SOLUTIONS OF SOME PROBLEMS IN LINEAR THEORY OF AN ELASTIC COSSERAT PLATE

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(Communicated by F. C. Auluck, F.N.A.

(Received 21 December 1974)

The linear theory of an elastic Cosserat plate developed by Green and Naghdi (1967) is applied to a few problems of practical interest. The relationship between the elastic constant α_3 of the Cosserat's surface and K_s of the Minxlin s theory has been established.

1. Introduction

A general non-linear dynamical theory of an elastic Cosserat surface has been given recently by Green *et al.* (1965) in order to overcome the approximate nature of the classical theory of plates and shells. The basic field equations and the explicit constitutive relations of the linear theory of an elastic Cosserat plate have been reproduced in a recent paper by Green and Naghdi (1967) in which a problem of an infinite Cosserat plate with a circular hole subjected to uniform bending at infinity has been solved as an example. This has opened the scope for solutions of various other problems to find the additional effects which are absent in the classical theory.

The constitutive equations of the linear theory (Green and Naghdi 1967) are such that the differential equations of the complete theory can be separated into those for purely extensional deformations and those for purely bending i.e., inextensional deformations. We employ in our present paper the latter part of the theory i.e., bending of the Cosserat plate in order to investigate the additional effects in comparison to those in the classical theory for bending of isotropic plates.

The following problems have been solved in order to investigate the additional effects.

(i) Bending of a circular plate with clamped edges carrying a uniformly distributed load spread in the form of a disc. The results for point load and for the load spread over the whole plate are derived as particular cases.

(ii) Bending of a circular plate with clamped edges and loaded with a uniformly distributed load in the form of a ring. The results for point load at the centre are deduced as a particular case.

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(iii) Bending of a square plate simply supported at the four edges under uniformly distributed load.

A relation of the elastic constant a_3 of the present theory (Green and Naghdi 1967) with that of the constant K_s introduced by Mindlin (1951) has been established. When $a_3 \to \infty$, the results obtained reduce to their counterparts in the classical theory. The numerical values of the transverse displacements for the present theory (Green and Naghdi 1967) and those for the classical theory, for a particular thickness parameter have been shown graphically for these problems.

2. Bending Theory and Basic Field Equations

General background of the theory (Green and Naghdi 1967) and the basic field equations required in this paper are reproduced here as follows :

Cosserat surface is a surface to every point of which a deformable vector called a director is assigned. The directors which are not necessarily along the normal to the surface possess the property that they remain invariant in length under rigid body motions.

Let an initially flat surface be referred to rectangular cartesian co-ordinates Z_i (i = 1, 2, 3) with unit base vectors $\vec{e_i}$. The outward unit normal to the initial flat surface is $\vec{e_3}$ and we assume in what follows, that the initial director $\vec{D}$ to the initial flat surface is coincident with $\vec{e_3}$. Then, when the (ordinary) monopolar displacements and the director displacements are infinitesimal and when $\vec{D} = \vec{e_3}$ i.e.,

$$D\alpha = 0,\ D_3 = 1 \quad (\alpha = 1, 2) \qquad \ldots \quad \ldots \quad \ldots \qquad (2{\cdot}1)$$

the kinematic variable, with reference to rectangular Cartesian co-ordinates are

$$e_{\alpha\beta} = \tfrac{1}{2}\,(U_{\alpha,\beta} + U_{\beta,\alpha}) \qquad \ldots \quad \ldots \quad \ldots \qquad (2{\cdot}2)$$

$$K_{\alpha\beta} = \bar{\delta}_{\alpha,\beta}\ ,\ K_{3\alpha} = \bar{\delta}_{3,\beta} \qquad \ldots \quad \ldots \quad \ldots \qquad (2{\cdot}3)$$

where comma denotes partial differentiation with respect to Z_α (α = 1. 2), Latin indices have the range 1, 2, 3, Greek indices take the values 1, 2 only; and

$$U_i = z_i - Z_i\,, \qquad \ldots \quad \ldots \quad \ldots \qquad (2{\cdot}4)$$

$$\delta_\alpha = d_\alpha = \bar{\delta}_\alpha - \beta_\alpha,\ \delta_3 = \bar{\delta}_3 = d_3 - 1 \qquad \ldots \quad \ldots \quad \ldots \qquad (2{\cdot}5)$$

$$\beta_\alpha = -\,U_{3,\alpha} \qquad \ldots \quad \ldots \quad \ldots \qquad (2{\cdot}6)$$

for infinitesimal deformations and in view of (2·1), z_i and $\vec{d}$ are the co-ordinates and the director respectively in the deformed state. In the absence of inertia forces and under isothermal conditions, we have

$$\left.\begin{aligned} K_{\alpha\beta} &= K_{(\alpha\beta)} + K_{[\alpha\beta]} \\ K_{(\alpha\beta)} &= K_{(\beta\alpha)} = \tfrac{1}{2}(\bar{\delta}_{\alpha,\beta} + \bar{\delta}_{\beta,\alpha}) = \tfrac{1}{2}(\delta_{\alpha,\beta} + \delta_{\beta,\alpha}) - U_{3,\alpha\beta} \\ K_{[\alpha\beta]} &= -K_{[\beta\alpha]} = \tfrac{1}{2}(\bar{\delta}_{\alpha\,\beta} - \bar{\delta}_{\beta,\alpha}) = \tfrac{1}{2}(\delta_{\alpha,\beta} - \delta_{\beta,\alpha}). \end{aligned}\right\} \quad (2\cdot7)$$

$$\left.\begin{aligned} M_{\alpha\beta} &= M_{(\alpha\beta)} + M_{[\alpha\beta]} \\ M_{(\alpha\beta)} &= \alpha_5\,\delta_{\alpha\beta}\,K_{\gamma\gamma} + (\alpha_6 + \alpha_7)\,K_{(\alpha\beta)} \\ M_{[\alpha\beta]} &= (\alpha_6 - \alpha_7)\,K_{[\alpha\beta]} \\ m_\alpha &= \alpha_3\,\delta_\alpha. \end{aligned}\right\} \quad (2\cdot8)$$

$$\left.\begin{aligned} M_{\beta\alpha,\alpha} + l_\beta &= m_\beta \\ N_{3\alpha,\alpha} + p = 0,\ N_{3\alpha} &= m_\alpha \end{aligned}\right\} \quad (2\cdot9)$$

where $\delta_{\alpha\beta}$ is Kronecker symbol; α_3, α_5, α_6 and α_7 are the elastic constants; $M_{(\alpha\beta)}$ and $M_{[\alpha\beta]}$ stand for the symmetric and anti-symmetric parts of $M_{\alpha\beta}$; $\{p_\beta, p = p_3\}$ and $\{l_\beta, l = l_3\}$ are respectively the assigned surface loads and the assigned director surface loads (or surface couples) each per unit area of the underformed surface of the plate.

