ON H-CONHARMONIC CURVATURE TENSOR IN KÄHLER MANIFOLDS

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Various authors (Mishra 1970 and Sinha 1972) have studied recurrence properties of curvature tensors in Kähler manifold. Recently Mishra and Pandey (1975) discussed birecurrence properties of curvature tensors in a Kähler manifold. In this paper, we shall define multi-recurrence and prove some results for H-conharmonic curvature tensor.

1. INTRODUCTION

We consider a $2n$-dimensional real manifold of differentiability class C^{r+1}. Let there be defined in M_{2n}, a vector valued linear function F, such that

$$\bar{\bar{X}} + X = 0 \qquad (1\cdot1)$$

for arbitrary vector field X, where

$$\bar{\bar{X}} \stackrel{\text{def}}{=} F X. \qquad (1\cdot2)$$

Then F is said to give an almost complex structure to M_{2n} and M_{2n} is called an almost complex manifold (Yano 1965).

Agreement 1·1—All the equations which follow hold for arbitrary vector fields $X, Y, Z, \ldots$, etc.

Let the almost complex manifold M_{2n} be also endowed with the Hermitian metric tensor g, that is

$$g(\bar{X}, \bar{Y}) = g(X, Y). \qquad (1\cdot3)$$

Then M_{2n} is called an almost Hermite manifold.

If we put

$$'F(X, Y) \stackrel{\text{def}}{=} g(\bar{X}, Y), \qquad (1\cdot4)$$

then the following hold :

$$'F(X, Y) + 'F(Y, X) = 0 \qquad (1\cdot5)$$

$$'F(\bar{X}, Y) + 'F(X, \bar{Y}) = 0 \qquad (1\cdot6)$$

$$'F(\bar{X}, \bar{Y}) - 'F(X, Y) = 0. \qquad (1\cdot7)$$

In an almost Hermite manifold, if

$$(\nabla F)(X, Y) = (D_Y F) X = 0 \qquad (1\cdot8)$$

where D is the Riemannian connexion in M_{2n}, then M_{2n} is called a Kähler manifold.

Let K be the curvature tensor of M_{2n}, given by

$$K(X, Y, Z) = D_X D_Y Z - D_Y D_X Z - D_{[X, Y]} Z. \quad (1 \cdot 9)$$

Let

$$'K(X, Y, Z, T) \stackrel{\text{def}}{=} g(K(X, Y, Z), T). \quad (1 \cdot 10)$$

Let Ric be the Ricci tensor of M_{2n}, given by

$$\text{Ric}(Y, Z) = (C_1^1 K)(Y, Z) \quad (1 \cdot 11)$$

where $(C_1^1 K)(Y, Z)$ is obtained by the contraction of $K(X, Y, Z)$ with respect to the first slot. Let us further put

for Ricci tensor, we also have

$$\text{Ric}(Y, Z) = g(R\, Y), Z); \; (C_1^1 R) \stackrel{\text{def}}{=} r, \quad (1 \cdot 12)$$

where r is the scalar curvature.

In the Kähler manifold, the follwoing equations hold :

$$'K(\bar{X}, \bar{Y}, Z, T) = 'K(X, Y, Z, T) = 'K(X, Y, \bar{Z}, \bar{T}), \quad (1 \cdot 13)$$

$$\text{Ric}(\bar{X}, \bar{Y}) = \text{Ric}(X, Y), \quad (1 \cdot 14)$$

$$\text{Ric}(\bar{X}, Y) = -\text{Ric}(X, \bar{Y}). \quad (1 \cdot 15)$$

H-conharmonic curvature tensor S of the Kähler manifold is defined by (Sinha 1972).

$$S(X, Y, Z) = K(X, Y, Z) + \frac{1}{2(n+2)} \Big[\text{Ric}(X, Z)\, Y - \text{Ric}(Y, Z)\, X + \text{Ric}(\bar{X}, Z)\, \bar{Y} - \text{Ric}(\bar{Y}, Z)\, \bar{X} + 2\,\text{Ric}(\bar{X}, Y)\, \bar{Z} + g(Z, X)\, R\, Y - g(Y, Z)\, R\, X + g(\bar{X}, Z)\, R\, \bar{Y} - g(\bar{Y}, Z)\, R\, \bar{X} + 2\, g(\bar{X}, Y) \Big].\, R\, \bar{Z} \quad (1 \cdot 16)$$

and

$$'S(X, Y, Z, T) \stackrel{\text{def}}{=} g(S(X, Y, Z), T). \quad (1 \cdot 17)$$

H-projective curvature tensor P of the Kähler manifold M_{2n} is given by (Mishra 1970).

