OPERATIONAL TECHNIQUES WITH CERTAIN DOUBLE GENERALIZED TRANSFORM

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In this paper, we present a double generalized transform of the Meijer's G-function and show how this transformation would lead to interesting operational techniques for augmenting parameters of the ${}_pF_q$ function.

1. Introduction

The single and double Euler transformations of the hypergeometric function ${}_pF_q$, are effective tools for augmenting its parameters. These transformations are given by Rainville (1965, p. 104) and Abdul Halim and Al-Salam (1963) respectively. A single application of the Laplace transformation (or its inverse) can augment only one parameter in the numerator (or denominator). Since it is readily seen that (cf., e.g. Erdelyi 1954, p. 219).

$$\int_0^\infty e^{-t}\, t^{\lambda-1}\, {}_pF_q\left[\begin{matrix} a_1,\ldots,a_p\,; \\ b_1,\ldots,b_q\,; \end{matrix}\; zt\right] dt = \Gamma(\lambda)\, {}_{p+1}F_q\left[\begin{matrix} \lambda,\, a_1,\ldots,a_p\,; \\ b_1,\ldots,b_q\,; \end{matrix}\; z\right] \qquad \ldots \quad \ldots \quad (1\cdot1)$$

where Re $(\lambda) > 0$, $p \leqslant q$, the equality holds if Re $(z) < 1$; and (Erdelyi 1954, p. 297)

$$\frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty} e^{t}\, t^{-\lambda}\, {}_pF_q\left[\begin{matrix} a_1,\ldots,a_p; \\ b_1,\ldots,b_q; \end{matrix}\; \frac{z}{t}\right] dt = \frac{1}{\Gamma(\lambda)}\, {}_pF_{q+1}\left[\begin{matrix} a_1,\ldots,a_p; \\ \lambda,\, b_1,\ldots,b_q; \end{matrix}\; z\right] \qquad \ldots \quad \ldots \quad (1\cdot2)$$

where $p \leqslant q+1$ and Re $(\lambda) > 0$.

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Recently Srivastava and Panda (1973, p. 309) have discussed a double Meijer transform of the generalized hypergeometric function in the form

$$\int_0^\infty \int_0^\infty x^{\alpha-1} y^{\beta-1} (x+y)^\sigma K_\mu [\lambda(x+y)] K_\nu [\lambda(x+y)]$$

$${}_pF_q\left[\begin{matrix} a_1,\ldots, a_p ; \\ b_1,\ldots, b_q ; \end{matrix} \; tx^{2s} y^{2k} (x+y)^{2r} \right] dxdy$$

$$= \frac{2^{\alpha+\beta+\sigma-3}\, \Gamma[\frac{1}{2}(\alpha+\beta \pm\mu\pm\nu+\sigma)]}{\lambda^{\alpha+\beta+\sigma}\,\Gamma(\alpha+\beta+\sigma)} B(\alpha,\beta) \times$$

$$\times {}_{p+6s+6k+4r}F_{q+4s+4k+2r}\left[\begin{matrix} a_1,\ldots, a_p, \triangle(2s,\alpha), \triangle(2k,\beta), \\ b_1,\ldots, b_q, \triangle(2s+2k, \alpha+\beta), \end{matrix}\right.$$

$$\left.\begin{matrix} \triangle[s+k+r, \frac{1}{2}(\alpha+\beta\pm\mu\pm\nu+\sigma)]; \\ \triangle[2s+2k+2r, \alpha+\beta+\sigma]; \end{matrix} \; \eta\eta' t\right], \quad \ldots \quad (1\cdot3)$$

where r, s, k, are non-negative integers; Re $(\alpha) > 0$, Re $(\beta) > 0$, Re $(\lambda) > 0$, Re $(\alpha+\beta\pm\mu\pm\nu+\sigma) > 0$; $\triangle(s, a)$ stands for a set of 's' parameters, $\frac{a}{s}, \frac{a+1}{s}, \ldots, \frac{a+s-1}{s}$; and $\eta = \frac{(2s)^{2s}(2k)^{2k}}{(2s+2k)^{2(s+k)}}$, $\eta' = \left(\frac{s+k+r}{\lambda}\right)^{2(s+k+r)}$

