RECURRENT FINSLER SPACES OF THIRD ORDER

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The theory of recurrent Finsler spaces was studied by Moor (1963) and Mishra and Pande (1968) considering the Berwald's curvature tensor field $H^i_{hjk}\ (x,\dot{x})$. Sinha and Singh (1970) have defined the recurrent Finsler spaces of second order and studied their properties. The object of the present paper is to define recurrent Finsler space of order 3 and to study the relations between the recurrent vector, birecurrent tensor and the tensor of order 3 which determines the third order recurrency in Finsler space.

1. Introduction

Let F_n be an n-dimensional Finsler space with the metric function $F(x, \dot{x})$ satisfying the required conditions (Rund 1959). The metric tensor $g_{ij}\ (x, \dot{x})$ is given by $\frac{1}{2}\ \dot{\partial}_i\ \dot{\partial}_j\ F^2\ (x,\ \dot{x})$, which is symmetric, positively homogeneous function of degree zero in the directional argument. If $T^i_j\ (x, \dot{x})$ is any tensor, the Berwald's covariant derivative of T^i_j is given by

$$T^i_{j\,(m)} = \partial_m\ T^i_j - (\dot{\partial}_k\ T^i_j)\ G^k_m + T^k_j\ G^i_{km} - T^i_k\ G^k_{jm} \qquad \ldots \qquad (1{\cdot}1)$$

where the connection coefficient $G^i_{jk}\ (x,\ \dot{x})$ is symmetric in the indices j and k.

The Berwald's curvature tensor $H^i_{hjk}\ (x,\ \dot{x})$ (Rund 1959) satisfies the following identities :

$$\{ H^r_{sik\,(l)} + H^r_{sli\,(k)} + H^r_{skl\,(i)}\}\ \dot{x}^s = 0 \qquad \ldots \qquad (1{\cdot}2)$$

$$\dot{\partial}_j\ H^r_{ik\,(l)} = H^r_{jik\,(l)} - H^r_{mk}\ G^m_{ijl} - H^r_{im}\ G^m_{kjl} + H^m_{ik}\ G^r_{mjl} \qquad \ldots \qquad (1{\cdot}3)$$

and

$$H^r_{jik\,(l)} + H^r_{jli\,(k)} + H^r_{jkl\,(i)} + H^m_{ik}\ G^r_{mjl} + H^m_{li}\ G^r_{mjk} + H^m_{kl}\ G^r_{mji} = 0 \quad \ldots \qquad (1{\cdot}4)$$

where we have

$$G^i_{hjk} \overset{def}{=} \dot{\partial}_k\ G^i_{hj}\ , \qquad G^i_{hjk}\ \dot{x}^k = 0$$

and H^r_{sjk} is skew-symmetric in j and k. The following commutation formulae are:

$$2\ X_{ij\,[(h)\,(k)]} = -\ (\dot{\partial}_r\ X_{ij})\ H^r_{hk} - X_{rj}\ H^r_{ihk} - X_{ir}\ H^r_{jhk} \qquad \ldots \qquad (1{\cdot}5)$$

$$(\dot{\partial}_k X^i_{jl})_{(h)} - \dot{\partial}_k (X^i_{jl(h)}) = X^i_{rl} G^r_{jkh} + X^i_{rj} G^r_{lkh} - X^r_{jl} G^i_{rkh} \quad \ldots \quad (1\cdot6)$$

$$2 X_{[(i)(k)](l)} = - \dot{\partial} X_{(l)} H^r_{sik} \dot{x}^s - (\dot{\partial}_r X) H^r_{sik(l)} \dot{x}^s \quad \ldots \quad \ldots \quad (1\cdot7)$$

$$X_{(l)(k)(l)} - X_{(k)(l)(l)} = - \dot{\partial}_r X_{(l)} H^r_{sik} \dot{x}^s - \dot{\partial}_r X_{(k)} H^r_{sil} \dot{x}^s - (\dot{\partial}_r X) H^r_{sik(l)} \dot{x}^s - X_{(r)} H^r_{kil}. \quad \ldots \quad \ldots \quad \ldots \quad (1\cdot8)$$

For a recurrent and birecurrent Finsler space we have respectively

$$H^i_{hjk(l)} = V_l H^i_{hjk}, \quad (V_{l \neq 0}) \quad \ldots \quad \ldots \quad \ldots \quad (1\cdot9)$$

and

$$H^i_{hjk(l)(m)} = a_{lm} H^i_{hjk}, \quad (a_{lm \neq 0}) \quad \ldots \quad \ldots \quad \ldots \quad (1\cdot10)$$

V_l and a_{lm} are respectively called the recurrent vector and birecurrent tensor.