$$P(X, Y, Z) = K(X, Y, Z) - \frac{1}{2(n+1)} \Big[\text{Ric}(Y, Z)\, X - \text{Ric}(X, Z)\, Y - \text{Ric}(Y, \bar{Z})\, \bar{X} + \text{Ric}(X, \bar{Z})\, \bar{Y} + 2\,\text{Ric}(X, \bar{Y})\, \bar{Z} \Big]. \quad (1 \cdot 18)$$

and

$$'P(X, Y, Z, T) \overset{\text{def}}{=\!=} g(P(X, Y, Z), T). \quad \ldots \quad (1{\cdot}19)$$

Bochner curvature tensor B of the Kähler manifold is given by (Tachibana, 1967)

$$B(X, Y, Z) = K(X, Y, Z) + \frac{1}{2(n+2)}\Big[\operatorname{Ric}(X, Z)\,Y - \operatorname{Ric}(Y, Z)\,X + (X, Z)\,R\,Y - g(Y, Z)\,Y\,X + \operatorname{Ric}(\bar{X}, Z)\,\bar{Y} - \operatorname{Ric}(\bar{Y}, Z)\,\bar{X} + {'F}(X, Z)\,R\,\bar{Y} - {'F}(Y, Z)\,R\,\bar{X} + 2\operatorname{Ric}(\bar{X}, Y)\,\bar{Z} + 2'F(X, Y)\,R\,\bar{Z}\Big] - \frac{r}{4(n+1)(n+2)}\Big[g(X, Z)\,Y - g(Y, Z)\,X + {'F}(X, Z)\,\bar{Y} - {'F}(Y, Z)\,\bar{X} + 2'F(X, Y)\,\bar{Z}\Big] \quad \ldots \quad (1{\cdot}20)$$

and

$$'B(X, Y, Z, T) \overset{\text{def}}{=\!=} g(B(X, Y, Z), T). \quad \ldots \quad (1{\cdot}21)$$

Let W, C, L and V be projective, conformal, conharmonic and concircular curvature tensors respectively, given by (Mishra 1965)

$$W(X, Y, Z) = K(X, Y, Z) - \frac{1}{2n-1}\Big[\operatorname{Ric}(Y, Z)\,X - \operatorname{Ric}(X, Z)\,Y\Big] \quad (1{\cdot}22)$$

$$C(X, Y, Z) = K(X, Y, Z) - \frac{1}{2(n-1)}\Big[\operatorname{Ric}(Y, Z)\,X - \operatorname{Ric}(X, Z)\,Y - g(X, Z)\,R\,Y + g(Y, Z)\,R\,X\Big] + \frac{r}{2(n-1)(2n-1)}\Big[g(Y, Z)\,X - g(X, Z)\,Y\Big] \quad (1{\cdot}23)$$

$$L(X, Y, Z) = K(X, Y, Z) - \frac{1}{2(n-1)}\Big[\operatorname{Ric}(Y, Z)\,X - \operatorname{Ric}(X, Z)\,Y - g(X, Z)\,R\,Y + g(Y, Z)\,R\,X\Big] \quad \ldots \quad (1{\cdot}24)$$

$$V(X, Y, Z) = K(X, Y, Z) - \frac{r}{2n(2n-1)}\Big[g(Y, Z)\,X - g(X, Z)\,Y\Big]. \quad (1{\cdot}25)$$

Let

$$'W(X, Y, Z, T) \stackrel{\text{def}}{=\!=} g(W(X, Y, Z), T) \quad \ldots \quad (1\cdot 26a)$$

$$'C(X, Y, Z, T) \stackrel{\text{def}}{=\!=} g(C(X, Y, Z), T) \quad \ldots \quad (1\cdot 26b)$$

$$'L(X, Y, Z, T) \stackrel{\text{def}}{=\!=} g(L(X, Y, Z), T) \quad \ldots \quad (1\cdot 26c)$$

$$'V(X, Y, Z, T) \stackrel{\text{def}}{=\!=} g(V(X, Y, Z), T). \quad \ldots \quad (1\cdot 26d)$$

2. Multi-recurrent Kähler Manifolds

In this section we shall define multi-recurrence in Kähler manifold M_{2n} analogous to recurrence and birecurrence in the same manifold as defined by Mishra (1970) and Mishra and Pandey (1975). The manifold will be called multi-recurrent Kähler manifold denoted by the same symbol M_{2n}.

