A CERTAIN CLASS OF ANALYTIC FUNCTIONS IN THE UNIT DISC

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Let $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k$ and $g(z) = z + \sum_{k=n+1}^{\infty} b_k z^k$ be analytic and satisfy

$$\left| \frac{f(z)}{\lambda f(z) + (1-\lambda) g(z)} - \alpha \right| < \alpha, \frac{1}{2} < \alpha \leqslant \frac{1}{1+\lambda};$$
$$0 \leqslant \lambda < 1; \text{ for } |z| < 1.$$

We determine in this paper the radius of starlikeness of $f(z)$ under the following conditions.

(i) $\operatorname{Re}\left\{\frac{g(z)}{z}\right\} > 0;$

(ii) $\operatorname{Re}\left\{\frac{g(z)}{z}\right\} \frac{1}{2}; n = 1$

(iii) $\operatorname{Re}\left\{\frac{z g'(z)}{g(z)}\right\} \beta (0 \leqslant \beta < 1).$

We also determine the radius of univalence of $f(z)$ under the following condition:

(iv) $g(z)$ is simply univalent.

We also find a lower bound for the radius of starlikeness of $f(z)$ under the conditions (i), (ii), (iii), and (iv) in the case when $\alpha \geqslant 1$; $0 \leqslant \lambda < 1$.

Let

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \text{ and } g(z) = z + \sum_{k=n+1}^{\infty} b_k z^k \text{ be analytic and satisfy}$$

$$\left| \frac{f(z)}{\lambda f(z) + (1-\lambda)\, g(z)} - \alpha \right| < \alpha; \tfrac{1}{2} < \alpha \leqslant \frac{1}{1+\lambda};$$
$$0 \leqslant \lambda < 1; \text{ for } |z| < 1.$$

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We propose to determine the radius of starlikeness of $f(z)$ under the following conditions

(i) $\mathrm{Re}\left\{\frac{g(z)}{z}\right\} > 0$;

(ii) $\mathrm{Re}\left\{\frac{g(z)}{z}\right\} > \frac{1}{2}$; $n = 1$;

(iii) $\mathrm{Re}\left\{\frac{zg'(z)}{g(z)}\right\} > \beta \quad (0 \leqslant \beta < 1)$

We also determine the radius of univalence of $f(z)$ under the assumption that $g(z)$ is simply univalent.

We also find a lower bound for the radius of starlikeness of $f(z)$ in the cases (i), (ii) and (iii) and give an estimate for the radius of univalence of $f(z)$ under the assumption $g(z)$ is univalent, when $\alpha \geqslant 1$ and $0 \leqslant \lambda < 1$. The results of Merkes and Causey (1970) and Padmanabhan (1972) are special cases of ours.

We need the following lemmas.

Lemma 1—If $\phi(z) = C_n z^n + C_{n+1} z^{n+1} + \ldots$ is analytic and satisfy $|\phi(z)| < 1$ for $|z| < 1$, then

(i) $|\phi(z)| \leqslant |z|^n$;

(ii) $|\phi'(z)| \leqslant \dfrac{n|z|^{n-1}[1-|\phi(z)|^2]}{1-|z|^{2n}}$.

PROOF: (i) is an iterated form of Schwarz lemma and (ii) is a result of Goluzin (1945).

Lemma 2—Suppose $G(z)$ is analytic for $|z| < 1$ and $G(0) = 1$. Let $|G(z) - \alpha| < \alpha$ where α is real and $\alpha > \frac{1}{2}$.

Let

$$G(z) = 1 + d_n z^n + d_{n+1} z^{n+1} + \ldots$$

Then

$$G(z) = \frac{1 + \phi(z)}{1 + \dfrac{1-\alpha}{\alpha}\phi(z)}$$

where $\phi(z)$ is analytic and satisfies

$$|\phi(z)| \leqslant |z|^n; \; |\phi'(z)| \leqslant \frac{n|z|^{n-1}[1-|\phi(z)|^2]}{1-|z|^{2n}} \text{ for } |z| < 1.$$

PROOF : Writing $h(z) = \dfrac{G(z) - \alpha}{\alpha}$ and setting

$$\phi(z) = \frac{h(z) - h(0)}{1 - h(z)\,h(0)}$$

we see that $\phi(z)$ and $h(z)$ are regular for $|z| < 1$.

