ZEROS OF A CROSS PRODUCT OF WHITTAKER FUNCTIONS I
$|m|$ LARGE

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The zeros of $W_{-k,m}(-\alpha\eta)\, W_{k,m}(\alpha) - W_{k,m}(\alpha\eta)\, W_{-k,m}(-\alpha)$, where $W_{k,m}(x)$ is the Whittaker function, are computed when $|m|$ is large.

The zeros of a cross product of Bessel functions have been computed by Chandrasekhar and Elbert (1953, 1954). Zeros of such functions are required in solutions of characteristic value problems in hydrodynamic stability. Zeros of a cross product of Whittaker functions

$$W_{-k,m}(-\alpha\eta)\, W_{k,m}(\alpha) - W_{-k,m}(-\alpha)\, W_{k,m}(\alpha\eta),\ (\eta < 1) \qquad (1)$$

have been obtained by Pathak and Prasad (in press). Using the asymptotic behaviour of $W_{k,m}(x)$ they derived the following approximate formula for the zeros of (1) neglecting terms $O(|m|^{-3})$.

$$\alpha = \beta + \frac{\beta^2}{\beta'} C_0 + \left\{ \frac{2}{(\beta')^2} C_0^2 - \frac{C_1}{\beta'} \right\} \beta^3$$
$$+ \left\{ \frac{5}{(\beta')^3} \left[\frac{1}{16} B_4 + C_2 \right] - \frac{5}{(\beta')^2} C_1 . C_2 + \frac{1}{\beta'} C_3 \right\} \beta^4. \qquad (2)$$

The various constants involved are defined as below :

$$A_0 = \eta^{2m},\ A_1 = \frac{1}{1-m},\ A_2 = \frac{1}{2-m},\ A_3 = \frac{1}{1+m},\ A_4 = \frac{1}{2+m},\ A_5 = 2k^2 + k^4,$$

$$A_6 = \frac{1}{16}(1 + 3k^2),\ A = 1 + A_1 A_3 A_5 + \tfrac{1}{2} A_5 (A_1 A_2 - A_3 A_4)$$

$$B_0 = \frac{k}{2}(1 + k^2),\quad B_1 = A_0 A_3 A_4 - A_1 A_2$$

$$B_2 = A_0 A_1 A_2 - A_3 A_4,\ B_3 = A_0 A_3 - A_1,\ B_4 = A_0 A_1 - A_3$$

$$B_5 = (A_0 - 1) A_1 A_3,\quad B_6 = \frac{\eta k}{32}(1 + \eta)$$

$$\beta = (A_0 - 1)\ A/\beta'$$

$$\beta' = \tfrac{1}{2} k\eta\, B_3 + \eta\, B_0 B_1 + \tfrac{1}{2} k\, B_4 + B_0 B_2 + \tfrac{1}{2} k^3 (1 + \eta)\, B_5$$

$$C_0 = \frac{\eta^2}{16}\, B_3 + \frac{1}{16}\, B_4 + \eta^2 A_6 B_1 + A_6 B_2$$

$$C_1 = \frac{\eta^2 k}{32}\, B_1 + B_6 B_5 + \frac{k}{32}\, B_2$$

$$C_2 = \frac{\eta^2}{16}\, B_3 + \eta^2 A_6 B_1 + A_6 B_2$$

$$C_3 = (\eta^4 B_1 + 2\eta^2 B_5 + B_5)/512.$$

For different values of k, m and η the zeros α_i are calculated by means of formula (2) (Tables I—III). The computation was made on IBM 7044 computer at I.I.T., Kanpur.

TABLE I

Zeros for m = 2.20

k/η	0·0	0·2	0·5	0·7	0·9
1·00	0·301811E10	0·250065E04	0·398887E02	0·173937E02	0·297805E02
1·40	0·552471E17	0·139029E04	0·198021E02	0·976053E01	0·133964E02
1·80	—0·320580E11	0·900441E03	0·144226E02	0·813400E01	0·917072E01
2·00	—0·199921E10	0·743578E03	0·134349E02	0·799116E01	0·837874E01
2·40	—0·887481E08	0·526676E03	0·128908E02	0·825589E01	0·784996E01
2·80	—0·146343E08	0·390852E03	0·133000E02	0·888337E01	0·799839E01
3·00	—0·758294E07	0·342495E03	0·136908E02	0·926790E01	0·820064E01
3·40	—0·272332E07	0·271948E03	0·146869E02	0·101192E02	0·874929E01
3·80	—0·126477E07	0·225455E03	0·158605E02	0·110396E02	0·941553E01
4·00	—0·921465E06	0·208450E03	0·164396E02	0·115166E02	0·977634E01
4·40	—0·539733E06	0·183299E03	0·178045E02	0·124937E02	0·105348E02
4·80	—0·349894E06	0·166730E03	0·191725E02	0·134929E02	0·113275E02
5·00	—0·290340E06	0·160793E03	0·198707E02	0·139985E02	0·117330E02
5·40	—0·210035E06	0·152417E03	0·212878E02	0·150190E02	0·125573E02
5·80	—0·160577E06	0·147579E03	0·227258E02	0·160490E02	0·133951E02
6·00	—0·142901E06	0·146171E03	0·234509E02	0·165667E02	0·138178E02
6·40	—0·116663E06	0·144920E03	0·249104E02	0·176068E02	0·146694E02
6·80	—0·986616E05	0·145390E03	0·263799E02	0·186517E02	0·155274E02
7·00	—0·917824E05	0·145989E03	0·271177E02	0·191757E02	0·159584E02
7·40	—0·810453E05	0·148124E03	0·285981E02	0·202261E02	0·168235E02
7·80	—0·732633E05	0·151094E03	0·300840E02	0·212794E02	0·176922E02
8·00	—0·701929E05	0·152832E03	0·300286E02	0·218070E02	0·181276E02