2. Recurrent Curvature Tensor Fields

In Finsler space F_n if Berwald's curvature tensor $H^i_{hjk}(x, \dot{x})$ satisfies the relation

$$H^s_{jhm(l)(k)(l)} = b_{ikl} H^s_{jhm}, \quad (b_{ikl} \neq 0) \quad \ldots \quad \ldots \quad \ldots \quad (2\cdot1)$$

where the recurrence tensor $b_{ikl}(x, \dot{x})$ is dependent on the positional coordinates as well as directional arguments, then it is called a recurrent Finsler space of order 3. Here we denote such a Finsler space by F_n^*.

From (2·1), we have

$$2 H^s_{jhm[(i)(k)](l)} = (b_{ikl} - b_{kil}) H^s_{jhm} \quad \ldots \quad \ldots \quad \ldots \quad (2\cdot2)$$

Using the commutation formula (1·7) in (2·2) for the curvature tensor H^s_{jhm} we obtain

$$2 b_{[ik]l} = - \frac{\partial}{\partial \dot{x}^r} H^s_{jhm(l)} H^r_{ik} - H^s_{rjhm} H^r_{ik(l)}. \quad \ldots \quad \ldots \quad (2\cdot3)$$

Differentiating (1·10) covariantly in the sense of Berwald, we get

$$2 H^s_{jhm[(i)(k)](l)} = 2 \{ a_{[ik]} H^s_{jhm} \}_{(l)} \quad \ldots \quad \ldots \quad \ldots \quad (2\cdot4)$$

with the help of eqns. (2·2), (2·3) and (2·4), we get

$$\{ b_{ikl} - a_{ik(l)} - b_{kil} - a_{kl(l)} \} H^s_{jhm} - 2 a_{[ik]} H^s_{jhm(l)} = 0 \quad \ldots \quad (2\cdot5)$$

Differentiating (2·3) covariantly with respect to x^n and using eqns. (1·9), (1·10) and (2·3), we get

$$(b_{ikl} - b_{kil})_{(n)} H^s_{jhm} + (b_{ikl} - b_{kil}) U_n H^s_{jhm}$$

$$= -(V_l H^s_{rjhm} + (\dot{\partial}_r V_l) H^s_{jhm})_{(n)} H^r_{ik}$$
$$- V_n H^r_{ik} (V_l H^s_{rjhm} + (\dot{\partial}_r V_l) H^s_{jhm})$$
$$- V_l V_n H^r_{ik} H^s_{rjhm} - a_{ln} H^r_{ik} H^s_{rjhm} \quad \ldots \quad (2{\cdot}6)$$

which reduces to the form

$$(b_{ikl} - b_{kil})_{(n)} H^s_{jhm} = -(V_{l(n)} + V_l V_n + a_{ln}) H^r_{ik} H^s_{rjhm}$$
$$- \{(\dot{\partial} V_l)_{(n)} + (\dot{\partial}_r V_l) V_n\} H^r_{ik} H^s_{jhm} \quad \ldots \quad (2{\cdot}7)$$

If the Berwald's curvature tensor H^s_{jhm} is independent of directional argument, then eqn. (2·7) yields

$$(b_{ikl} - b_{kil})_{(n)} + \{(\dot{\partial}_r V_l)_{(n)} + (\dot{\partial}_r V_l) V_n\} H^r_{ik} = 0 \quad \ldots \quad (2{\cdot}8)$$

Thus we have the following theorems :

Theorem 2·1—In recurrent Finsler space $F_n{}^*$ the recurrent tensor $b_{ikl}(x, \dot{x})$ satisfies (2·5).

Theorem 2·2—In $F_n{}^*$ if the Berwald's curvature tensor $H^s_{jhm}(x, \dot{x})$ is independent of $\dot{x}^l$, then (2·8) holds.

Taking the interchange of indices 1 and n in (2·7) and putting $\dot{\partial}_r H^s_{jhm}=0$ we obtain

$$\{b_{[ik]\,l\,(n)} - b_{[ik]\,n\,(l)}\} H^s_{jhm}$$
$$+ \{\dot{\partial}_r V_{[l\,(n)]} + \dot{\partial}_r V_{[l\,V\,n]}\} H^r_{ik} H^s_{jhm} = 0. \quad \ldots \quad (2{\cdot}9)$$