We see that $\phi(z)$ satisfies the hypothesis of lemma because

$$|h(z)| < 1 \text{ for } |z| < 1.$$

Substituting we get the result required.

Lemma 3—If $g(z) = z + b_2 z^2 + \ldots$ is analytic and univalent for $|z| < 1$ then we have

$$\left|\frac{g(z_2) - g(z_1)}{z_2 - z_1}\right| \geqslant \frac{|g(z_1)|\,|g(z_2)|\,(1 - r^2)}{r^2} \quad \text{for } |z_1| = |z_2| = r.$$

PROOF : May be found in Goluzin (1946).

Lemma 4—(i) If $p(z) = 1 + c_1 z + c_2 z^2 + \ldots$ be regular for $|z| < 1$ and satisfies Re $\{p(z)\} > \frac{1}{2}$ for $|z| < 1$.

Then

$$\text{Re}\left\{\frac{zp'(z)}{p(z)}\right\} \geqslant \begin{cases} \dfrac{-r}{1+r}\,; & |z| = r < \dfrac{1}{3} \\ -\dfrac{(\sqrt{2} - \sqrt{1 - r^2})^2}{1 - r^2}\,; & |z| = r;\ 1/3 \leqslant r \leqslant \sqrt{8\sqrt{2} - 11}. \end{cases}$$

(ii) If $q(z) = 1 + d_n z^n + \ldots$ be regular for $|z| < 1$ and

$$\text{Re } [q(z)] > 0 \text{ for } |z| < 1;$$

then

$$\left|\frac{zq'(z)}{q(z)}\right| \leqslant \frac{2nr}{1 - r^2}\,; \quad |z|^n = r < 1.$$

PROOF : (i) can be found in Merkes and Causey (1970).
(ii) can be found in Shah (1972).

Theorem 1*a*—Let

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k$$

and

$$g(z) = z + \sum_{k=n+1}^{\infty} b_k z^k$$

be regular and satisfy

$$\left| \frac{f(z)}{\lambda f(z) + (1-\lambda) g(z)} - \alpha \right| < \alpha$$

where

$$\frac{1}{2} < \alpha \leqslant \frac{1}{1+\lambda}\,; \quad 0 \leqslant \lambda < 1. \quad \text{for } |z| < 1.$$

Let

$$\operatorname{Re} \left\{ \frac{g(z)}{z} \right\} > 0 \text{ for } |z| < 1.$$

Then $f(z)$ is starlike for $|z| < R^{1/n}$ where R is the least positive root of the equation

$$(t-\lambda) r^3 + r^2 \; [2n(t-\lambda) - (1-\lambda) - n(1-t)] - r \; [(t-\lambda)$$
$$+ zn(1-\lambda) + n(1-t)] + (1-\lambda) = 0$$

where

$$t = \frac{1-\alpha}{\alpha}$$

and the bound is sharp.

PROOF: Let

$$h(z) = \frac{f(z)}{\lambda f(z) + (1-\lambda) g(z)}\,.$$

Then $h(z)$ satisfies $|h(z) - \alpha| < \alpha$ for $|z| < 1$; $h(z)$ is regular and we have by Lemma 1

$$h(z) = \frac{1 + \phi(z)}{1 + t\phi(z)}\,; \; t = \frac{1-\alpha}{\alpha}$$

where $\phi(z)$ is regular and satisfies $|\phi(z)| \leqslant |z|^n$ and

$$|\phi'(z)| \leqslant \frac{n\,|z|^{n-1}(1 - |\phi(z)|^2)}{1 - |z|^{2n}}\,.$$

$$f(z) = \frac{(1-\lambda) g(z)\, h(z)}{[1 - \lambda h(z)]}\,;$$

let $g(z) = zp(z)$;

$$\therefore \frac{zf'(z)}{f(z)} = 1 + \frac{zp'(z)}{p(z)} + \frac{zh'(z)}{h(z)\;[1 - \lambda h(z)]}$$