$N_{\alpha\beta}$, $M_{(\alpha\beta)}$, $N_{3\alpha}$, m_α, $M_{3\alpha}$ and m_3 are given by

$$\left.\begin{aligned} &\int_{-\frac{1}{2}h}^{+\frac{1}{2}h} \sigma_{\alpha\beta}\,dZ_3, \quad \int_{-\frac{1}{2}h}^{+\frac{1}{2}h} \sigma_{\alpha\beta}\,Z_3\,dZ_3, \quad \int_{-\frac{1}{2}h}^{+\frac{1}{2}h} \sigma_{\alpha 3}\,dZ_3 \\ &\int_{-\frac{1}{2}h}^{+\frac{1}{2}h} \sigma_{3\alpha}\,dZ_3, \quad \int_{-\frac{1}{2}h}^{+\frac{1}{2}h} \sigma_{\alpha 3}\,Z_3\,dZ_3, \quad \int_{-\frac{1}{2}h}^{+\frac{1}{2}h} \sigma_{33}\,dZ_3 \end{aligned}\right\} \quad (2\cdot10)$$

in the order listed where σ_{ij} denote the Cartesian components of the symmetric stress tensor and h is the thickness of the plate.

The system of differential equations for the bending theory are

$$\left.\begin{aligned} \nabla^2\phi &= -\frac{p}{\alpha_3} \\ \nabla^2\chi &= \left(\frac{\alpha_3}{D}\right)\phi \\ \left[\nabla^2 - \frac{1}{\lambda^2}\right]\psi &= 0 \end{aligned}\right\} \quad (2\cdot11)$$

where $$\lambda = \left(\frac{k}{\alpha_3}\right)^{\frac{1}{2}} \quad (2\cdot12)$$

$$K = \tfrac{1}{2}\left[(1-\gamma)\, D + (a_6 - a_7)\right] \quad \ldots \quad (2\cdot13)$$

$$\delta_a = \phi_{;a} + \epsilon_{a\gamma}\, \psi_{;\gamma} \quad \ldots \quad (2\cdot14)$$

$$\bar{\delta}_a = \chi_{;a} + \epsilon_{a\gamma}\, \psi_{;\gamma} \quad \ldots \quad (2\cdot15)$$

$$\chi = \phi - u_3 \quad \ldots \quad (2\cdot16)$$

D and γ represent flexural rigidity and Poisson's ratio respectively, ∇^2 is the two-dimesional Laplacian and $\epsilon_{a\gamma}$ is the two-dimensional permutation symbol

$$\epsilon_{11} = \epsilon_{22} = 0,\ \epsilon_{12} = -\epsilon_{21} = 1. \quad \ldots \quad (2\cdot17)$$

Equations (2·11) represent a sixth-order system of differential equations in the unknowns u_3 and δ_a.

For an isotropic plate

$$a_5 = \gamma D \quad \ldots \quad (2\cdot18)$$

$$a_6 + a_7 = D\,(1-\gamma) \quad \ldots \quad (2\cdot19)$$

3. Circular Plate Clamped at the Edges with Uniformly Distributed Load in the Form of A Disc.

Let W be the total load uniformly distributed in the form of a disc of radius $b<a$ in the circular plate of radius a (Fig. 1) and p be the intensity of the load given in the form

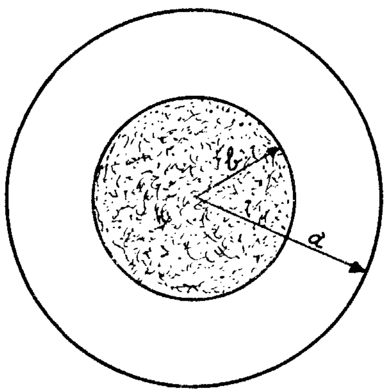

Fig. 1.

$$p = h\, q\,(r) \quad \ldots \quad (3\cdot1)$$

so that

$$q\,(r) = \frac{W}{\pi\, b^2\, h} \quad \text{for } r \leqslant b \quad \ldots \quad (3\cdot2)$$

$$= 0 \qquad \text{for } r > b \quad \ldots \quad (3\cdot3)$$

Thus for $r \leqslant b, \int_0^r r\,q\,(r)\,dr = \dfrac{Wr^2}{2\,\pi\,b^2\,h}$ .. .. .. (3·4)

and for $r > b, \int_0^r r\,q\,(r)\,dr = \int_0^b rq\,(r)\,dr + \int_b^r r\,q\,(r)\,dr = \dfrac{W}{2\pi h}$ .. (3·5)

From first and second equations of (2·11) the differential equation to be solved is

$$\nabla^4 x = -\frac{h}{D}\,q\,(r) \quad .. \quad .. \quad .. \quad (3{\cdot}6)$$

As this is an axially symmetric problem, eqn. (3·6) can be written as

$$\frac{1}{r}\,\frac{d}{dr}\left[\,r\,\frac{d}{dr}\left\{\frac{1}{r}\,\frac{d}{dr}\left(\,r\,\frac{dx}{dr}\right)\right\}\right] = -\frac{h}{D}\,q\,(r) \quad .. \quad .. \quad 3{\cdot}7$$

Using (3·4) and (3·5), the solution of (3·7) is given by

$$x_1 = -\frac{W}{4\pi D}\left\{\frac{r^2}{2}\log r - \frac{r^2}{2}\right\} + \frac{D_1}{2}\left\{\frac{r^2}{2}\log r - \frac{r^2}{2}\right\}$$

$$+\frac{D_2}{4}\,r^2 + D_3\log r + D_4 \quad (\text{for } r > b) \quad .. \quad (3{\cdot}8)$$

and $$x_2 = -\frac{W}{64\,\pi\,b^2\,D}\,r^4 + \frac{C_1}{2}\left\{\frac{r^2}{2}\;\log\,r - \frac{r^2}{2}\right\} + \frac{C_2}{4}\,r^2$$

$$+ C_3\log r + C_4\,(\text{for } r \leqslant b\,). \quad .. \quad .. \quad .. \quad (3{\cdot}9)$$

where C_1 to C_4 and D_1 to D_4 are the arbitrary constants to be determined.

