ON GENERATING FUNCTIONS FOR A GENERALIZED POLYNOMIAL

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Seven generating functions for a generalized polynomial defined by Bhonsle and Agarwal (1963) have been obtained. The generating functions given recently by Gupta and Goyal (1971, 1972) follow as special cases of our main results.

1. Introduction

Bhonsle and Agarwal (1963) have defined the following generalized polynomial :

$$f_n^{(k,\alpha,\beta)} \quad (a_p \; ; b_q \; ; x) \equiv$$

$$\frac{(1+\alpha+\beta, n)}{n\,!} \quad {}_{p+k+1}F_{q+k+1}\left[\begin{matrix} -n, a_p, \triangle(k, n+\alpha+\beta+1) \\ b_q, \; \triangle(k+1, \alpha+\beta+1) \end{matrix} \quad \frac{4k^k x}{(k+1)^{k+1}}\right] \tag{1·0}$$

If $k = 1$ and $\alpha + \beta + 1 = 2\nu$, (1·0) is reduced to a new class of polynomial defined and represented by Cohen (1969) as given below :

$$H_n\,(\,\nu; a_p; b_o, b_q; x\,) \equiv \frac{(2\nu, n)}{n\,!}$$

$$_{p+2}F_{q+2}\left[\begin{matrix} -n, 2\nu + n, a_p \\ \nu + \tfrac{1}{2}, b_0\,, b_q \end{matrix} \; ; x \right] \text{ when } b_0 = \nu \quad .. \quad .. \tag{1·1}$$

where $_pF_q$ is generalized hypergeometric function and n, p, q are non-negative integers.

Further we shall require the generalized Kämpè-de-Fèriet function which is defined as follows :

$$F^{\,p,\lambda,\mu}_{\,q,\nu,\sigma}\left[\begin{matrix} a_p : b_\lambda \; ; c_\mu \\ d_q : e_\nu \; ; f_\sigma \end{matrix} \;\middle|\; x, y \right]$$

$$= \sum_{m,n=0} \frac{\{(a_p, m+n)\} \; \{(b_\lambda, m)\} \; \{(c_\mu, n)\} \;\; x^m y^u}{\{(d_q, m+n)\} \; \{(e_\nu, m)\} \; \{(f_\sigma, n)\} \;\; m\,!\,n!} \quad .. \quad .. \tag{1·2}$$

The double infinite series represented by (1·2) is absolutely convergent, if one of the following sets of conditions is satisfied :

(i) $1+q+\nu-p-\lambda>0,\ 1+q+\sigma-p-\mu>0$ for all x and y

(ii) $1+q+\nu-p-\lambda=0,\ 1+q+\sigma-p-\mu=0,$
$|x|+|y|<\min(1, 2^{q-p+1})$.

2. Notations

The following notations will be used in this paper :

$$\triangle(k, 1+a+\beta+n)=\prod_{j=0}^{k-1}\left(\frac{1+a+\beta+n+j}{k}\right) \quad \cdots \quad (2\cdot0)$$

$$\{\triangle(k, 1+a+\beta-nk), k'\}=\prod_{j=0}^{k-1}\left\{\frac{\Gamma\left(\frac{1+a+\beta+j}{k}-n+k'\right)}{\Gamma\left(\frac{1+a+\beta+j}{k}-n\right)}\right\} \quad (2\cdot1)$$

$$\phi(k)\equiv\frac{1}{\sqrt{2\pi}}\left(\frac{k+1}{k}\right)^{a+\beta+\frac{1}{2}}\frac{\prod_{j=0}^{k}\Gamma\left(\frac{1+a+\beta+j}{k+1}\right)}{\prod_{j=0}^{k-1}\Gamma\left(\frac{1+a+\beta+j}{k}\right)} \quad \cdots \quad (2\cdot2)$$

$\phi(k)=1$, when $k=1$ and $a+\beta+1=2\nu$.