$\operatorname{Re}\,(p(z)) > 0$ for $|z| < 1$;

$$\therefore \ \mathrm{Re}\left\{\frac{zp'(z)}{p(z)}\right\} \geqslant -\left|\frac{zp'(z)}{p(z)}\right| \geqslant -\frac{2nr}{1-r^2}; \quad r = |z|^n < 1 \ \text{[Lemma 4]}.$$

$$\mathrm{Re}\ \frac{zh'(z)}{h(z)\ (1-\lambda h(z))} \leqslant \left|\frac{zh'(z)}{h(z)\ (1-\lambda h(z))}\right|$$

$$= \left|\frac{(1-t)\,z\phi'(z)}{[1+\phi(z)]\,[(1-\lambda)+(t-\lambda)\,\phi(z)]}\right|$$

$$\leqslant \frac{(1-t)\,nr\,(1-|\phi(z)|^2)}{(1-|\phi(z)|)\,(1-\lambda-(t-\lambda)\,|\phi(z)|)\,(1-r^2)}\,.$$

We have used the fact that

$$|\phi'(z)| \leqslant [n\,|z|^{n-1}\,(1-|\phi(z)|^2)]/(1-|z|^{2n}).$$

$$\therefore \quad \mathrm{Re}\ \left\{\frac{zf'(z)}{f(z)}\right\} \geqslant 1-\frac{2nr}{1-r^2}-\frac{(1-t)\,nr}{(1-r)\,(1-\lambda-(t-\lambda)\,r)} \geqslant 0$$

provided

$$Q(r) \equiv (t-\lambda)\,r^3 + r^2\ [-(1-\lambda)+2n(t-\lambda)-n(1-t)]$$
$$+\,r\,[-(t-\lambda)-2n(1-\lambda)-n(1-t)]+1-\lambda \geqslant 0.$$

$$Q(0) = (1-\lambda) > 0; \quad Q(1) < 0.$$

Therefore there exists a least positive root R of the equation $Q(r) = 0$.

$\therefore$ $f(z)$ is starlike for $|z| < R^{1/n}$.

We show that the bound is sharp.

Let

$$g(z) = \frac{z(1-z^n)}{1+z^n}.$$

We have

$$p(z) = \frac{g(z)}{z} = \frac{1-z^n}{1+z^n}.$$

$$\frac{zp'(z)}{p(z)} = \frac{-2n\,R}{1-R^2} \text{ at } z^n = R; \text{ let } h(z) = \frac{1-z^n}{1-tz^n} \quad t = \frac{1-\alpha}{\alpha},$$

We have

$$\frac{zh'(z)}{h(z)\,(1-\lambda h(z))} = \frac{-nR(1-t)}{(1-R)\,(1-\lambda-(t-\lambda)\,R)} \text{ at } z^n = R.$$

If

$$f(z) = \frac{(1-\lambda)\ g(z)\,h(z)}{[1-\lambda h(z)]},$$

we see that

$$\frac{zf'(z)}{f(z)} = 0 \text{ at } z^n = R$$

and this shows that the bound R is sharp.