From Second equation of (2·11) and (3·8) we get

$$\phi_1 = \frac{D}{a_3}\left[-\frac{W}{2\pi D}\,\log r + D_1\log\,r + D_2\right]. \quad .. \quad .. \quad (3{\cdot}10)$$

Using (2·16), (3·8) and (3·10) we obtain

$$(U_3)_1 = \frac{D}{a_3}\left[-\frac{W}{2\pi D}\log r + D_1\log\,r + D_2\right] + \frac{W}{4\pi D}\left\{\frac{r^2}{2}\,\log r - \frac{r^2}{2}\right\}$$

$$-\frac{D_1}{2}\left\{\frac{r^2}{2}\log r-\frac{r^2}{2}\right\}-\frac{D_2}{4}r^2-D_3\log r-D_4\ (\text{for } r>b) \qquad (3{\cdot}11)$$

From Second equation of (2·11) and (3·9) we have

$$\phi_2=\frac{D}{\alpha_3}\left[-\frac{Wr^2}{4\pi b^2 D}+2C_1\log r+C_2\right] \quad .. \quad .. \quad .. \qquad (3{\cdot}12)$$

and from (2·16), (3·9) and (3·12) we get

$$(U_3)_2=\frac{D}{\alpha_3}\left[-\frac{Wr^2}{4\pi b^2 D}+2C_1\log r+C_2\right]+\frac{Wr^4}{64\pi b^2 D}$$

$$-C_1\left\{\frac{r^2}{2}\log r-\frac{r^2}{2}\right\}-\frac{C_2}{4}r^2-C_3\log r-C_4\ (\text{for } r\leqslant b) \quad .. \qquad (3{\cdot}13)$$

At $r=0$, $(U_3)_2$ should be finite, so that from (3·13) we have

$$C_1=C_3=0 \quad .. \quad .. \quad .. \qquad (3{\cdot}14)$$

Thus eqn. (3·13) reduces to

$$(u_3)_2=\frac{D}{\alpha_3}\left[-\frac{Wr^2}{4\pi b^2 D}+C_2\right]+\frac{Wr^4}{64\pi b^2 D}-\frac{C_2}{4}r^2-C^4\ (\text{for } r\leqslant b) \qquad (3{\cdot}15)$$

For the plate clamped along the boundary $r=a$, the continuity and boundary conditions of the problem require that

$$\left.\begin{array}{l}
\text{(i)}\ \left(\dfrac{d\chi_1}{dr}\right)_{r=b}=\left(\dfrac{d\chi_2}{dr}\right)_{r=b}, \quad \text{(ii)}\ \left(\dfrac{d^2\chi_1}{dr^2}\right)_{r=b}=\left(\dfrac{d^2\chi_2}{dr^2}\right)_{r=b}, \\
\text{(iii)}\ (u_3)_{1(r=b)}=(u_3)_{2(r=b)} \quad \text{(iv)}\ \dfrac{d}{dr}\Big[(u_3)_1\Big]_{r=b}=\dfrac{d}{dr}\Big[(u_3)_2\Big]_{r=b}, \\
\text{(v)}\ \left(\dfrac{d\chi_1}{dr}\right)_{r=a}=0, \quad \text{(vi)}\ \Big[(u_3)_1\Big]_{r=a}=0, \quad \text{(vii)}\ \left[\dfrac{d\chi_2}{dr}\right]_{r=0}=0, \\
\text{(viii)}\ \dfrac{d}{dr}\Big[(u_3)_2\Big]_{r=0}=0, \quad \text{(ix)}\ (N_{3\theta})_{r=a}=0.
\end{array}\right\} \qquad (3{\cdot}16)$$

Using these conditions in (3·8), (3·9), (3·11), (3·14) and (3·15), the constants are found to be

$$\left.\begin{aligned}
C_1 &= 0\\
C_2 &= \frac{W}{2\pi D}\left[\log a - \log b + \frac{b^2}{4a^2}\right]\\
C_3 &= 0\\
C_4 &= \frac{D}{a_3}\left[-\frac{W}{4\pi D} + C_2 + \frac{W}{2\pi D}\log b - D_2\right] + \frac{Wb^2}{64\pi D}\\
&\quad - \frac{C_2}{4}b^2 - \frac{Wb^2}{8\pi D}\left(\log b - 1\right) + \frac{D_2}{4}b^2 + D_3\log b + D_4\\
D_1 &= 0\\
D_2 &= \frac{W}{2\pi Da}\left[a\log a - \frac{a}{2} + \frac{b^2}{4a}\right]\\
D_3 &= -\frac{Wb^2}{16\pi D}\\
D_4 &= \frac{D}{a_3}\left[-\frac{W}{2\pi D}\log a + \frac{W}{2\pi Da}\left\{a\log a - \frac{a}{2} + \frac{b^2}{4a}\right\}\right]\\
&\quad - \frac{Wa^2}{16\pi D} - \frac{Wb^2}{32\pi D} + \frac{Wb^2}{16\pi D}\log a.
\end{aligned}\right\} \qquad (3{\cdot}17)$$

If $\{M_{rr}, M_{\theta\theta}, M_{r\theta}, M_{\theta r}\}$ and $\{N_{3r}, N_{3\theta}\}$ are the physical components of $M_{\alpha\beta}$ and $N_{3\alpha}$ respectively referred to polar co-ordinates and remembering that ϕ, χ and ψ are functions of r only, we obtain from equations