Definition 2·1—The Kähler manifold is said to be a Multi-recurrent 'Kähler manifold M_{2n}, if

$$(\nabla \ldots \nabla \nabla K)(X, Y, Z, U_1, U_2, \ldots, U_t) = A(U_1, U_2, \ldots, U_t)\, K(X, Y, Z), \quad \ldots \quad (2\cdot 1)$$

where A is non-vanishing $\overset{\infty}{C}$ 1 – form and $t \leqslant r + 1$, the differentiability class of the manifold.

The manifold M_{2n} is said to be Ricci multi-recurrent, if

$$(\nabla \ldots \nabla \nabla \text{ Ric})(Y, Z, U_1, U_2, \ldots U_t) = A(U_1, U_2, \ldots, U_t) \text{ Ric}(Y, Z). \quad \ldots \quad (2\cdot 2)$$

From (1·11) and (2·2), we have the following :

$$(\nabla \ldots \nabla \nabla R)(Z, U_1, U_2, \ldots, U_t) = A(U_1, U_2, \ldots, U_t)\, R Z(Z), \quad \ldots \quad (2\cdot 3)$$

$$(\nabla \ldots \nabla \nabla r)(U_1, U_2, \ldots, U_t) = A(U_1, U_2, \ldots, U_t)\, r. \quad (2\cdot 4)$$

Here we note that (2·2) is obtainable from (2·1). Hence if the manifold is multi-recurrent then it is implied that it is Ricci multi-recurrent also, for the same recurrence parameter.

Let Q, a vector-valued trilinear function, be any one of the curvature tensors S, P, B, W, C, L or V, then M_{2n} is said to be Q-multi-recurrent, if

$$(\nabla \ldots \nabla \nabla Q)(X, Y, Z, U_1, U_2, \ldots, U_t) = A(U_1, U_2, \ldots, U_t)\, Q(X, Y, Z). \quad (2\cdot 5)$$

We know (Mishra 1965) that the manifold is said to be symmetric, if

$$(\nabla K)(X, Y, Z, U) = 0 \quad \ldots \quad (2\cdot 6a)$$

which gives

$$(\nabla \ldots \nabla \nabla\ K)\ (X, Y, Z, U_1, U_2, \ldots, U_t) = 0. \qquad (2\cdot6b)$$

It is said to be Ricci-symmetric, if

$$(\nabla\ \mathrm{Ric})\ (Y, Z, U) = 0, \qquad (2\cdot7a)$$

from which we have the following :

$$(\nabla \ldots \nabla \nabla\ \mathrm{Ric})\ (Y, Z, U_1, U_2, \ldots, U_t) = 0 \qquad (2\cdot7b)$$

$$(\nabla \ldots \nabla \nabla\ R)\ (Z, U_1, U_2, \ldots U_t) = 0 \qquad (2\cdot7c)$$

$$(\nabla \ldots \nabla \nabla\ r)\ (U_1, U_2, \ldots U_t) = 0. \qquad (2\cdot7d)$$

Generally, it is said to be Q-symmetric, if

$$(\nabla\ Q)\ (X, Y, Z, U) = 0 \qquad (2\cdot8a)$$

which gives

$$(\nabla \ldots \nabla \nabla\ Q)\ (X, Y, Z, U_1, U_2, \ldots U_t) = 0. \qquad (2\cdot8b)$$

3. H-Conharmonic Curvature Tensor

Various properties of H-conharmonic curvature tensor have been studied by Sinha (1972). Here, we will mainly study its multi-recurrence properties.