Here we present a double generalized transform of the Meijer's G-function and show how this transformation would lead to interesting operational techniques for augmenting parameters of the ${}_pF_q$ function. Indeed, we first obtain the double integral transformation

$$\int_0^\infty \int_0^\infty x^{\alpha-1} y^{\beta-1} (x+y)^\sigma \; G^{m,n}_{p,q}\left[(x+y) \left| \begin{matrix} a_1,\ldots, a_p \\ b_1,\ldots, bq \end{matrix}\right.\right] \times$$

$${}_PF_Q\left[\begin{matrix} \alpha_1,\ldots, \alpha_P; \\ \beta_1,\ldots, \beta_Q; \end{matrix} \; \delta x^{2s} y^{2k} (x+y)^{2r}\right] dxdy$$

$$= B(\alpha,\beta) . \frac{\prod_{j=1}^{m} \Gamma(B_j) \prod_{j=1}^{n} \Gamma(1-A_j)}{\prod_{j=m+1}^{q} \Gamma(1-B_j) \prod_{j=n+1}^{p} \Gamma(A_j)} \times$$

$${}_{P+(2s+2k)(q+1)+2rq}F_{Q+(2s+2k)(p+1)+2rp}\left[\begin{matrix} \alpha_1,\ldots, \alpha_P, \\ \beta_1,\ldots, \beta_Q, \end{matrix}\right.$$

$$\left.\begin{matrix} \triangle(2s,\alpha), \triangle(2k,\beta), \triangle(2r+2s+2k, B_1),\ldots, \triangle(2r+2s+2k, B_q); \\ \triangle(2s+2k, \alpha+\beta), \triangle(2r+2s+2k, A_1),\ldots, \triangle(2r+2s+2k, A_p); \end{matrix} \; \delta\eta\eta'\right] \quad (1\cdot4)$$

where r, s, k are non-negative integers, Re $(\alpha) > 0$, Re $(\beta) > 0$, Re $(1 - a_j - \alpha - \beta - \sigma) > 0$, Re $(1 - b_j - \alpha - \beta - \sigma) > 0$; and

$$A_j = a_j + \alpha + \beta + \sigma \; ; \qquad j = 1, \ldots, p.$$
$$B_j = b_j + \alpha + \beta + \sigma \; ; \qquad j = 1, \ldots, q.$$

and

$$\eta = \frac{(2s)^{2s}\,(2k)^{2k}}{(2s+2k)^{2(s+k)}}, \; \eta' = \frac{(2r+2s+2k)^{(2r+2s+2k)\,(q-p)}}{(-1)^{(2r+2s+2k)\,(q-m-n)}}.$$

2. Derivation of Formula (1·4)

By appealing to the familiar result (Erdelyi 1953, p. 177)

$$\int_0^\infty \int_0^\infty \psi(x+y)\, x^{\alpha-1}\, y^{\beta-1}\, dxdy$$
$$= B(\alpha, \beta) \int_0^\infty \psi(z)\, z^{\alpha+\beta-1}\, dz, \quad \ldots \quad \ldots \qquad (2\cdot1)$$

Re $(\alpha) > 0$, Re $(\beta) > 0$.

the left-hand side of (1·4) within the domain of convergence, after changing the order of integration and summation, equals to

$$\sum_{i=0}^{\infty} \frac{\prod_{j=1}^{P} (\alpha_j)_i}{\prod_{j=1}^{Q} (\beta_j)_i} \cdot \delta^i \cdot \frac{\Gamma(2si+\alpha)\;\Gamma(2ki+\beta)}{\Gamma(2si+2ki+\alpha+\beta)}$$

$$\times \int_0^\infty z^{2ri+2si+2ki+\sigma+\alpha+\beta-1}\, G^{m,n}_{p,q}\left[z \,\middle|\, \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} \right] dxdy, \qquad (2\cdot2)$$