(2·7), (2·8), third equation of (2·9), (2·14), (2·15), (2·17) to (2·19)

$$M_{rr} = \gamma D\nabla^2\chi + (1-\gamma)D\frac{d^2\chi}{dr^2} \qquad (3{\cdot}18)$$

$$M_{\theta\theta} = \gamma D\nabla^2\chi + (1-\gamma)D\left\{\frac{1}{r}\frac{d\chi}{dr}\right\} \qquad (3{\cdot}19)$$

$$(M_{r\theta} + M_{\theta r}) = (1-\gamma)D\left[\frac{1}{r}\frac{d\psi}{dr} - \frac{d^2\psi}{dr^2}\right] \qquad (3{\cdot}20)$$

$$(M_{r\theta} - M_{\theta r}) = (a_6 - a_7)\nabla^2\psi \qquad (3{\cdot}21)$$

$$N_{3_r} = \alpha_3 \frac{d\phi}{dr} \quad \cdots \quad (3\cdot22)$$

$$N_{3\theta} = -\alpha_3 \frac{d\psi}{dr} \quad \cdots \quad (3\cdot23)$$

Thus, fron (3·8), (3·9), (3·10), (3·12), (3·17), to (3·19) and (3·22) we obtain

$$\left.\begin{aligned} M_{rr} &= \frac{W}{4\pi}\left[(1+\gamma)\left\{\log\frac{a}{b}+\frac{b^2}{4a^2}\right\}-\frac{\gamma+3}{4}\frac{r^2}{b^2}\right] && (3\cdot24)\\ M_{\theta\theta} &= \frac{W}{4\pi}\left[(1+\gamma)\left(\log\frac{a}{b}+\frac{b^2}{4a^2}\right)-\frac{1+3\gamma}{4}\frac{r^2}{b^2}\right] && (3\cdot25)\\ N_{3_r} &= -\frac{Wr}{2\pi b^2}\,. && (3\cdot26)\end{aligned}\right\}\ (\text{for } r\leqslant b)$$

$$\left.\begin{aligned} M_{rr} &= \frac{W}{4\pi}\left[-1+(1+\gamma)\left\{\frac{b^2}{4a^2}-\log\frac{r}{a}\right\}+\frac{1-\gamma}{4}\frac{b^2}{r^2}\right] && (3\cdot27)\\ M_{\theta\theta} &= \frac{W}{4\pi}\left[-\gamma+(1+\gamma)\left\{\frac{b^2}{4a^2}-\log\frac{r}{a}\right\}-\frac{1-\gamma}{4}\frac{b^2}{r^2}\right] && (3\cdot28)\\ N_{3_r} &= -\frac{W}{2\pi r}\,. && (3\cdot29)\end{aligned}\right\}\ (\text{for } r>b)$$

From (3·11), (3·15) and (3·17) we get

$$(u_3)_1 = \frac{W}{2\pi\alpha_3}\log\frac{a}{r}+\frac{W}{16\pi D}(b^2+2r^2)\log\frac{r}{a}+ + \frac{W}{16\pi D}(a^2-r^2)\left(1+\frac{b^2}{2a^2}\right) \quad (\text{for } r>b) \quad (3\cdot30)$$

$$(u_3)_2 = \frac{W}{4\pi\alpha_3}\left(2\log\frac{a}{b}+1-\frac{r^2}{b^2}\right)+\frac{W}{16\pi D}(b^2+2r^2)\log\frac{b}{a} + \frac{W}{64\pi D}\left(4a^2-3b^2+\frac{r^4}{b^2}-2\frac{b^2}{a^2}r^2\right) \quad (\text{for } r\leqslant b) \quad \cdots \quad (3\cdot31)$$

If $\alpha_3 \to \infty$, (3·30) and (3·3) reduce to

$$(u_3)_1 = \frac{W}{16\pi D}(b^2+2r^2)\log\frac{r}{a}+\frac{W}{16\pi D}(a^2-r^2)\left(1+\frac{b^2}{2a^2}\right) \quad \cdots \quad (3\cdot32)$$

$$(u_3)_2 = \frac{W}{16\pi D}(b^2+2r^2)\log\frac{b}{a}+\frac{W}{64\pi D}\left(4a^2-3b^2+\frac{r^4}{b^2}-2\frac{b^2}{a^2}r^2\right) \quad (3\cdot38)$$

The values of $(u_3)_1$ and $(u_3)_2$ determined by Gupta (1972) in terms of k_s the shear constant introduced by Mindlin, are

$$(u_3)_1 = \frac{Wk_s}{2\,\pi\,h\,G}\,\log\frac{a}{r} + \frac{W}{16\pi D}(b^2 + 2r^2\,\log\frac{r}{a} + \frac{W}{16\pi D}\,(a^2 - r^2)\left(1 + \frac{b^2}{2a^2}\right) \quad \ldots \quad (3{\cdot}34)$$

$$(u_3)_2 = \frac{Wk_s}{4\pi hG}\left(1 - \frac{r^2}{b^2} + 2\,\log\frac{a}{b}\right) + \frac{W}{16\pi D}\,(b^2 + 2r^2)\,\log\frac{b}{a} + \frac{W}{64\pi D}\left(4a^2 - 3b^2 + \frac{r^4}{b^2} - 2\,\frac{b^2}{a^2}\,r^2\right) \quad \ldots \quad \ldots \quad (3{\cdot}35)$$

Where

$$G = \frac{E}{2\,(1 + \gamma)} \quad \ldots \quad \ldots \quad \ldots \quad (3{\cdot}36)$$

and E stands for Young's modulus.

Comparing the results (3·30), (3·31), (3·34) and (3·35) we establish that

$$\alpha_3 = \frac{hG}{k_s}\,. \quad \ldots \quad \ldots \quad \ldots \quad (3{\cdot}37)$$

We now seek the solution of the third equation of (2·11).

It can be verified to be given by

$$\psi = A\,I_0\left(\frac{r}{\lambda}\right) + B\,k_0\left(\frac{r}{\lambda}\right) \quad \ldots \quad \ldots \quad \ldots \quad (3{\cdot}38)$$

where $I_0\left(\frac{r}{\lambda}\right)$ and $k_0\left(\frac{r}{\lambda}\right)$ are modified Bessel's functions of the first and second kind respectively and A, B are constants to be evaluated.