Theorem 3·1—If the manifold is H-conharmonic multi-recurrent, we have

$$\begin{aligned}
&A\ (U_1, U_2, \ldots U_t)\ S\ (X, Y, Z) \\
&\quad = (\nabla \ldots \nabla \nabla\ K)\ (X, Y, Z, U_1, U_2, \ldots, U_t) \\
&\quad +[(\nabla \ldots \nabla \nabla\ \mathrm{Ric})\ (X, Z, U_1, U_2, \ldots, U_t)\ Y \\
&\quad - (\nabla \ldots \nabla \nabla\ \mathrm{Ric})\ (Y, Z, U_1, U_2, \ldots, U_t)\ X \\
&\quad + (\nabla \ldots \nabla \nabla\ \mathrm{Ric})\ (\bar{X}, Z, U_1, U_2, \ldots, U_t)\ \bar{Y} \\
&\quad - (\nabla \ldots \nabla \nabla\ \mathrm{Ric})\ (\bar{Y}, Z, U_1, U_2, \ldots, U_t)\ \bar{X} \\
&\quad +2(\nabla \ldots \nabla \nabla\ \mathrm{Ric})\ (\bar{X}, Y, U_1, U_2, \ldots, U_t)\ \bar{Z} \\
&\quad + (\nabla \ldots \nabla \nabla\ R)\ (Y, U_1, U_2, \ldots, U_t)\ g\ (Z, X) \\
&\quad - (\nabla \ldots \nabla \nabla\ R)\ (X, U_1, U_2, \ldots, U_t)\ g\ (Z, Y) \\
&\quad + (\nabla \ldots \nabla \nabla\ R)\ (\bar{Y}, U_1, U_2, \ldots, U_t)\ g\ (\bar{X}, Z) \\
&\quad - (\nabla \ldots \nabla \nabla\ R)\ (\bar{X}, U_1, U_2, \ldots, U_t)\ g\ (\bar{Y}, Z) \\
&\quad +2(\nabla \ldots \nabla \nabla\ R)\ (Z, U_1, U_2, \ldots, U_t)\ g\ (\bar{X}, Y)]/2\ (n+2). \qquad (3\cdot1)
\end{aligned}$$

Proof : Since the manifold is H-conharmonic multi-recurrent, substituting S for the general curvature tensor Q in (2·5). we have

$$(\nabla \cdots \nabla \nabla S)(X, Y, Z, U_1, U_2, \ldots, U_t) = A(U_1, U_2, \ldots, U_t)\, S(X, Y, Z). \quad \ldots \quad (3\cdot 2)$$

Differentiating (1·16) t-times covariantly, we get

$$\begin{aligned}
&(\nabla \cdots \nabla \nabla\, S)(X, Y, Z, U_1, U_2, \ldots U_t) \\
&= (\nabla \cdots \nabla \nabla\, K)(X, Y, Z, U_1, U_2, \ldots, U_t) + \\
&\quad [(\nabla \cdots \nabla \nabla\, \mathrm{Ric})(X, Z, U_1, U_2, \ldots, U_t)\, Y - \\
&\quad (\nabla \cdots \nabla \nabla\, \mathrm{Ric})(Y, Z, U_1, U_2, \ldots, U_t)\, X + \\
&\quad (\nabla \cdots \nabla \nabla\, \mathrm{Ric})(\bar{X}, Z, U_1, U_2, \ldots, U_t)\, \bar{Y} - \\
&\quad (\nabla \cdots \nabla \nabla\, \mathrm{Ric})(\bar{Y}, Z, U_1, U_2, \ldots, U_t)\, \bar{X} + \\
&\quad 2(\nabla \cdots \nabla \nabla\, \mathrm{Ric})(\bar{X}, Y, U_1, U_2, \ldots, U_t)\, \bar{Z} + \\
&\quad (\nabla \cdots \nabla \nabla\, U)(Y, U_1, U_2, \ldots, U_t)\, g(Z, X) - \\
&\quad (\nabla \cdots \nabla \nabla\, R)(X, U_1, U_2, \ldots, U_t)\, g(Z, Y) + \\
&\quad (\nabla \cdots \nabla \nabla\, R)(\bar{Y}, U_1, U_2, \ldots, U_t)\, g(\bar{X}, Z) - \\
&\quad (\nabla \cdots \nabla \nabla\, R)(\bar{X}, U_1, U_2, \ldots, U_t)\, g(\bar{Y}, Z) + \\
&\quad 2(\nabla \cdots \nabla \nabla\, R)(\bar{Z}, U_1, U_2, \ldots, U_t)\, g(\bar{X}, Y)]/2(n+2). \quad \ldots \quad \ldots \quad (3\cdot 3)
\end{aligned}$$

Comparing (3·2) and (3·3), we have (3·1).

Theorem 3·2—A multi-recurrent Kähler manifold is H-conharmonic multi-recurrent.