TABLE II

Zeros of (1) for M = 2.40

k/η	0·0	0·2	0·5	0·7	0·9
1·00	0·731463E08	0·191754E05	0·397805E03	0·154726E03	0·289807E03
1·40	0·178198E12	0·118301E05	0·115334E03	0·416299E02	0·700130E02
1·80	—0·111272E12	0·101666E05	0·570758E02	0·209996E02	0·308407E02
2·00	—0·143000E10	0·969099E04	0·449243E02	0·171477E02	0·234152E02
2·40	—0·232362E08	0·878666E04	0·322834E02	0·136142E02	0·161788E02
2·80	—0·253584E07	0·780775E04	0·267280E02	0·124932E02	0·132448E02
3·00	—0·122466E07	0·730418E04	0·252593E02	0·123569E02	0·125163E02
3·40	—0·380328E06	0·632261E04	0·237924E02	0·125438E02	0·118439E02
3·80	—0·161144E06	0·542566E04	0·235272E02	0·130960E02	0·117861E02
4·00	—0·113446E06	0·501928E04	0·236777E02	0·134562E02	0·118966E02
4·40	—0·629484E05	0·429632E04	0·243541E02	0·142864E02	0·122974E02
4·80	—0·391522E05	0·368868E04	0·253765E02	0·152168E02	0·128605E02
5·00	—0·319233E05	0·342447E04	0·259784E02	0·157080E02	0·131838E02
5·40	—0·223972E05	0·296614E04	0·273133E02	0·167279E02	0·138901E02
5·80	—0·166776E05	0·258884E04	0·287775E02	0·177848E02	0·146546E02
6·00	—0·146647E05	0·242620E04	0·295458E02	0·183236E02	0·150531E02
6·40	—0·117103E05	0·214514E04	0·311375E02	0·194170E02	0·158746E02
6·80	—0·970710E04	0·191407E04	0·327863E02	0·205269E02	0·167214E02
7·00	—0·894652E04	0·181438E04	0·336273E02	0·210867E02	0·171523E02
7·40	—0·776389E04	0·164178E04	0·353358E02	0·222139E02	0·180256E02
7·80	—0·690857E04	0·149948E04	0·370726E02	0·233495E02	0·189115E02
8·00	—0·657081E04	0·143795E04	0·379495E02	0·239198E02	0·193583E02

TABLE III
Zeros for M = 2.80

k/η	0.0	0.2	0.5	0.7	0.9
1·00	0·336456E07	0·144469E06	0·122417E05	0·576206E04	0·107265E05
1·40	0·740119E08	0·793582E05	0·172643E04	0·596910E03	0·981985E03
1·80	0·553405E14	0·100410E06	0·560843E03	0·155236E03	0·230249E03
2·00	—0·397515E12	0·130002E06	0·383877E03	0·984698E02	0·139405E03
2·40	—0·381917E08	0·238197E06	0·221778E03	0·520064E02	0·674538E02
2·80	—0·128837E07	0·434086E06	0·151560E03	0·347848E02	0·415134E02
3·00	—0·435626E06	0·572007E06	0·130368E03	0·301915E02	0·346138E02
3·40	—0·905740E05	0·932868E06	0·102041E03	0·247451E02	0·262921E02
3·80	—0·302121E05	0·139737E07	0·846732E02	0·220131E02	0·218477E02
4·00	—0·195189E05	0·165992E07	0·785251E02	0·212297E02	0·204374E02
4·40	—0·950604E04	0·222058E07	0·696136E02	0·203914E02	0·186121E02
4·80	—0·538051E04	0·279188E07	0·638787E02	0·202003E02	0·176571E02
5·00	—0·422340E04	0·306843E07	0·618648E02	0·202698E02	0·173983E02
5·40	—0·278138E04	0·358179E07	0·590999E02	0·206426E02	0·171868E02
5·80	—0·196854E04	0·402170E07	0·576121E02	0·212433E02	0·172712E02
6·00	—0·169346E04	0·420918E07	0·572339E02	0·216069E02	0·173948E02
6·40	—0·130139E04	0·451507E07	0·570463E02	0·224296E02	0·177640E02
6·80	—0·104419E04	0·472937E07	0·574577E02	0·233507E02	0·182583E02
7·00	—0·948527E03	0·480381E07	0·578414E02	0·238399E02	0·135416E02
7·40	—0·802039E03	0·489276E07	0·588947E02	0·248634E02	0·191654E02
7·80	—0·697819E03	0·491076E07	0·602575E02	0·259351E02	0·198498E02
8·00	—0·657039E03	0·489673E07	0·610334E02	0·264855E02	0·202102E02

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