As ψ is finite at $r = 0$, B should be zero and equation (3·38) thus takes the form

$$\psi = A\,I_0\left(\frac{r}{\lambda}\right)\cdot \quad \ldots \quad \ldots \quad \ldots \quad (3{\cdot}39)$$

From (3·23) and (3·39) we have

$$N_{3\theta\theta} = -\,\alpha_3\,A\,\frac{d}{dr}\left[\,I_0\left(\frac{r}{\lambda}\right)\right]\cdot \quad \ldots \quad \ldots \quad \ldots \quad (3{\cdot}40)$$

Using the boundary condition $(\,N_{3\theta}\,)_{r=a} = 0$ in (3·40), we get $A = 0$.

As $A = B = 0$, the solution (3·38) reduces to

$$\psi = 0. \quad \ldots \quad \ldots \quad \ldots \quad (3{\cdot}41)$$

Using third equation of (2·7), third equation of (2·8), (2·14) and (3·41), we find that

$$M_{[rr]} = M_{[\theta\theta]} = M_{[r\theta]} = 0 \quad \text{.. .. ..} \quad (3\cdot42)$$

The case of a point load 'W' placed at the centre of the plate can be deduced by making the radius of the disc $b \to 0$. Thus the results (3·27) to (3·30) reduces to

$$M_{rr} = -\frac{W}{4\pi}\left[1 + (1+\gamma)\log\frac{r}{a}\right] \quad \text{.. .. ..} \quad (3\cdot43)$$

$$M_{\theta\theta} = -\frac{W}{4\pi}\left[\gamma + (1+\gamma)\log\frac{r}{a}\right] \quad \text{.. .. ..} \quad (3\cdot44)$$

$$N_{3r} = -\frac{W}{2\pi r} \quad \text{.. .. ..} \quad (3\cdot45)$$

$$u_3 = \frac{W}{2\pi\alpha_3}\log\frac{a}{r} + \frac{Wr^2}{8\pi D}\log\frac{r}{a} + \frac{W}{16\pi D}(a^2 - r^2) \quad \text{..} \quad (3\cdot46)$$

Making $\alpha_3 \to \infty$, these results reduce to those of classical theory (Timoshanko *et al.* 1959).

In order to determine the additional effects numerically at a specific point due to the presence of the new elastic constant α_3, we take $\gamma = 0\cdot3$, $\frac{1}{r_s} = 0\cdot85$ the value indicated by Cowper (1966) so that using (3·36) and (3·37), α_3 is given by

$$\alpha_3 = 0\cdot327\ hE \quad \text{.. .. ..} \quad (3\cdot47)$$

Now taking $\frac{r}{a} = 0\cdot1$ and $\frac{h}{a} = 0\cdot2$ and using (3·46)

we get $u_3 = \frac{Wa^2}{\pi D}(0\cdot07189)$ .. .. .. (3·48)

whereas the corresponding value of u_3 in classical plate theory is found to be

$$u_3 = \frac{Wa^2}{\pi D}(0\cdot0129). \quad \text{.. .. ..} \quad (3\cdot49)$$

Further the results of bending of a circular plate with clamped edges, carrying a uniformly distributed load spread over the whole plate can be obtained by making the radius of the disc $b \to a$. Thus from (3·24) to (3·26) we obtain

$$M_{rr} = \frac{p}{16}\left[a^2 - 3r^2 + \gamma(a^2 - r^2)\right] \quad \text{.. .. ..} \quad (3\cdot50)$$

$$M_{\theta\theta} = \frac{p}{16}\left[\gamma(a^2 - 3r^2) + a^2 - r^2\right] \quad \text{.. .. ..} \quad (3\cdot51)$$

$$N_{3r} = -\frac{1}{2}\,p\,r \quad \text{.. .. ..} \quad (3\cdot52)$$

and from (3·31) we get

$$u_3 = \frac{p}{64D}(a^2 - r^2)^2 + \frac{p}{4\alpha_3}(a^2 - r^2) \quad \ldots \quad \ldots \quad \ldots \quad (3{\cdot}53)$$

If $\alpha_3 \to \infty$, (3·53) reduces to

$$u_3 = \frac{p}{64D}(a^2 - r^2)^2 \quad \ldots \quad \ldots \quad \ldots \quad (3{\cdot}54)$$

which is in agreement with the deflection equation in the classical plate theory (Timoshanko *et al.* 1959).

The behaviour of the deflections (3·53) and (3·54) for the particular thickness parameter $\frac{h}{a} = 0{\cdot}01$ with $\gamma = 0{\cdot}3$ and $\alpha_3 = 0{\cdot}327\ hE$ is represented graphically, in Fig. 2.

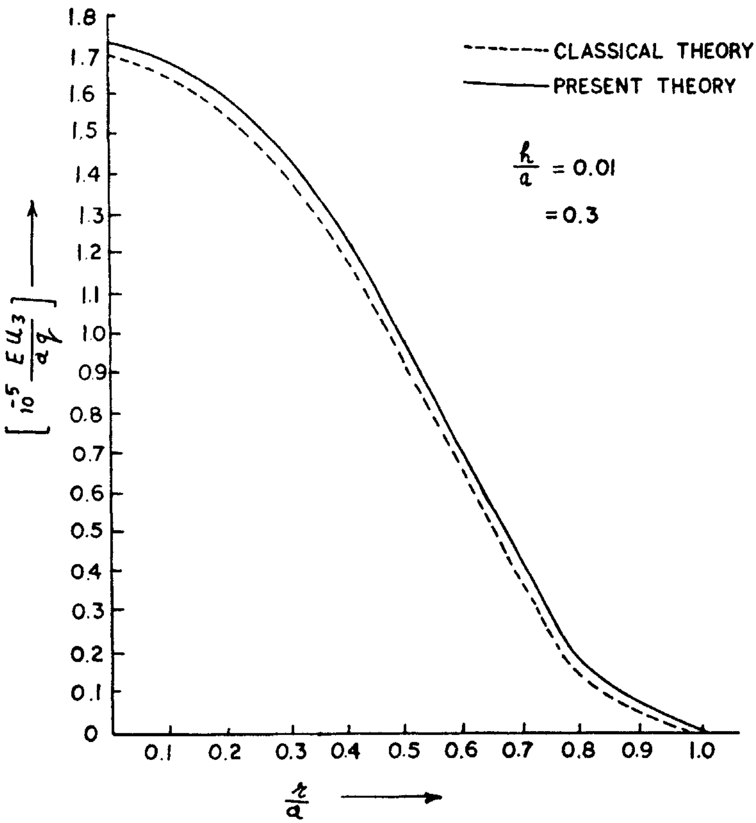

FIG. 2.