now by recalling the definition of the Mellin transform $F(s)$ of a function $f(x)$ as

$$F(S) = \mathcal{M}\left\{ f(x) \; ; \; S \right\} = \int_0^\infty x^{s-1} f(x)\, dx,$$

where s is a complex number; we have (Erdelyi 1954, p. 337)

$$\mathcal{M}\left\{ G^{m,n}_{p,q}\left[x \,\middle|\, \begin{matrix} a_p \\ b_q \end{matrix} \right] : S \right\} = \frac{\prod_{j=1}^{m} \Gamma(b_j+s) \prod_{j=1}^{n} \Gamma(1-a_j-s)}{\prod_{j=m+1}^{q} \Gamma(1-b_j-s) \prod_{j=n+1}^{p} \Gamma(a_j+s)} \qquad (2\cdot3)$$

provided

$$-\min_{1\leqslant j\leqslant m}\text{Re } b_j < \text{Re } s < 1 - \max_{1\leqslant j\leqslant m}\text{Re } a_j$$

if we make use of (2·3) in (2·2), we get the right-hand side of (1·4).

3. Integral Operators

In terms of the linear operator

$$O^{a_i, b_i}_{\alpha,\beta,\sigma}\left\{\quad\right\} = \left[\frac{B(\alpha,\beta)\prod_{j=1}^{m}\Gamma(b_j+\alpha+\beta+\sigma)\prod_{j=1}^{n}\Gamma(1-a_j-\alpha-\beta-\sigma)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-\alpha-\beta-\sigma)\prod_{j=n+1}^{p}\Gamma(a_j+\alpha+\beta+\sigma)}\right]^{-1}\times$$

$$\int_0^\infty\int_0^\infty u^{\alpha-1}\, v^{\beta-1}\,(u+v)^\sigma\, G^{m,n}_{p,q}\left[(u+v)\,\Big|\,\begin{matrix}a_p\\ b_q\end{matrix}\right]\left\{\quad\right\} du dv \tag{3·1}$$

formula (1·4) with $r = 0$ yields

$$O^{a_i, b_i}_{\alpha,\beta,\sigma}\left\{{}_PF_Q\left[\begin{matrix}\alpha_1,\ldots,\alpha_P;\\ \beta_1,\ldots,\beta_Q;\end{matrix}\ \delta\, u^{2s}\, v^{2k}\right]\right\}$$

$$= {}_{P+(2s+2k)(q+1)}F_{Q+(2s+2k)(p+1)}\left[\begin{matrix}\alpha_1,\ldots,\alpha_P,\ \triangle(2s,\alpha),\\ \beta_1,\ldots,\beta_Q),\ \triangle(2s+2k,\alpha+\beta),\end{matrix}\right.$$

$$\left.\begin{matrix}\triangle(2k,\beta),\ \triangle(2s+2k;B_1),\ldots,\triangle(2s+2k,B_q);\\ \triangle(2s+2k,A_1),\ldots,\triangle(2s+2k,Ap);\end{matrix}\ \delta T\right] \quad \ldots \quad \ldots \tag{3·2}$$

where

$$T = \frac{(2s)^{2s}\,(2k)^{2k}\,(2s+2k)^{(2s+2k)(q-p-1)}}{(-1)^{(2s+2k)(q-m-n)}} \quad \ldots \quad \ldots \quad \ldots \tag{3·3}$$

s, k are non-negative integers, and Re $(\alpha) > 0$, Re $(\beta) > 0$, etc..