PROOF : Since the manifold is multi-recurrent, we have from (3·2),

$$\begin{aligned}
&(\nabla \cdots \nabla \nabla S)(X, Y, Z, U_1, U_2, \ldots, U_t) \\
&\quad = A(U_1, U_2, \ldots, U_t)\, [K(X, Y, Z) + \\
&\qquad + \{\mathrm{Ric}(X, Z)\, Y - \mathrm{Ric}(Y, Z)\, X + \\
&\qquad + \mathrm{Ric}(\bar{X}, Z)\, \bar{Y} - \mathrm{Ric}(\bar{Y}, Z)\, \bar{X} + \\
&\qquad + 2\, \mathrm{Ric}(\bar{X}, Y)\, Z + g(Z, X)\, R\, Y - \\
&\qquad - g(Y, Z)\, R\, X + g(\bar{X}, Z)\, R\, \bar{Y} - \\
&\qquad - g(\bar{Y}, Z)\, R\, \bar{X} + 2\, g(\bar{X}, Y)\, R\, Z\}/2(n+2)]. \\
&\quad = A(U_1, U_2, \ldots, U_t)\, S(X, Y, Z). \quad \ldots \quad (3\cdot 4)
\end{aligned}$$

which shows that the manifold is H-conharmonic multi-recurrent.

Theorem 3·3—For the Kähler manifold, if any two of the following conditions hold, then the third also holds, for the same recurrence parameter,

(1) it is H-conharmonic multi-recurrent,

(2) it is H-projective multi-recurrent,

(3) it is Ricci-multi-recurrent.

PROOF : From eqns. (1·16), (1·18), (1·15) and (1·5), we have

$$S(X, Y, Z) = P(X, Y, Z) - \frac{1}{2(n+1)(n+2)} [\mathrm{Ric}(X, Z) Y - \mathrm{Ric}(Y, Z) X + \mathrm{Ric}(\bar{X}, Z) \bar{Y} - \mathrm{Ric}(\bar{Y}, Z) \bar{X} + 2 \mathrm{Ric}(\bar{X}, Y) \bar{Z}] + \frac{1}{2(n+2)} [g(Z, X) R Y - g(Z, Y) R X + F(X, Z) R \bar{Y} - F(Y, Z) R \bar{X} - 2 'F(X, Y) R \bar{Z}]. \quad \ldots \quad \ldots \quad (3\cdot5)$$

Hence, we have

$$(\nabla \ldots \Delta \Delta S)(X, Y, Z, U_1, U_2, \ldots, U_t)$$
$$= (\nabla \ldots \nabla \nabla P)(X, Y, Z, U_1, U_2, \ldots, U_t)$$
$$- \frac{1}{2(n+1)(n+2)} [(\nabla \ldots \nabla \nabla \mathrm{Ric})(X, Z, U_1, D_2, \ldots, U_t) Y -$$
$$- (\nabla \ldots \nabla \nabla \mathrm{Ric})(Y, Z, U_1, U_2, \ldots, U_t) X$$
$$+ (\nabla \ldots \nabla \nabla \mathrm{Ric})(\bar{X}, Z, U_1, U_2, \ldots, U_l) \bar{Y}$$
$$- (\nabla \ldots \nabla \nabla \mathrm{Ric})(\bar{Y}, Z, U_1, U_2, \ldots, U_t) \bar{X}$$
$$+ 2(\nabla \ldots \nabla \nabla \mathrm{Ric})(\bar{X}, Y, U_1, U_2, \ldots, U_t) \bar{Z}]$$
$$+ \frac{1}{2(n+2)} [(\nabla \ldots \nabla \nabla R)(Y, U_1, U_2, \ldots, U_t) g(Z, X)$$
$$- (\nabla \ldots \nabla \nabla R)(X, U_1, U_2), \ldots, U_t) g(Z, Y)$$
$$+ (\nabla \ldots \nabla \nabla R)(\bar{Y}, U_1, U_2, \ldots, U_t)' F(X, Z)$$
$$- (\nabla \ldots \nabla \nabla R)(\bar{X}, U_1, U_2, \ldots, U_t)' F(Y, Z)$$
$$+ 2(\nabla \ldots \nabla \nabla R)(Z, U_1, U_2, \ldots, U_t)' F(X, Y)]. \quad \ldots \quad (3\cdot6)$$

Multiplying (3·5) by $A(U_1, U_2, \ldots, U_t)$ throughout, subtracting the resulting equation from (3·6) and then using the fact that the manifold is *H*-conharmonic multi-recurrent and *H*-projective multi-recurrent, we have

$$(\nabla \ldots \nabla \nabla \mathrm{Ric})(Y, Z, U_1, U_2, \ldots, U_t) = A(U_1, U_2, \ldots, U_t) \mathrm{Ric}(Y, Z)$$

for arbitrary vector fields X, Y, $Z, \ldots$, etc. Similarly, we can prove the remaining two cases also.