4. Circular Plate Clamped at the Edges with Uniformly distributed load in the form of a Ring

Let W be the total load applied along the concentric ring of radius $b < a$ on the circular plate of radius a, and p be the intensity of the load (Fig. 3) given by

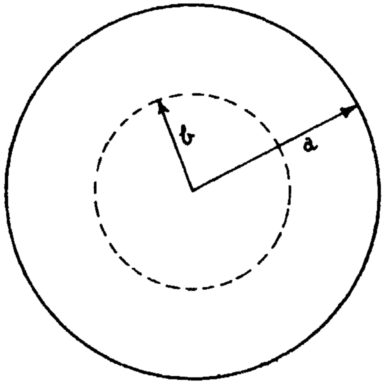

Fig. 3.

$$p = h\, q\,(r) \qquad \ldots \qquad \ldots \qquad \ldots \qquad (4{\cdot}1)$$

so that
$$\int_o^r r\, q\,(r)\, dr = 0 \quad (\text{for } r < b) \qquad \ldots \qquad \ldots \qquad \ldots \qquad (4{\cdot}2)$$

and
$$\int_o^r r\, q\,(r)\, dr = \frac{W}{2\pi h} \quad (\text{for } r \geqslant b). \qquad \ldots \qquad \ldots \qquad \ldots \qquad (4{\cdot}3)$$

Proceeding as in section 3, we obtain

$$M_{rr} = -\frac{W}{4\pi} + \frac{W}{8\pi}\frac{b^2}{r^2}(1-\gamma) + \frac{1+\gamma}{8\pi} W\left(\frac{b^2}{a^2} - 2\log\frac{r}{a}\right) (\text{for } r \geqslant b) \qquad \ldots \qquad \ldots \qquad \ldots \qquad (4{\cdot}4)$$

$$M_{\theta\theta} = -\frac{\gamma W}{4\pi} - \frac{1-\gamma}{8\pi} W \frac{b^2}{r^2} + \frac{(1+\gamma)\, W}{8\pi}\left(\frac{b^2}{a^2} - 2\log\frac{r}{a}\right) (\text{for } r \geqslant b) \qquad \ldots \qquad \ldots \qquad \ldots \qquad (4{\cdot}5)$$

$$N_{3_r} = \frac{-W}{2\pi r} \ (\text{for } r \geqslant b) \qquad \ldots \qquad \ldots \qquad \ldots \qquad (4{\cdot}6)$$

$$M_{rr} = M_{\theta\theta} = \frac{W}{8\pi}(1+\gamma)\left(\frac{b^2}{a^2} - 1 + 2\log\frac{a}{b}\right) \text{ (for } r < b) \quad .. \quad (4\cdot7)$$

$$\mathcal{N}_{3r} = 0 \quad (\text{for } r < b) \quad .. \quad .. \quad .. \quad (4\cdot8)$$

$$(u_3)_1 = \frac{W}{2\pi\alpha_3}\log\frac{a}{r} + \frac{W}{8\pi D}(b^2 + r^2)\log\frac{r}{a} + \frac{W}{16\pi D}(a^2+b^2)\left(1 - \frac{r^2}{a^2}\right)$$
$$(\text{for } r \geqslant b) \quad .. \quad .. \quad .. \quad (4\cdot9)$$

$$(u_3)_2 = \frac{W}{2\pi\alpha_3}\log\frac{a}{b} + \frac{W}{8\pi D}(b^2 + r^2)\log\frac{b}{a} + \frac{W}{16\pi D}\frac{(a^2 - b^2)(a^2 + r^2)}{a^2}$$
$$(\text{for } r < b) \quad .. \quad .. \quad .. \quad (4\cdot10)$$

If $\alpha_3 \to \infty$, (4·9) and (4·10) reduce to

$$(u_3)_1 = \frac{W}{8\pi D}(b^2 + r^2)\log\frac{r}{a} + \frac{W}{16\pi D}(a^2+b^2)\left(1 - \frac{r^2}{a^2}\right) \text{ (for } r \geqslant b) \quad (4\cdot11)$$

$$(u_3)_2 = \frac{W}{8\pi D}(b^2 + r^2)\log\frac{b}{a} + \frac{W}{16\pi D}\frac{(a^2 - b^2)(a^2 + r^2)}{a^2} \text{ (for } r < b) \quad (4\cdot12)$$

The values of $(u_3)_1$ and $(u_3)_2$ determined by Gupta (1972) in terms of k_s the shear constant introduced by Mindlin, are

$$(u_3)_1 = \frac{Wks}{2\pi hG}\log\frac{a}{r} + \frac{W}{8\pi D}(b^2 + r^2)\log\frac{r}{a} + \frac{W}{16\pi D}(a^2+b^2)\left((1 - \frac{r^2}{a^2}\right)$$
$$(\text{for } r \geqslant b) \quad .. \quad ... \quad .. \quad (4\cdot13)$$

$$(u_3)_2 = \frac{Wks}{2\pi hG}\log\frac{a}{b} + \frac{W}{8\pi D}(b^2 + r^2)\log\frac{b}{a} + \frac{W}{16\pi D}\frac{(a^2 - b^2)(a^2 + r^2)}{a^2}$$
$$(\text{for } r < b) \quad .. \quad .. \quad .. \quad (4\cdot14)$$

Comparing the results (4·9), (4·10), (4·13) and (4·14) we again find that

$$\alpha_3 = \frac{hG}{k_s} \quad .. \quad .. \quad .. \quad (4\cdot15)$$

The case of a point load W placed at the centre of the plate can be deduced by making the radius of the ring $b \to 0$.