We also notice that

$$O^{a_i, b_i}_{\alpha,\beta,\sigma}\left\{1\right\} = 1. \quad \ldots \quad \ldots \quad \ldots \tag{3·4}$$

then in general

$$O^{a_i, b_i}_{\alpha,\beta,\sigma}\left\{u^M\, v^N\,(u+v)^L\right\}$$

$$= \frac{(\alpha)_M\ (\beta)_N}{(\alpha+\beta)_{M+N}} \frac{\prod_{j=1}^{q} (b_j+\alpha+\beta+\sigma)_{M+N+L}}{\prod_{j=1}^{p} (a_j+\alpha+\beta+\sigma)_{M+N+L}} (-1)^{(M+N+L)(m+n-q)} \quad .. \quad (3{\cdot}5)$$

provided M, N, L are non-negative integers, such that $(M+N+L)$ is an even integer, and $0 \leqslant m \leqslant q$, $0 \leqslant n \leqslant p$, $p+q < 2\ (m+n)$ and particularly :

$$O_{\alpha,\beta,\sigma}^{a_i,\, b_i} \left\{ u^N\, v^N \right\}$$

$$= \frac{(\alpha)_N\ (\beta)_N}{(\alpha+\beta)_{2N}} \frac{\prod_{j=1}^{q} (b_j+\alpha+\beta+\sigma)_{2N}}{\prod_{j=1}^{p} (a_j+\alpha+\beta+\sigma)_{2N}} . \quad .. \quad .. \quad (3{\cdot}6)$$

4. Applications

In this section we apply the above operational relationships, to obtain linear, bilinear and bilateral generating functions, and certain reduction formulae.

Some Generating Relations

In the Bateman's generating function [6, p. 256]

$$_0F_1\left[- ;\ 1+\alpha\ ;\ \tfrac{1}{2}(x-1)\,t \right]\ _0F_1\left[- ;\ 1+\beta\ ;\ \tfrac{1}{2}(x+1)\,t \right]$$

$$\sum_{n=0}^{\infty} \frac{t^n}{(1+\alpha)_n\,(1+\beta)_n}\ P_n^{(\alpha,\,\beta)}(x) \quad .. \quad .. \quad (4{\cdot}1)$$

if we replace t by tuv, taking u and v as variables and operating both sides by

$$O_{a,\, b,\, -a-b}^{a_i,\, b_i}\ , \quad .. \quad .. \quad .. \quad (4{\cdot}2)$$

we get

$$\sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n \prod_{j=1}^{q} \left\{ (\tfrac{1}{2}\,b_j)_n\,(\tfrac{1}{2}(b_j+1))_n \right\}\ P_n^{(\alpha,\,\beta)}(x)\ t^n}{(\tfrac{1}{2}(a+b))_n\,(\tfrac{1}{2}(a+b+1))_n\,(1+\alpha)_n\,(1+\beta)_n \prod_{j=1}^{p} \left\{ (\tfrac{1}{2}\,a_j)_n (\tfrac{1}{2}(a_j+1))_n \right\}}$$

$$= F\left[\begin{array}{l} a, b, \triangle(2, b_1), .., \triangle(2, bq) : — ; — ; \\ \triangle(2, a+b), \triangle(2, a_1), .., \triangle(2, a_p) : 1+\alpha; 1+\beta ; \end{array} \frac{1}{2}(x-1)\, t, \frac{1}{2}(x+1)\, t \right] \quad .. \quad .. \quad (4{\cdot}3)$$

where $F(x, y)$ denotes a generalized Appell's function and this formula, evidently provides a generalization to the known result (Srivastava and Panda 1973)

$$\sum_{n=0}^{\infty} \frac{(a)_n\, (b)_n\, (c \pm \mu \pm \gamma)_n\, P_n^{(\alpha,\beta)}(x)\, t^n}{(1+\alpha)_n\, (1+\beta)_n\, (\frac{1}{2}(a+b))_n\, (\frac{1}{2}(a+b+1))_n\, (c)_n\, (c+\frac{1}{2})_n}$$

$$= F\left[\begin{array}{l} a, b, c \pm \mu \pm \gamma : — ; — ; \\ \triangle(2, a+b), \triangle(2, 2c) : 1+\alpha; 1+\beta ; \end{array} \frac{1}{2}(x-1)\, t, \frac{1}{2}(x+1)\, t \right]. \quad (4{\cdot}4)$$

A similar application of the operator (4·2) to the generating relation (Srivastava 1971, p. 74)

