ON THE SET OF POLYNOMIALS $G_n^\lambda(x)$

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The polynomials $G_n^\lambda(x)$ considered in this paper are defined by

$$e^t(1-tx)^\lambda = \sum_{n=0}^{\infty} G_n^\lambda(x)\frac{t^n}{n!}$$

For these polynomials we establish a bilateral generating relation. We also obtained expansions of Hermite polynomials $H_n(x)$ and Legendre polynomial $P_n(x)$ in terms of $G_n^\lambda(x)$. Lastly a finite sum for $G_n^\lambda(xy)$ in terms of $G_n^\lambda(x)$ is derived.

1. INTRODUCTION

In a recent paper Prabhakar and Rekha (*in press*) have obtained generating functions and some other results on polynomials $G_n^\lambda(x)$ defined by the generating relation

$$\sum_{n=0}^{\infty} G_n^\lambda(x)\frac{t^n}{n!} = e^t(1-tx)^\lambda. \qquad ...(1.1)$$

These polynomials possess a number of properties analogous to those possessed by most of the standard polynomials such as Laguerre polynomials $L_n^\alpha(x)$, Hermite polynomials $H_n(x)$, Bassel polynomials $Y_n(x)$, Charlier polynomials $C_n(x, a)$ (see Rainville 1960 and Erdélyi 1953), biorthogonal polynomials $Z_n^\alpha(x)$ and $Y_n^\alpha(x)$ (see Prabhakar 1970, 1971), and the polynomials set $L_n^{\alpha,\beta}(x)$ studied by Prabhakar and Rekha (1972). In this paper we derive for $G_n^\lambda(x)$, a bilateral generating function and give expansions of $H_n(x)$ and the Legendre polynomials $P_n(x)$ in terms of $G_n^\lambda(x)$.

2. A BILATERAL GENERATING FUNCTION

We prove the relation

$$(1-tx)^{\lambda+c}[1-t(x+y)]^{-c}\exp.\ t+(1-tx)\ {}_2F_0\left[\begin{matrix}c,-\lambda\\ -,-\end{matrix};\frac{xyt}{1-t(x+y)}\right]$$

$$\cong \sum_{n=0}^{\infty} {}_2F_0\left[\begin{matrix}-n,c\\ -,-\end{matrix};-y\right]G_n^\lambda(x)\,t^n \qquad ...(2.1)$$

where $\cong$ indicates the divergence of the series in the generating relation.

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Prabhakar and Rekha (*in press*) have established the generating relation

$$(1-t)^{-c}\, {}_2F_0\left[\begin{matrix} c, -\lambda \\ -, - \end{matrix}; \frac{xt}{1-t}\right] \cong \sum_{k=0}^{\infty} \frac{(c)_k G_k^{\lambda}(x)\, t^k}{k\,!} \qquad \ldots(2.2)$$

This can be proved by using hypergeometric representation (Prabhakar and Rekha *in press*).

$$G_n^{\lambda}(x) = {}_2F_0(-n, -\lambda; - ; -x) \qquad \ldots(2.3)$$

which can be obtained from (1.1).

Now in (2.2) replacing t by $yt/(1-tx)$ and multiplying both sides by $(1-tx)^{\lambda} \exp . t + (1-tx)$, we get

$$(1-tx)^{\lambda}\left[1 - \frac{yt}{1-tx}\right]^{-c} \exp . t + (1-tx)\, {}_2F_0\left[\begin{matrix} c, -\lambda \\ -, - \end{matrix}; \frac{xyt}{1-t(x+y)}\right]$$

$$\cong \sum_{k=0}^{\infty} \frac{(c)_k (1-tx)^{\lambda-k} y^k t^k}{k\,!} G^{\lambda}(x) . \exp . t + (1-tx)$$

$$\cong \sum_{n,k=0}^{\infty} \frac{(c)_k y^k t^k (n+k)\,!\, G_{n+k}^{\lambda}(x)\, t^n}{n\,!\, k\,!},$$

using (2.5) of Prabhakar and Rekha (*in press*)

$$\cong \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(c)_k y^k t^k n\,!\, G_n^{\lambda}(x) t^{n-k}}{k\,!\,(n-k)\,!}$$

$$\cong \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(c)_k (-n)_k (-y)^k}{k\,!} G_n^{\lambda}(x)\, t^n$$

$$\cong \sum_{n=0}^{\infty} {}_2F_0\left[\begin{matrix} -n, c \\ -, - \end{matrix}; -y\right] G_n^{\lambda}(x)\, t^n.$$

This is the required bilateral generating relation.

