}{dT}\right)L_R$$

$$L_r = \frac{\bar{G}}{4\beta\left(1+\frac{\gamma+1}{r}\bar{G}\right)^{1/2}}\left[\frac{2\beta}{(1+\beta u)u} - \frac{1+2\beta u}{u^2(1+\beta u)} - \frac{\beta(1+2\beta u)}{u(1+\beta u)^2} - \frac{2\beta^2}{1+\beta u} + \frac{2\beta}{u}\right]$$

$$L_R = \frac{\bar{G}}{4\beta\left(1+\frac{\gamma+1}{R}\bar{G}\right)^{1/2}}\left[\frac{2\beta}{(1+\beta U)U} - \frac{1+2\beta U}{U^2(1+\beta U)} - \frac{\beta(1+2\beta U)}{U(1+\beta U)^2} - \frac{2\beta^2}{1+\beta U} + \frac{2\beta}{U}\right].$$

$$\bar{G} = G/c_0^2 .$$

In the above, βu and βU are the same as defined in eqn. (4·6) and R, G, H are functions of τ (equal to T at the shock) and follow from the solution of eqn. (3·5). When the solution of the above equation, $T = T(r)$, say, is substituted in the characteristic relation (4·6), we obtain the shock locus $t_s = t_s(r)$.

5. Comparison with the Existing Theories

First of all we notice that when βu is small, eqn. (4·5) immediately leads to

$$t - \tau = \frac{G\beta}{c_o^2}\left[\frac{1}{\beta u} - \frac{1}{\beta U} - 2 \ln (r/R)\right] \quad \ldots \quad \ldots \quad \ldots \quad (5\cdot 1)$$

and since, in this approximation, $G \sim c_0\, ru \sim c_0\, RU$ and $\beta\, u \sim \frac{\gamma+1}{4}\,(G/rc_0^2)$, we recover precisely eqn. (3·7) of Whitham (1956), namely,

$$t - \tau = \frac{r - R}{c_0} - k\, F(\tau) \log (r/R) \quad \ldots \quad (5\cdot 2)$$

where

$$k = \frac{\gamma + 1}{2\, \gamma c_0} \text{ and } F(\tau) = \frac{\gamma\, u}{c_0}\, r.$$

Thus, Whitham's weak shock theory gives for the problem under consideration the same results as the present theory in the limit $\beta\, u \rightarrow 0$.

In an earlier paper, Lighthill (1948) considered the uniform expansion of a spherical piston in addition to some other aerodynamic phenomena involving weak shocks. From an analytic treatment of this problem, he arrived at the general result that the pressure near a piston with arbitrary motion can be obtained correctly from the linear solution for the velocity potential provided the exact non-linear form of the Bernoulli equation is employed. Here we consider the uniform expansion of a sphere so that its motion is given by $r = c_0 a\, t$, where a is a constant. Following Lighthill's study, we obtain from the non-linear Bernoulli equation the equation for the square of the speed of sound :

$$\begin{aligned} \frac{c^2}{c_0^2} &= 1 + \frac{\gamma - 1}{2}\; \frac{3 - a}{1 + a}\, a^2 \\ &= 1 + \frac{\gamma - 1}{2}\, a^2 \left[\, 3 - 4a + 4\, a^2 + \ldots \,\right] \quad \ldots \quad (5\cdot 3) \end{aligned}$$

where $a < 1$.

For this piston motion, eqn. (3·5) gives

$$\frac{c^2}{c_0^2} = 1 + {}^{3} \left(\gamma - 1\right) a^2 \left(1 - \frac{a}{c}\right) \left(1 - \frac{a}{3c}\right) \quad \ldots \quad (5\cdot 4)$$

which, when solved iteratively for c/c_0, gives (5·3) with an error o(a^5). Thus we have a good agreement of the present theory with Lighthill's solution.

Now we turn to the comparision of the present theory with the more recent theory of 'relatively undistorted waves, proposed by Varley and Cumberbatch (1965), which is closely connected. We notice that, according to this theory, $f_1 = h\,(\rho) + \frac{1}{2}\, u^2$ and $f_2 = \rho\, u\, r^2$ both remain constant along the forward characteristic (cf. eqn. (2·3) of Varley and Cumberbatch) while we assume that only $r\ (h\ (\rho) + \frac{1}{2}\ u^2)$ remains constant along the positive characteristic. The latter assumption, we note, is motivated by our eqn. (2·3). We have also considered the (high frequency) pulsating sphere problem by the present method and compared the results with the asymptotic solution of Varley and Cumberbatch (1966) in powers of ω^{-1}. We omit the details and remark that the present theory and the theory of relatively undistorted waves are both accurate to first order only for the high frequency waves.

It is concluded that the theory of Kirkwood and Bethe for weak spherical shocks compares favourably with most of the existing theories but is not accurate to second order.

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