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{(-\alpha-\beta)_n} H_n^{\alpha-n, \beta-n} \left[\rho, \sigma, x\right] t^n$$

$$= F_2 \left[\lambda, -\alpha, \rho; -\alpha-\beta, \sigma ; -t, xt \right] \quad .. \quad .. \quad (4{\cdot}5)$$

involving the generalized Ricc polynomial defined by

$$H_n^{(\alpha, \beta)}\left(\zeta, p; \nu\right) = \binom{\alpha+n}{n} {}_3F_2\left[\begin{array}{l} -n, \alpha+\beta+n+1, \zeta; \nu \\ \alpha+1, p ; \end{array}\right], n \geqslant 0, \quad (4{\cdot}6)$$

would yield the formula

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n\, (a)_n\, (b)_n \prod_{j=1}^{q} \left\{(\frac{1}{2} b_j)_n\, (\frac{1}{2}(b_j+1))_n\right\}}{(-\alpha-\beta)_n\, (\frac{1}{2}(a+b))_n\, (\frac{1}{2}(a+b+1))_n \prod_{i=1}^{p} \left\{(\frac{1}{2} a_j)_n\, (\frac{1}{2}(a_j+1))_n\right\}} \times$$

$$H_n^{(\alpha-n, \beta-n)}\left[\rho, \sigma, x\right] t^n$$

$$= F\left[\begin{array}{l} \lambda, a, b, \triangle(2, b_1), .., \triangle(2, bq) : -\alpha ; -\rho ; \\ \triangle(2, a+b), \triangle(2, a_1), .., \triangle(2, a_p) : -\alpha-\beta ; \sigma ; \end{array} -t, xt \right]. \quad (4{\cdot}7)$$

We recall the familiar generating function (Rainville 1965, p. 202)

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{(1+\alpha)_n} L_n^{(\alpha)}(x)\, z^n = (1-z)^{-\lambda}\, {}_1F_1\left[\begin{array}{l} \lambda ; \\ 1+\alpha; \end{array} \frac{-xz}{1-z}\right]. \quad .. \quad (4{\cdot}8)$$

In this formula if we replace first x by xt and multiply both sides by $t^{\sigma-1}$ and take their Laplace transform using formula (1·1) in conjunction with the definition

$$L_n^{(\alpha)}(x) = \binom{\alpha+n}{n} {}_1F_1\left[\begin{matrix} -n\,; \\ 1+\alpha\,; \end{matrix}\; x\right],\; n \geqslant 0. \qquad \ldots \qquad \ldots \qquad (4\cdot9)$$

In the resulting formula involving hypergeometric function ${}_2F_1$ on both sides, we replace x by xuv and apply the operator (4·2). This evidently lead us to the result

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}\; {}_{2q+4}F_{2p+3}\left[\begin{matrix} -n, a, b, \sigma, \triangle(2, b_1), \ldots, \triangle(2, b_q)\,; x \\ 1+\alpha, \triangle(2, a+b), \triangle(2, a_1), \ldots, \triangle(2, a_p)\,; \end{matrix}\right]$$

$$= (1-z)^{-\lambda}\; {}_{2p+4}F_{2p+3}\left[\begin{matrix} \lambda, a, b, \sigma, \triangle(2, b_1), \ldots, \\ 1+\alpha, \triangle(2, a+b), \triangle(2, a_1), \ldots, \end{matrix}\right.$$

$$\left.\begin{matrix} \triangle(2, b_q)\,; \\ \triangle(2, a_p)\,; \end{matrix}\; \frac{-xz}{1-z}\right] \qquad \ldots \qquad \ldots \qquad (4\cdot10)$$

which is of the form

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}\; {}_{p+1}F_q\left[\begin{matrix} -n, \alpha_1, \ldots, \alpha_p\,; \\ \beta_1, \ldots, \beta_q\,; \end{matrix}\; x\right] z^n$$

$$= (1-z)^{-\lambda}\; {}_{p+1}F_q\left[\begin{matrix} \lambda, \alpha_1, \ldots, \alpha_p\,; \\ \beta_1, \ldots, \beta_q\,; \end{matrix}\; \frac{-xz}{1-z}\right], \qquad \ldots \qquad \ldots \qquad (4\cdot11)$$

a formula due to Chaundy (1943, p. 62). Incidentally Chaundy's result can also be established directly by the application of finite difference operators $\triangle$ and E, defined as

$$E_\alpha f(\alpha) = f(\alpha+1)$$
$$\triangle_\alpha f(\alpha) = f(\alpha) - f(\alpha+1)$$
$$\triangle_\alpha \equiv 1 - E_\alpha$$