Theorem 3·4—For the Kähler manifold, if any two of the following conditions hold, the third also holds for the same recurrence parameter :

(1) it is H-conharmonic multi-recurrent,

(2) it is Bochner multi-recurrents,

(3) it is Ricci multi-recurrent.

PROOF : From (1·16) and (1·20), we have

$$S(X,Y,Z) = B(X,Y,Z) + \frac{r}{4(n+1)(n+2)}[g(X,Z)Y - g(Y,Z)X + 'F(X,Z)\bar{Y} - 'F(Y,Z)\bar{X} + 2'F(X,Y)Z].$$

The remaining part follows the pattern of the Theorem 3·3

Theorem 3·5—For the Kähler manifold, if any two of the following conditions hold, the third also holds for the same recurrence parameter :

(1) it is H-conharmonic multi-recurrent,

(2) it is projective multi-recurrent,

(3) it is Ricci multi-recurrent.

PROOF : In the Kähler manifold, we have

$$S(X,Y,Z) = W(X,Y,Z) - \frac{5}{2(2n-1)(n+2)}[\mathrm{Ric}(X,Z)Y - \mathrm{Ric}(Y,Z)X] + \frac{1}{2(n+2)}\mathrm{Ric}(X,Z)\bar{Y} - \mathrm{Ric}(\bar{Y},Z)X + 2\,\mathrm{Ric}(X,Y)Z + g(Z,X)RY - g(Y,Z)R\bar{Y} + g(X,Z) - g(\bar{Y},Z)R\,X + 2g(X,Y)RZ.$$

The remaining part is obvious.

Theorem 3·6—For the Kähler manifold, if any two of the following conditions hold, the third also holds for the same recurrence parameter:

(1) it is H-conharmonic multi-recurrent,

(2) it is conformal multi-recurrent,

(3) it is Ricci multi-recurrent.

PROOF : We have

$$S(X,Y,Z) = C(X,Y,Z) - \frac{3}{2(n-1)(n+2)}[\mathrm{Ric}(X,Z)Y - \mathrm{Ric}(Y,Z)X + g(X,Z)RY - g(Y,Z)RX)]$$

$$+\frac{1}{2(n+2)}[\mathrm{Ric}\ X, Z)\bar{Y}-\mathrm{Ric}(\bar{Y}, Z)X+2\,\mathrm{Ric}(X, Y)Z+$$
$$g(X, Z)R\bar{Y}-RXg(\bar{Y}, Z)+2RZg(X, Y)-\frac{r}{2(n-1)(2n-1)}$$
$$[g(Y.Z)X-g(X, Z)Y].$$

The remaining part follows the pattern of the theorem (3·3).

Theorem 3·7—For the Kähler manifold, if any two of the following conditions hold, the third also holds for the same recurrence parameter:

(1) it is *H*-conharmonic multi-recurrent,

(2) it is conharmonic multi-recurrent,

(3) it Ricci multi-recurrent.

Proof : We have in a Kähler manifold,

$$S(X, Y, Z)=L(X. Y. Z)-\frac{3}{2(n-1)(n+2)}\Big[\mathrm{Ric}(X, Z)Y$$
$$-\mathrm{Ric}(Y, Z)X+g(X, Z)RY-g(Y, Z)RX\Big]$$
$$+\frac{1}{2(n+2)}\Big[\mathrm{Ric}(\bar{X}, Z)\bar{Y}-\mathrm{Ric}(\bar{Y}, Z)\bar{X}+2\,\mathrm{Ric}(\bar{X}, Y)Z$$
$$+g(\bar{X}, Z)-g(\bar{Y}, Z)R\bar{Y}+2g(X, \bar{X}, Y)R\bar{X}\Big].$$

The rest part follows the pattern of the previous theorems.