Differentiating covariantly the identity (1·4) twice with respect to x^p and x^q in the sense of Berwald's and using equations (1·10) and (2·1), we obtain

$$b_{lpq} H^r_{jik} + b_{kpq} H^r_{jli} + b_{ipq} H^r_{jkl}$$
$$+ a_{pq} (H^m_{ik} G^r_{mjl} + H^m_{li} G^r_{mjk} + H^m_{kl} G^r_{mji})$$
$$+ H^m_{ik} G^r_{mjl\,(p)\,(q)} + H^m_{ll} G^r_{mjk\,(p)\,(q)}$$
$$+ H^m_{kl} G^r_{mji\,(p)\,(p)} = 0. \quad \ldots \quad (2{\cdot}10)$$

Eliminating $H^m_{ik} G^r_{mjl} + H^m_{ll} G^r_{mjk}$ from equations (1·4) and (2·10) and using (1·9) we get

$$(b_{lpq} - V_l a_{pq}) H^r_{jik} + (b_{kpq} - V_k a_{pq}) H^r_{jli}$$

$$+ (b_{ipq} - V_{ipq})\ H^r_{jkl} + H^m_{ik}\ G^r_{mjl\,(p)\,(q)}$$

$$+ H^m_{li}\ G^r_{mjk\,(p)\,(q)} + H^m_{kl}\ G^r_{mji\,(p)\,(q)} = 0. \qquad \ldots \qquad \ldots \qquad (2{\cdot}11)$$

Again, differentiating (1·2) covariantly twice with respect to x^p and x^q and using equation (2·1), we get

$$b_{lpq}\ H^r_{ik} + b_{kpq}\ H^r_{li} + b_{ipq}\ H^r_{kl} = 0. \qquad \ldots \qquad \ldots \qquad \ldots \qquad (2{\cdot}12)$$

Differentiating (2·12) covariantly with respect to x^n and using equations (1·9) and (2·12), we obtain

$$b_{lpq\,(n)}\ H^r_{ik} + b_{kpq\,(n)}\ H^r_{li} + b_{ipq\,(n}\ H^r_{kl} = 0 \qquad \ldots \qquad \ldots \qquad (2{\cdot}13)$$

and multiplying (2·12) by $\dot{x}^i$ and using the homogeneity property of H^i_{jk}, we get

$$b_{lpq}\ H^r_k - b_{kpq}\ H^r_l + b_{ipq}\ H^r_{kl}\ \dot{x}^i = 0. \qquad \ldots \qquad \ldots \qquad \ldots \qquad (2{\cdot}14)$$

Thus we have

Theorem 2·3—In $F_n{}^*$, equations (2·9), (2·11) and (2·13) necessarily holds.

Differentiating (1·3) twice covariantly with respect to x^p and x^q and using eqns. (1·9), (1·10) and (2·1), we get

$$\{\dot{\partial}_j\ (V\ \ H^r_{ik}\ )\}_{(p)\,(p)} = b_{lpq}\ H^r_{jik} - a_{pq}\ (H^r_{mk}\ G^m_{ijl}$$

$$+ H^r_{lm}\ G^m_{kjl} - H^m_{ik}\ G^r_{mjl}) - H^r_{mk}\ G^m_{ijl\,(p)\,(q)}$$

$$- H^r_{im}\ G^m_{kjl\,(p)\,(q)} + H^m_{ik}\ G^r_{mjl\,(p)\,(q)} \qquad \ldots \qquad \ldots \qquad \ldots \qquad (2{\cdot}15)$$

which yields in the form, after simplifications.

$$\{V_l\ a_{pq} + V_{l\,(p)\,(q)} - b_{lpq}\}\ H^r_{jik} + (\dot{\partial}_j\ V_l)_{(p)\,(q)}\ H^r_{ik}$$

$$= H^m_{ik}\ G^r_{mjl\,(p)\,(q)}\ H^r_{mk}\ G^m_{ijl\,(p)\,(q)}$$

$$- H^r_{im}\ G^m_{kjl\,(p)\,(q)}\,. \qquad \ldots \qquad \ldots \qquad \ldots \qquad (2{\cdot}16)$$

Thus

Theorem 2·4— In $F_n{}^*$, the recurrent vector and birecurrent tensor satisfies (2·16).

References

Moor, A. (1969). Untersuchungen über Finsler-Raüme vo rekurrenter Krummung. *Tensor (N.S.)*. **13.**

Mishra, R. S. and Pande, H. D. (1968). Recurrent Finsler spaces. *J. Indian math. Soc.*, **32,** 17-22.

Rund, H. (1959). The Differential Geometry of Finsler Spaces. Springer-Verlag, Berlin.

Sinha, B. B. and Singh, S. P. (1970). On recurrent spaces of second order in Finsler spaces. *Yakohama math. J.*, **18,** 27-32.

Walker. A. G. (1950). On Ruse's spaces of recurrent curvature. *Proc. Lond. math. Soc.*, **50,** 36-64.