Thus the results (4·4) to (4·6) and (4·9) reduce to

$$M_{rr} = -\frac{W}{4\pi}\left[(1 + (1+\gamma)\log\frac{r}{a}\right] \quad .. \quad .. \quad (4\cdot16)$$

$$M_{\theta\theta} = -\frac{W}{4\pi}\left[(\gamma + (1+\gamma)\log\frac{r}{a}\right] \quad .. \quad .. \quad (4\cdot17)$$

$$N_{3r} = -\frac{W}{2\pi r} \quad \ldots \quad (4\cdot18)$$

$$u_3 = \frac{W}{2\pi a_3}\log\frac{a}{r} + \frac{W}{8\pi D}r^2\log\frac{r}{a} + \frac{W}{16\pi D}a^2\left(1-\frac{r^2}{a^2}\right) \quad \ldots \quad (4\cdot19)$$

If $a_3 \to \infty$, (4·19) reduces to

$$u_3 = \frac{W}{8\pi D}r^2\log\frac{r}{a} + \frac{W}{16\pi D}a^2\left(1-\frac{r^2}{a^2}\right) \quad \ldots \quad (4\cdot20)$$

which is in agreement with deflection equation in the classical plate theory (Timoshanko *et al.* 1959). The behaviour of the deflections (4·19) and (4·20) for the particular thickness parameter $\frac{h}{a} = 0{\cdot}1$ with $\gamma = 0{\cdot}3$ and $a_3 = 0{\cdot}327\, hE$ is represented graphically in Fig. 4.

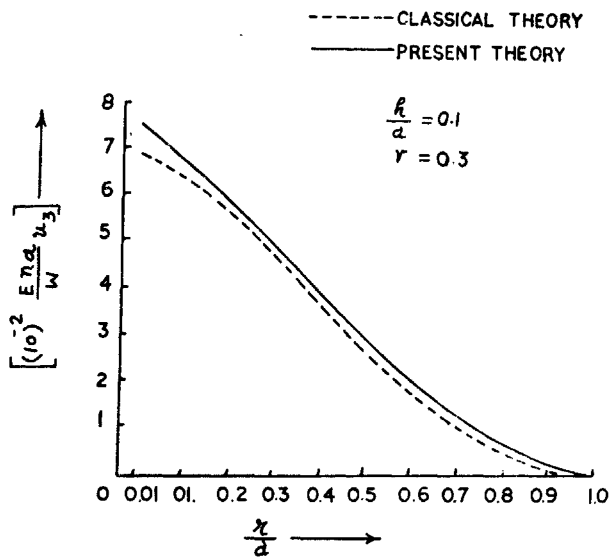

FIG. 4.

5. Bending of a Square Plate Simply Supported at the four Edges, Under Uniformly Distributed Load.

We take the uniform load distribution p in the form

$$p = q_0 \sin\frac{\pi x}{a}\ \text{Sin}\ \frac{\pi y}{a} \quad \ldots \quad (5\cdot1)$$

over a square plate of length a (Fig. 5), where q_0 is the load at the centre of the plate.

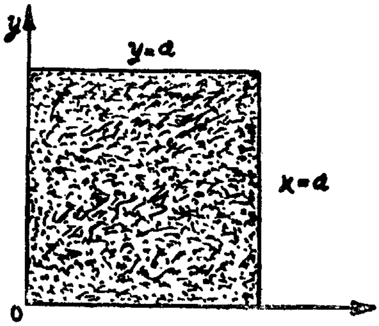

FIG. 5.

The boundary conditions of the problem are

$$\left.\begin{aligned} M_{xx} &= 0 \text{ at } x = 0 \text{ and } x = a, \\ M_{yy} &= 0 \text{ at } y = 0 \text{ and } y = a, \\ N_{3x} &= 0 \text{ at } y = 0 \text{ and } y = a, \end{aligned}\right\} \quad (5\cdot2)$$

From (2·7), (2·8), third equation of (2·9), (2·14), (2·15) and (2·17) to (2·19) we obtain

$$M_{xx} = \gamma D \nabla^2 \chi + D(1-\gamma)\left(\frac{\partial^2 x}{\partial x^2} + \frac{\partial^2 \psi}{\partial x \partial y}\right) \quad (5\cdot3)$$

$$M_{yy} = \gamma D \nabla^2 \chi + D(1-\gamma)\left(\frac{\partial^2 \chi}{\partial y^2} - \frac{\partial^2 \psi}{\partial x \partial y}\right) \quad (5\cdot4)$$

$$M_{xy} = D(1-\gamma)\frac{\partial^2 \chi}{\partial x \partial y} + \alpha_6 \frac{\partial^2 \psi}{\partial y^2} - \alpha_7 \frac{\partial^2 \psi}{\partial x^2} \quad (5\cdot5)$$

$$N_{3x} = a_3\, \delta_x = a_3\left[\frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y}\right] \quad (5\cdot6)$$

$$N_{3y} = a_3\, \delta_y = a_3\left[\frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x}\right] \quad (5\cdot7)$$

Also from first equation of (2·11), and (5·1), the first differential equation becomes,

$$\nabla^2 \phi = -\frac{1}{a_3} q_0 \operatorname{Sin}\frac{\pi x}{a} \operatorname{Sin}\frac{\pi y}{a}. \quad (5\cdot8)$$

Solution of (5·8) is given by

$$\phi = \frac{a^2 q_0}{2\pi^2 a_3} \operatorname{Sin}\frac{\pi x}{a} \operatorname{Sin}\frac{\pi y}{a}. \quad (5\cdot9)$$

From Second equation of (2·11) and (5·9) the second differential equation reduces to

$$\nabla^2 \chi = \frac{a^2 q_0}{2\pi^2 D} \operatorname{Sin} \frac{\pi x}{a} \operatorname{Sin} \frac{\pi y}{a} \quad \ldots \quad \ldots \quad \ldots \quad (5{\cdot}10)$$

the solution of which is given by ..

$$\chi = -\frac{a^4 q_0}{4D\pi^4} \operatorname{Sin} \frac{\pi x}{a} \operatorname{Sin} \frac{\pi y}{a}. \quad \ldots \quad \ldots \quad \ldots \quad (5{\cdot}11)$$

It can be easily seen that the solution of the third differential equation in (2·11) is of the form.