$$\theta\equiv\frac{(k+1)^{k+1}}{k^k} \quad \cdots \quad \cdots \quad \cdots \quad (2\cdot3)$$

3. Main Results Proved

(a) *Simple Generating Functions*

(i) $$\sum_{n=0}^{\infty}\phi(k)\frac{(\lambda, n)\ \{(1-b_q, n)\}}{\{(1-a_p, n)\}}$$
$$\times f_n^{(k, a-n, \beta-nk)}(a_p-n, c_r; b_q-n, d_s; x)\ t^n$$
$$=F^{1, q+k+1, r}_{0, p+k, s}\left[\begin{matrix}\lambda : 1-b_q, \triangle(k+1, -a-\beta); c_r \\ - : 1-a_p, \triangle(k, -a-\beta); d_s\end{matrix}\ \middle|\ -\theta t, 4(-1)^{p-q+1}xt\right] \quad \cdots \quad \cdots \quad \cdots \quad (3\cdot0)$$

(ii) $$\sum_{n=0}^{\infty} \frac{(\lambda, n)}{n\,!} f_m^{(k,\alpha,\beta)} (-n, a_p ; b_q ; x)\, t^n$$

$$= (1-t)^{-\lambda} f_m^{(k,\alpha,\beta)} \left(\lambda, a_p ; b_q ; -\frac{xt}{1-t} \right) \quad \ldots \quad (3\cdot1)$$

(b) *Bilinear Generating Function*

(iii) $$\sum_{n=0}^{\infty} \phi(k) \frac{\{(1-b_q, n)\}\, (-1)^{(q-p+1)}\, n\,!}{\{(1-a_p, n)\}}$$

$$\times f_n^{(k,\alpha-n,\beta-nk)} (a_p - n; b_q - n; x)\, C_n^b(y) \left(\frac{t}{4}\right)^n$$

$$= (\sigma)^{-2b} F_{p+k,1,1}^{q+k+3,0,0} \left[R \,\middle|\, \frac{(-1)^{q-p+1}\, \theta t\,(xt - y + \sigma)}{8\sigma^2}, \right.$$

$$\left. \frac{(-1)^{q-p+1}\, \theta t\,(xt - y - \sigma)}{8\sigma^2} \right] \quad \ldots \quad (3\cdot2)$$

where $\sigma = (1 - 2xyt + x^2 t^2)^{\frac{1}{2}}$ and

$$R = \begin{matrix} 1-b_q, \triangle(k+1, -\alpha-\beta), 2b, b+\frac{1}{2}; -; - \\ 1-a_p, \triangle(k, -\alpha-\beta) : b+\frac{1}{2}; b+\frac{1}{2} \end{matrix}$$

(iv) $$\sum_{k=0}^{\infty} \frac{(\lambda, k)}{k\,!} f_m^{(k',\alpha,\beta)} (-k, a_p ; b_q ; x)$$

$$\times f_n^{(k'',\alpha',\beta')} (\lambda + k, c_r ; d_s ; y)\, t^k$$

$$= \frac{(1+\alpha+\beta, m)}{m\,!} \times \frac{(1+\alpha'+\beta', n)}{n\,!} \times (1-t)^{-\lambda}$$

$$\times F_{0,k'+q+k,k''+s+1}^{1,k'+p+1,k''+r+1} \left[\begin{matrix} \lambda : -m, a_p, \triangle(k', 1+\alpha+\beta+m); \\ - : b_q, \triangle(k', +1+\alpha+\beta); \end{matrix} \right.$$

$$\begin{matrix} -n, c_r, \triangle(k'', 1+\alpha'+\beta'+n) \\ d_s, \triangle(k''+1, 1+\alpha'+\beta') \end{matrix} ; \frac{-4k'^{k'} xt}{(k'+1)^{k'+1}(1-t)},$$

$$\left. \frac{4k''^{k''} y}{(k''+1)^{k''+1}(1-t)} \right] \quad \ldots \quad \ldots \quad \ldots \quad (3\cdot3)$$