Theorem 3·8—For the Kähler manifold, if any two of the following conditions hold, the third also holds, for the same recurrence parameter:

(1) it is *H*-conharmonic multi-recurrent,

(2) it is concircular multi-recurrent,

(3) it is Ricci multi-recurrent.

Proof : We have

$$S(X,Y,Z)=V(X,Y,Z)+\frac{1}{2(n+2)}\Big[\mathrm{Ric}(X, Z)Y-\mathrm{Ric}(Y, Z)X+$$
$$\mathrm{Ric}(\bar{X}, Z)\bar{Y}-\mathrm{Ric}(\bar{Y}, Z)\bar{X}+2\,\mathrm{Ric}(\bar{X}, Y)Z+g(Z, X)RY$$
$$-g(Y, Z)RX+g(X, Z)R\bar{Y}-g(\bar{Y}, Z)R\bar{\bar{X}}$$
$$+2g(X, Y)RZ\Big]$$
$$+\frac{r}{2n(2n-1)}\Big[g(Y, Z)X-g(X, Z)Y\Big].$$

The remaining part is obvious.

Theorem 3·9—A symmetric M_{2n} is H-conharmonic symmetric.

PROOF: If the manifold is symmetric then it is implied that it is Ricci symmetric also. Using this fact in the equation (3·3), we have the statement.

Theorem 3·10—If the multi-recurrent Kähler manifold is H-conharmonic symmetric and H-projective symmetric, then it is Ricci symmetric, for the same recurrence parameter.

PROOF : Using eqns. (2·6) and (2·7) in eqn. (3.6), we have the statement.

Theorem 3·11—If the multi-recurrent Kähler manifold is H-conharmonic symmetric and Bochner symmetric or projective symmetric or conformal symmetric or conharmonic symmetric or concircular symmetric, then it is Ricci symmetric for the same recurrence parameter.

PROOF : The statement follows the pattern of the proof of the Theorem 3·10.

Note : We can state the Theorems 3·10 and 3·11 in a more general way as follows :

In the multi-recurrent Kähler manifold, if any two of the following conditions hold, then the third also holds for the same recurrence parameter :

(1) it is H-conharmonic symmetric,

(2) it is H-projective or Bochner or projective or conformal or conharmonic or concircular symmetric,

(3) it is Ricci symmetric.

4. RECURRENCE PARAMETER

In this section we establish some relations amongst recurrence parameters.

We know (Mishra 1965) that the manifold is said to be recurrent, if

$$(\nabla K)(X, Y, Z, S) = a(S)\, K(X, Y, Z) \qquad (4\cdot1)$$

and it is said to be birecurrent (Mishra and Pandey 1975), if

$$(\nabla\nabla K)(X, Y, Z, S, T) = A'(S, T)\, K(X, Y, Z). \quad \ldots \quad \ldots \qquad (4\cdot2)$$

We (Mishra and Pandey 1975) have seen that

$$A'(S, T) = (\nabla a)(S, T) + a(S)\, a(T). \quad \ldots \quad \ldots \qquad (4\cdot3)$$

For a tri-recurrent manifold, we have

$$(\nabla\nabla\nabla K)(X, Y, Z, S, T, U) = A''(S, T, U)\, K(X, Y, Z). \quad \ldots \qquad (4\cdot4)$$

Differentiating (4·2) covariantly and using (4·2) and (4·1) in the resulting equation, we have

$$(\nabla\nabla\nabla K)(X, Y, Z, S, T, U) = [(\nabla A'(S, T, U) + A'(S, T)\, a(U)] \, K(X, Y, Z). \quad \ldots \quad (4·5)$$

From (4·4) and (4·5), we have

$$A''(S, T, U) = (\nabla A')(S, T, U) + A'(S, T)\, a(U). \quad \ldots \quad (4·6)$$

Hence taking account of the equations (4·3) and (4·6), we have in a Kähler manifold.

Theorem 4·1—For the multi-recurrence parameter, $A(U_1, U_2, —, U_t) = (\nabla A^*)(U_1, U_2, \ldots, U_{t-1}, U_t) + A^*(U_1, U_2 \ldots, U_{t-2}, U_{t-1})\, a(U_t)$. .. (4·7)
where

$$(\nabla \ldots \nabla\nabla K)(X, Y, Z, U_1, U_2, \ldots, U_{t-1}) = A^*(U_1, U_2, \ldots, U_{t-1})\, K(X, Y, Z).$$

The proof is obvious.

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