$$\psi = B\, K_2 \left(\frac{\sqrt{x^2 + y^2}}{\lambda} \right) \operatorname{Sin} \left(2 \tan^{-1} \frac{y}{x} \right) \quad \ldots \quad \ldots \quad (5{\cdot}12)$$

where $k_2 \left(\frac{\sqrt{x^2 + y^2}}{\lambda} \right)$ is the modified Bessel function of the second kind and B is a constant to be determined. Using the boundary condition in third of (5·2) we find that B vanishes so that

$$\psi = 0 \quad \ldots \quad \ldots \quad \ldots \quad (5{\cdot}13)$$

throughout the region of the plate, and thus from (5·6) and (5·7) we get

$$N_{3x} = \frac{a\, q_0}{2\, \pi} \operatorname{Cos} \frac{\pi\, x}{a} \operatorname{Sin} \frac{\pi\, y}{a} \quad \ldots \quad \ldots \quad \ldots \quad (5{\cdot}14)$$

$$N_{3y} = \frac{a\, q_0}{2\, \pi} \operatorname{Sin} \frac{\pi\, x}{a} \operatorname{Cos} \frac{\pi\, y}{a}. \quad \ldots \quad \ldots \quad \ldots \quad (5{\cdot}15)$$

From third in (2·7), third in (2·8), (2·14), (2·17) and (5·13) we find that $M_{[xx]} = M_{[yy]} = M_{[xy]} = 0.$ .. .. .. (5·16)

Again from (5·3) to (5·5), (5·11) and (5·13) we obtain

$$M_{xx} = M_{yy} = \frac{a^2\, q_0}{4\, \pi^2} (1 + \gamma) \operatorname{Sin} \frac{\pi\, x}{a} \operatorname{Sin} \frac{\pi\, y}{a} \quad \ldots \quad (5{\cdot}17)$$

and

$$M_{xy} = -(1 - \gamma) \frac{a^2\, q_0}{4\, \pi^2} \operatorname{Cos} \frac{\pi\, x}{a} \operatorname{Cos} \frac{\pi\, y}{a}. \quad \ldots \quad (5{\cdot}18)$$

Now using (2·16), (5·9) and (5·11) we get

$$u_3 = \frac{q_0\, a^2}{2\pi^2\, \alpha_3} \operatorname{Sin} \frac{\pi\, x}{a} \operatorname{Sin} \frac{\pi\, y}{a} + \frac{q_0\, a^4}{4D\, \pi^4} \operatorname{Sin} \frac{\pi\, x}{a} \operatorname{Sin} \frac{\pi\, y}{a}. \quad \ldots \quad (5{\cdot}19)$$

If $\alpha_3 \rightarrow \infty$, (5·19) reduces to

$$u_3 = \frac{q_0 a^4}{4D\pi^4} \text{ Sin } \frac{\pi x}{a} \text{ Sin } \frac{\pi y}{a} \qquad \ldots \quad \ldots \quad \ldots \qquad (5{\cdot}20)$$

which is in agreement with the result of the classical plate theory. (Jaeger 1965).

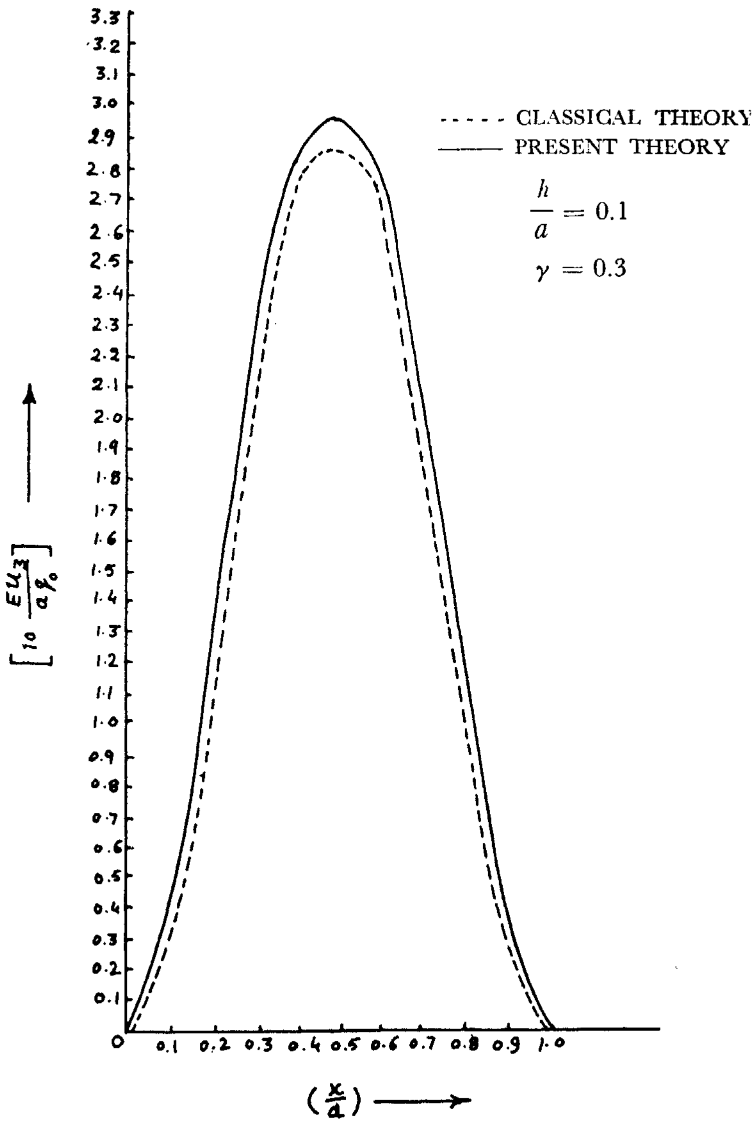

FIG. 6.

The numerical values of the deflections (5·19) for the present theory and (5·20) for the classical theory along the diagonal of the square plate for the particular thinckness parameter $\frac{h}{a} = 0 \cdot 1$ with $\gamma = 0 \cdot 3$ and $\alpha_3 = 0 \cdot 327hE$ are represented graphically in Fig. 6.

6. Conclusion

A relation between the elastic constant α_3 in the present theory (Green and Naghdi 1967) and the shear constant K_s due to Mindlin (1951) has been established. If $\alpha_3 \rightarrow \infty$, the results obtained for all these problems in the present paper reduce to their counterparts in the classical theory. The numerical values of the deflections for a particular thickness parameter for these problems have been shown graphically.

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