(v) $$\sum_{n=0}^{\infty} \phi(k) \frac{(m+n)!\{(1-b_q, n)\} \quad (-1)^{(q-p+1)n}}{\{(1-a_p, n)\}} \left(\frac{t}{4}\right)^n$$

$$\times f_n^{(k, a-n, \beta-nk)} (a_p - n ; b_q - n ; x) \; P_{m+n}^{\rho-n, \delta-n} (y)$$

$$= (1+\rho+\delta+m, m) \left(\frac{(y-1)}{2}\right)^m \left(\frac{(y-1)}{y+1}\right)^\delta (1+ux)^{\rho+\delta+m}$$

$$F_{0, p+k, 1}^{1, q+k+1, 1} \left[\begin{matrix} -\rho-\delta-m : 1-b_q, \triangle(k+1, -a-\beta); -\delta-m \\ - : \triangle(k, -a-\beta), 1-a_p ; -\rho-\delta-2m \end{matrix} \middle| \frac{(-1)^{q-p+1} u\,\theta}{4(1+ux)}, \frac{2}{(1-y)(1+ux)} \right] \quad .. \quad (3\cdot4)$$

where $u = \dfrac{(y+1)t}{2}$.

(vi) $$\sum_{n=0}^{\infty} \phi(k) \frac{\{(1-b_q, n)\} \; (-1)^{n(-p+q+1)} n!}{\{(1-a_p, n)\}}$$

$$f_n^{(k, a-n, \beta-nk)} (a_p - n ; b_q - n ; x) \; L_n^{(a')} (y) \left(\frac{t}{4}\right)^n$$

$$= (1-xt)^{-1-a'} \exp(y) \, F_{0, p+k, 1}^{1, q+k+1, 0}$$

$$\left[\begin{matrix} 1+a' : 1-b_q, \triangle(k+1, -a-\beta); - \\ - : 1-a_p, \triangle(k, -a-\beta); 1+a' \end{matrix} \middle| \frac{\theta t(-1)^{q-p}}{4(1-xt)}, \frac{-y}{1-xt} \right] \quad .. \quad (3\cdot5)$$

where $L_n^{(a')} (x)$ is Laguerre polynomial.

(vii) $$\sum_{n=0}^{\infty} (\phi k) \frac{\{(1-b_q, n)\} \; (-1)^{(-p+q+1)n} \; n!}{\{(1-a_p, n)\}}$$

$$\times f_n^{(k, a-n, \beta-nk)} (a_p - n ; b_q - n ; x) \; P_n^{(a, -n, \beta'-n)} (y) \left(\frac{t}{4}\right)^n$$

$$= \left(1 + \frac{y+1}{2} xt\right)^{a'} \left(1 + \frac{y-1}{2} xt\right)^{\beta'}$$

$$\times F_{p+k, 0, 0}^{q+k+1, 1, 1} \left[\begin{matrix} 1-b_q, \triangle(k+1, -a-\beta) : -\beta'; -a' \\ 1-a_p, \triangle(k, -a-\beta) : - ; - \end{matrix} \middle| \frac{(-1)^{q-p+1} (y+1)\theta t}{4[2+(y+1)xt]}, \frac{(-1)^{q-p+1} (y-1)\theta t}{4[2+(y-1)xt]} \right] \quad .. \quad .. \quad (3\cdot6)$$

4. Proofs

Proof for (3·0)

With the help of (1·0), the left-hand side of (3·0)

$$= \sum_{n=0}^{\infty} \sum_{k'=0}^{n} \phi(k) \frac{(\lambda, n)\{(1-b_q, n)\}}{\{(1-a_p, n)\}\{(b_q-n, k')\}}$$
$$\times \frac{(1+\alpha+\beta+n-nk, n)(-n, k')\{(a_p-n, k')\}}{\{\triangle(k+1, 1+\alpha+\beta-n-nk), k'\}}$$
$$\times \frac{\{\triangle(k, 1+\alpha+\beta-nk), k'\}\{(c_r, k')\}}{\{(d_s, k')\} \; n!k'!} \left(\frac{4x}{\theta}\right)^{k'} t^n .$$