Therefore we have

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}\; {}_{p+1}F_q\left[\begin{matrix} -n, \alpha_1+\alpha, \ldots, \alpha_p+\alpha\,; \\ \beta_1+\alpha, \ldots, \beta_q+\alpha\,; \end{matrix}\; x\right] z^n$$

$$= \frac{\prod_{i=1}^{q} \Gamma(\beta_i+\alpha)}{\prod_{i=1}^{p} \Gamma(\alpha_i+\alpha)}\, x^{-\alpha} \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}\, z^n\, (1-E_\alpha)^n\, \frac{\prod_{i=1}^{p} \Gamma(\alpha_i+\alpha)}{\prod_{i=1}^{q} \Gamma(\beta_i+\alpha)}\, x^\alpha$$

$$= \frac{\prod_{i=1}^{q} \Gamma(\beta_i + a)}{\prod_{i=1}^{p} \Gamma(a_i + a)} x^{-a} \{1 - (1 - E_a) z\}^{-\lambda} \frac{\prod_{i=1}^{p} \Gamma(a_i + a)}{\prod_{i=1}^{q} \Gamma(\beta_i + a)} x^{a}$$

$$= \frac{\prod_{i=1}^{q} \Gamma(\beta_i + a)}{\prod_{i=1}^{p} \Gamma(a_i + a)} x^{-a} (1 - z)^{-\lambda} \sum_{r=0}^{\infty} \frac{(\lambda)_r}{r!} \frac{\prod_{i=1}^{p} \Gamma(a_i + a + r)}{\prod_{i=1}^{q} \Gamma(\beta_i + a + r)}$$

$$\times\ x^{a+r} \frac{z^r}{(1 - z)^r} (-1)^r$$

$$= (1 - z)^{-\lambda} {}_{p+1}F_{q} \left[\begin{matrix} \lambda, a_1 + a, .., a_p + a ; \\ \beta_1 + a, .., \beta_q + a ; \end{matrix} \frac{-xz}{1 - z} \right].$$

On putting $a = 0$, we at once arrive at (4·11).

Bilinear and Bilateral Generating Functions

Let us consider the bilinear generating function (Rainville 1965, p. 212)

$$(1 - z)^{-1-a} \exp \left[\frac{-(x + y) z}{1 - z} \right] {}_0F_1 \left[\begin{matrix} - ; \\ 1 + a ; \end{matrix} \frac{x y z}{(1 - z^2)} \right]$$

$$= \sum_{n=0}^{\infty} \frac{n!}{(1 + a)_n} L_n^{(a)}(x) L_n^{(a)}(y) z^n \qquad \ldots \qquad (4·12)$$

known as the Hille-Hardy formula.

Making use of the definition (4·9), we first replace y by yt and multiply both sides by $t^{\sigma-1}$ and take their Laplace transform. In the resulting formula that would involve the sum

$$\sum_{n=0}^{\infty} L_n^{(a)}(x) \, {}_2F_1 \left[-n, \sigma ; 1 + a ; y \right] z^n .$$