3. Expansions in Terms of $G_n^{\lambda}(x)$

We now give expansions of the polynomials $H_n(x)$ and $P_n(x)$ in finite series of $G_n^{\lambda}(x)$. We first obtain such an expansion of x^n.

From the generating relation (1.1), we get

$$(1 - tx)^\lambda = e^{-t} \sum_{n=0}^{\infty} \frac{G_n^\lambda(x)\, t^n}{n\,!}$$

i.e.,

$$\sum_{n=0}^{\infty} \frac{(-\lambda)_n\, x^n\, t^n}{n\,!} = \sum_{n=0}^{\infty} \frac{(-1)^n\, t^n}{n\,!} \sum_{k=0}^{\infty} \frac{G_k^\lambda(x)\, t^k}{k\,!}$$

$$\sum_{n=0}^{\infty} \frac{(-\lambda)_n x^n t^n}{n\,!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^{n-k}\, G_k^\lambda(x)\, t^n}{(n-k)\,!\,k\,!}.$$

Equating coefficients of t^n, we obtain

$$x^n = \sum_{k=0}^{n} \frac{(-1)^n (-n)_k}{k\,!\,(-\lambda)_n} \times G_k^\lambda(x). \qquad \ldots(3.1)$$

On using [Erdélyi 1953, 10.13 (19)]

$$\sum_{n=0}^{\infty} H_n(x)\, t^n/n\,! = \exp(2xt - t^2)$$

and (3.1), it is not difficult to obtain the following expansion for Hermite polynomials:

$$H_n(x) = \sum_{k=0}^{n} {}_4F_0\left[\begin{matrix} -\frac{1}{2}(n-k), -\frac{1}{2}(n-k-1), \frac{1}{2}(1+\lambda-n), \frac{1}{2}(2+\lambda-n) \\ -\,,\quad -\,,\quad -\,,\quad - \end{matrix}; -4\right]$$

$$\times \frac{(-1)^n\, 2^n\, G_k^\lambda(x)\, (-n)_k}{k\,!\,(-\lambda)_n}.$$

In a similar way from the generating relation [Erdélyi 1953, 10.10 (39)]

$$\sum_{n=0}^{\infty} P_n(x)\, t^n = (1 - 2xt + t^2)^{-1/2} - 1 < x < 1, \quad |t| < 1$$

and (3.1), one would get

$$P_n(x) \cong \sum_{k=0}^{n} {}_4F_1\left[\begin{matrix} -\frac{1}{2}(n-k), -\frac{1}{2}(n-k-1), \frac{1}{2}(1+\lambda-n), \frac{1}{2}(2+\lambda-n) \\ \frac{1}{2} - n \end{matrix}; -4\right]$$

$$\times \frac{(\frac{1}{2})_n 2^n\, G_k^\lambda(x)\, (-n)_k}{k\,!\,n\,!\,(-\lambda)_n}$$

4. A Finite Sum

Since

$$e^{t}(1-txy)^{\lambda}=e^{t}\,{}_1F_0(-\lambda\,;\,-\,;\,xyt)$$
$$=e^{yt}\,e^{(1-y)\,t}\times{}_1F_0(-\lambda;\,-\,;\,xyt)$$

we get

$$\sum_{n=0}^{\infty}\frac{G_n^{\lambda}(xy)\,t^n}{n\,!}=\sum_{n=0}^{\infty}\frac{(1-y)^n\,t^n}{n\,!}\times\sum_{k=0}^{\infty}\frac{G_k^{\lambda}(x)\,y^k\,t^k}{k\,!}$$

using (1.1) of Prabhakar and Rekha (*in press*).

Equating coefficients of t^n

$$G_n^{\lambda}(xy)=\sum_{k=0}^{n}\frac{(-1)^k(-n)_k(1-y)^{n-k}\,G_k^{\lambda}(x)\,y^k}{k\,!}$$

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