By virtue of well-known formulae (Rainville 1963, p. 58, eqn. (3); Erdelyi 1954, p, 3, eqn. (3) and p. 4, eqn. (11) we get

$$\left(\frac{2\pi k}{k+1}\right)^{\frac{1}{2}} \left(\frac{k}{k+1}\right)^{\alpha+\beta} \phi(k) \frac{\prod_{j=1}^{q} \Gamma(b_j)}{\prod_{j=1}^{p} \Gamma(a_j)} \times$$

$$\sum_{n=0}^{\infty} \sum_{k'=0}^{n} \frac{(\lambda, n) \prod_{j=1}^{p} \Gamma(a_j-n+k') \prod_{j=1}^{k-1} \Gamma\left(\frac{1+\alpha+\beta+j}{k}-n+k'\right)}{\prod_{j=1}^{q} \Gamma(b-n+k') \prod_{j=0}^{k} \Gamma\left(\frac{1+\alpha+\beta+j}{k+1}-n+k'\right)}$$
$$\frac{\{(c_r, k')\}}{\{(d_s, k')\} k'!(n-k')!} \left(-\frac{4x}{\theta}\right)^{k'} [(-1)^{q-p}\theta t]^n$$

using lemma [Reinville 1963, p. 57, eqn. (2)]

$$= \left(\frac{2\pi k}{k+1}\right)^{\frac{1}{2}} \left(\frac{k}{k+1}\right)^{\alpha+\beta} \phi(k) \frac{\prod_{j=1}^{q} \Gamma(b_j)}{\prod_{j=0}^{p} \Gamma(a_j)} \times$$

$$\sum_{n=0, k=0}^{\infty} \frac{(\lambda, n+k') \prod_{j=1}^{p} \Gamma(a_j-n) \prod_{j=0}^{k-1} \Gamma\left(\frac{1+\alpha+\beta+j}{k}-n\right)}{\prod_{j=1}^{q} \Gamma(b_j-n) \prod_{j=0}^{k} \Gamma\left(\frac{1+\alpha+\beta+j}{k+1}-n\right)}$$
$$\times \frac{\{(c_r, k')\}}{\{(d_s, k')\} \; n!k'!} \left[4(-1)^{p-q+1} xt\right]^{k'} \left[(-1)^{p-q} \theta t)\right]^n$$

Again using [Erdelyi 1953, p. 3, eqn. (3)], (2·2) and then interpreting with the help of (1·2) we immeditely get the right-hand side of (3·0)

Proof for (3·1)

With the help of (1·0) and Rainville 1963, p. 57, eqn. (2) and p. 58, eqn. (3) the left-hand side of (3·1)

$$= \frac{(1+\alpha+\beta, m)}{m\,!} \sum_{k'=0}^{\infty} \frac{(\lambda, k')\,(-m, k')\,\{(a_p, k')\}\,\{\triangle\,(k, 1+\alpha+\beta+m), k'\}}{\{(b_q, k')\}\,\{\triangle\,(k+1+\alpha+\beta), k'\}\,k\,!} \times \left[-\frac{4xt}{\theta}\right]^{k'} \sum_{n=0}^{\infty} \frac{(\lambda+k')_n\, t^n}{n\,!}$$

Apply Rainville 1963, p. 58, eqn. (1) then by virtue of (1·0) we get the right-hand side of (3·1).