On its right-hand side, we replace y by yuv and apply the operator (4·2). Thus we obtain the bilateral generating function

$$\sum_{n=0}^{\infty} L_n^{(a)}(x) \, {}_{2q+4}F_{2p+3} \left[\begin{matrix} -n, a, b, \sigma, \triangle(2, b_1), .., \triangle(2, b_q); \\ 1 + a, \triangle(2, a + b), \triangle(2, a_1), .., \triangle y(2, a_p); \end{matrix} \right] z^n$$

$$= (1 - z)^{-1-a} \exp. \left[\frac{-xz}{1 - z} \right] F \left[\begin{matrix} -n, a, b, \sigma, \\ \triangle(2, a + b), \end{matrix} \right.$$

$$\left.\begin{matrix} \triangle\,(2, b_1), \ldots, \triangle\,(2, b_q) : - ; - ; & \dfrac{xyz}{(1-z)^2}, \dfrac{yz}{z-1} \\ \triangle\,(2, a_1), \ldots, \triangle\,(2, a_p) : 1 + \alpha ; - ; & \end{matrix}\right] \quad (4{\cdot}14)$$

Reduction Formulae

If in the known relationship (4. Erdelyi 1953, p. 185),

$${}_0F_1\,[- ; \rho ; z]\;{}_0F_1\,[- ; \sigma; z] = {}_2F_3\left[\begin{matrix} \triangle\,(2, \rho + \sigma - 1\,); \\ \rho, \sigma, \rho + \sigma - 1\;; \end{matrix}\; 4z\right] \quad (4{\cdot}15)$$

we replace z by zuv and operate upon both sides by the operator in (4·2) we shall get our first reduction formula in the form

$$F\left[\begin{matrix} a, b, \triangle\,(2, b_1), \ldots, \triangle\,(2, b_q) : - ; - ; \\ \triangle\,(2, a + b), \triangle\,(2, a_1), \ldots, \triangle\,(2, a_p) : \rho ; \sigma ; \end{matrix}\; z, z\right]$$

$$= {}_{2q+4}F_{2p+5}\left[\begin{matrix} \triangle\,(2, \rho + \sigma - 1\,), a, b, \triangle\,(2, b_1), \ldots, \triangle\,(2, b_q)\;; \\ \rho, \sigma, \rho + \sigma - 1, \triangle\,(2, a + b), \triangle\,(2, a_1), \ldots, \triangle\,(2, a_p)\,; \end{matrix}\; 4z\right] \quad (4{\cdot}16)$$

Similar application of the operator (4·2) to the known results (3) and (7) of Erdelyi (1953, p. 186) would result in the reduction formulae

$$F\left[\begin{matrix} a, b, \triangle\,(2, b_1), \ldots, \triangle\,(2, b_q) : - ; - ; \\ \triangle\,(2, a + b), \triangle\,(2, a_1), \ldots, \triangle\,(2, a_p) : \rho ; \rho ; \end{matrix}\; z, -z\right]$$

$$= {}_{4q+4}F_{4p+7}\left[\begin{matrix} \triangle\,(2, a), \triangle\,(2, b), \triangle\,(4, b_1), \ldots, \triangle\,(4, b_q)\;; \\ \rho, \triangle\,(2, \rho), \triangle\,(4, a + b), \triangle\,(4, a_1), \ldots, \triangle\,(4, a_p)\;; \end{matrix}\; -4z^2\right] \quad (4{\cdot}17)$$

and

$$F\left[\begin{matrix} a, b, \triangle\,(2, b_1), \ldots, \triangle\,(2, b_q) : - ; - ; \\ \triangle\,(2, a + b), \triangle\,(2, a_1), \ldots, \triangle\,(2, a_p) : \rho, \sigma; \rho, \sigma; \end{matrix}\; z, -z\right]$$

$$= {}_{4q+7}F_{4p+12}\left[\begin{matrix} \triangle\,(3, \rho + \sigma - 1\,), \triangle\,(2, a), \triangle\,(2, b), \\ \rho, \sigma, \triangle\,(2, \rho), \triangle(2, \sigma), \triangle\,(2, \rho + \sigma - 1\,), \end{matrix}\right.$$

$$\left.\begin{matrix} \triangle(4, b_1), \ldots, \triangle(4, b_q); \\ \triangle(4, a + b), \triangle(4, a_1), \ldots, \triangle(4, a_p); \end{matrix}\; \frac{-27}{4}\, z^2\right] \quad (4{\cdot}18)$$

respectively.

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