Proof for (3·2)

Proceeding as in (3·0) the left-hand side of (3·2)

$$= \sum_{n=0}^{\infty} \frac{\{(1-b_q, n)\}\,\{\triangle\,(k+1, -\alpha-\beta), n\}\,(-1)^{(q-p)\,n}}{\{(1-a_p, n)\}\,\{\triangle\,(k, -\alpha-\beta), n\}} \times \left\{\sum_{k'=0}^{\infty} \frac{(n+k.)\,!}{n\,!\,k'\,!}\, C^{b}_{n+k'}\,(y)\,(xt)^{k'}\right\} \left(\frac{\theta t}{4}\right)^n .$$

Now on using the formulae (Rainville 1963, p. 280, eqn. (23) and p. 279, eqn. (17) we get

$$(\sigma)^{-2b} \sum_{n=0}^{\infty} \frac{\{(1-b_q, n)\}\,\{\triangle\,(k+1, -\alpha-\beta), n\}}{\{(1-a_p, n)\}\,\{\triangle\,(k, -\alpha-\beta), n\}} \left[\frac{(-1)^{q-p}\;\theta t}{4\sigma}\right]^n \times \left[\frac{y-xt+\sigma}{2\sigma}\right]^n \frac{(2\,b, n)}{n\,!} \sum_{s=0}^{n} \frac{(-n, s)\,(\frac{1}{2}-b-n, s)}{(b+\frac{1}{2}, s)\,s\,!} \left[\frac{y-xt-\sigma}{y-xt+\sigma}\right]^s$$

By applying the formula [Rainville 1963, p. 23, eqn. (3), p. 58, eqn. (3) and p. 57, eqn. (2); Erdèlyi 1953, p. 3, eqn. (3)] and making use of (1·2) the required result follows.

Proof for (3·3)

(3·3) can be easily worked out with the help of known results [Rainville 1963, p. 23, eqn. (3), p. 58, eqn. (3) and p. 57, eqn. (2)].

Proof for (3·4)

By virtue of well-known results [Rainville 1963, p. 255 eqn. (9), p. 60, eqn. (4) and p. 45, eqn. (1)] the left-hand side of (3·4) reduces to the following expression

$$(1+\rho+\delta+m, m)\left(\frac{y-1}{2}\right)^m\left(\frac{y-1}{y+1}\right)^\delta$$

$$\times\sum_{k'=0}^{\infty}\frac{(-\rho-\delta-m, k')\,(-\delta-m, k')}{(-\rho-\delta-2m, k')\,k'\,!}\left(\frac{2}{1-y}\right)^{k'}$$

$$\sum_{n=0}^{\infty}\phi(k)\;\frac{(-\rho-\delta-m+k', n)\{(1-b_q, n)\}\,(-1)^{n(-p+q+1)}}{2^{2n}\{(1-a_p, n)\}}\times$$

$$\times f_n^{(k,\alpha-n,\beta-nk)}(a_p-n\,;b_q-n\,;x)\left(-\frac{y+1}{2}\;t\right)^n .$$

Now making use of (3·0) we obtain

$$(1+\rho+\delta+m, m)\left(\frac{y-1}{2}\right)^m\left(\frac{y-1}{y+1}\right)^\delta\times$$

$$\times\sum_{k'=0}^{\infty}\frac{(-\rho-\delta-m, k')\,(-\delta-m, k')}{(-\rho-\delta-2m, k')\;k'\,!}\left(\frac{2}{1-y}\right)^{k'}$$

$$\times\;F\begin{matrix}1, q+k+1, 0\\ 0, p+k, 0\end{matrix}\left[\begin{matrix}-\rho-\delta-m+k' : 1-b_q, \triangle(k+1, -\alpha-\beta);-\\ - : \triangle(k, -\alpha-\beta), 1-a_p\;;-\end{matrix}\;\frac{(-1)^{-p+q+1}\,u\theta}{4}, -ux\right].$$

Expanding with the help of (1·2) and using [Rainville 1963, p. 58, eqn. (1), p. 73, eqn. (1) and p. 23, eqn. (3) we get the required result.

Proof for (3·5)

Proceeding as for (3·0), left-hand side of 3·5) becomes

$$\sum_{n=0}^{\infty}\frac{\{(1-b_q, n)\}\{\triangle(k+1, -\alpha-\beta), n\}\;(-1)^{n(q-p)}}{\{(1-a_p, n)\}\{\triangle(k, -\alpha-\beta), n\}}$$

$$\times\left[\sum_{k'=0}^{\infty}\frac{(n+k')\,!}{n\,!\,k'\,!}\;L_{n+k'}^{(\alpha')}(y)\,(xt)^{k'}\right]\left(\frac{\theta t}{4}\right)^n .$$

Using [Rainville 1963, p. 211, eqn. (9) and p. 200, eqn. (1)] we obtain

$$(1-xt)^{-a'-1}\sum_{n=0}^{\infty}\frac{\{(1-b_q\,,n)\}\ \{\triangle(k+1,-\alpha-\beta),n\}\ (-1)^{(q-p)n}\ (1+\alpha',n)}{\{(1-a_p\,,n)\}\ :\ \{\triangle(k,-\alpha-\beta),n\}\ n\,!}$$

$$\times\ \exp\left(\frac{-xyt}{1-xt}\right){}_1F_1\left[-n;\ 1+\alpha'\ ;\ \frac{y}{1-xt}\right]\times\left[\frac{\theta\, t}{4\,(1-xt)}\right]^n$$

Lastly applying Rainville 1963, p. 125 and Erdèlyi 1953 p. 3, eqn. (3) and then interpreting with the help of (1·2) the required result follows.

Proof for (3·6)

Proceeding as in (3·1), the left-hand side of (3·6) becomes

$$\sum_{n=0}^{\infty}\frac{\{(1-b_q\,,n)\}\ \{\triangle(k+1,-\alpha-\beta),n\}}{\{(1-a_p\,,n)\}\ \{\triangle(k,\ -\alpha-\beta),\ n\}}\ \frac{(-1)^{n(q-p)}}{}$$

$$\left[\sum_{k'=0}^{\infty}\frac{(n+k')\,!}{n\,!\ k'\,!}\ P_{n+k'}^{\alpha'-n-k',\,\beta-n-k'}(y)\ (xt)^{k'}\right]\times\left(\frac{\theta\, t}{4}\right)^n.$$

Using [Manocha and Sharma 1966, p. 460, eqn. (7)] and [Rainville 1963, p. 255, eqn. (8)] the above expression takes the following form

$$\left(1+\frac{y+1}{2}\,x\,t\right)^{\alpha'}\left(1+\frac{y-1}{2}\,xt\right)^{\beta'}\times$$

$$\sum_{n=0}^{\infty}\frac{\{(1-b_q\,,n)\}\ \{\triangle(k+1,-\alpha-\beta),n\}\ (1+\beta'-n,n)}{\{(1-a_p\,,n)\}\ \{\triangle(k,-\alpha-\beta),n\}\ \ n\,!}\times$$

$${}_2F_1\left[\begin{matrix}-n,\ \alpha'\\ 1+\beta'-n\end{matrix};\ \frac{(y+1)\ \{2+(y-1)\ xt\}}{(y-1)\ \{2+(y+1)\ xt\}}\right]\times$$

$$\left[\frac{(-1)^{q-p}\,(y-1)\ \theta\, t}{8+4\,(y-1)\ x\,t}\right]^n$$

Applying Rainville 1963, p. 57, eqn. (2) and Erdèlyi 1953, p. 3, eqn. (3), we obtain the required result after interpreting with the help of (1·2).

5. Particular Cases

Putting $k=1$ and $\alpha+\beta+1=2\nu$ in (3·0) to (3·2) and (3·4) to (3·6), we get the generating functions for a new class of polynomial when $b_0=\nu$, recently obtained by Gupta and Goyal.

Further putting $k' = k'' = 1$ and $\alpha + \beta + 1 = 2\nu = \alpha' + \beta' + 1$ in (3·3) we obtain the generating function concerning with the product of $H_n\ (\nu; a_p; b_0, b_q; x)$ when $b_0 = \mu$ and $d_0 = \nu$, already established by Gupta and Goyal (1971, 1972).

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