Theorem 1*b*—Suppose $f(z) = z + a_{n+1} z^{n+1} + \ldots$ and

$$g(z) = z + b_{n+1} z^{n+1} + \ldots$$

are regular and satisfy

$$\left| \frac{f(z)}{\lambda f(z) + (1-\lambda) g(z)} - \alpha \right| < \alpha;$$

for some $\alpha \geqslant 1$ and for $|z| < 1$

where $0 \leqslant \lambda < 1$;

and

$$\text{Re} \left\{ \frac{g(z)}{z} \right\} > v; \quad 0 \leqslant v < \tfrac{1}{2}.$$

Then $f(z)$ is univalent and starlike for $|z| < R^{1/n}$ where R is the least positive root of the equation

$$\begin{aligned}
&(1-2v)\ (\alpha-1)\ (\alpha-1+\alpha\lambda) r^4 + r^3\ [2v(\alpha-1)\ (\alpha-1+\alpha\lambda) \\
&\quad + (1-2v)\ (2\alpha^2\lambda - \alpha\lambda) + 2n(1-v)\ (\alpha-1)\ (\alpha-1+\alpha\lambda) \\
&\quad + (1-2v)\ (\alpha-1)(2\alpha-1)\, n] + r^2\ [n(\alpha-1)(2\alpha-1) \\
&\quad - (\alpha-1)\ (\alpha-1+\alpha\lambda) - (1-2v)\,\alpha^2(1-\lambda) + 2v(2\alpha^2\lambda - \alpha\lambda) \\
&\quad + 2n\ (1-v)(2\alpha^2\lambda - \alpha\lambda) - \alpha n(1-2v)(2\alpha-1) + r\,[\alpha\lambda - 2\alpha^2\lambda \\
&\quad - \alpha n(2\alpha-1) - \alpha^2(1-\lambda)(2v + 2n - 4nv)] + \alpha^2(1-\lambda) = 0.
\end{aligned}$$

PROOF: Let

$$h(z) = \frac{f(z)}{\lambda f(z) + (1-\lambda) g(z)}; \ = 1 + c_n z^n + \ldots.$$

$h(z)$ satisfies $|h(z) - \alpha| < \alpha$; for $|z| < 1$.

Therefore by Lemma 1,

$$h(z) = \frac{1 + \phi(z)}{1 + t\phi(z)}; \ t = \frac{1-\alpha}{\alpha}$$

$\phi(z)$ is regular and satisfies $|\phi(z)| \leqslant |z|^n$; and

$$|\phi'(z)| \leqslant \frac{n|z|^{n-1}(1 - |\phi(z)|^2)}{1 - |z|^{2n}}; \text{ for } |z| < 1.$$

Now

$$f(z) = \frac{(1-\lambda)\, zp(z)\, h(z)}{[1-\lambda h(z)]}$$

where $g(z) = zp(z)$.

$$\therefore \frac{zf'(z)}{f(z)} = 1 + \frac{zp'(z)}{p(z)} + \frac{zh'(z)}{h(z)\,(1-\lambda h(z))} .$$

$$\therefore \left| \frac{zf'(z)}{f(z)} - 1 \right| \leqslant \frac{2n\,|z|^n (1-v)}{(1-|z|^n)\,(1+(1-2v)\,|z|^n)}$$

$$+ \left| \frac{zh'(z)}{h(z)} \right| \frac{1}{|1-\lambda h(z)|}. \text{ (Shah 1972).}$$

We know

$$|\phi'(z)| \leqslant n\,|z|^{n-1}\,[1-|\phi(z)|^2]/(1-|z|^{2n}).$$

Substituting the expressions for $\phi(z)$ and $\phi'(z)$ in terms of $h(z)$ and simplifying we get

$$|h'(z)| \leqslant \frac{\alpha}{(2\alpha-1)} \frac{n\,|z|^{n-1}}{1-|z|^{2n}} \left\{ \left(\frac{2\alpha-1}{\alpha}\right) \left[2\mathrm{Re}\, h(z) - \frac{|h(z)|^2}{\alpha} \right] \right\} .$$

$$\therefore \left| \frac{z\,h'(z)}{h(z)} \right| \leqslant \frac{n\,|z|^n}{1-|z|^{2n}} \left\{ 2 - \frac{1-|z|^n}{\alpha \left[1 + \frac{1-\alpha}{\alpha}\,|z|^n \right]} \right\}$$

as

$$|h(z)| \geqslant \frac{1-|z|^n}{1+\frac{\alpha-1}{\alpha}\,|z|^n}$$

from the repesentation.

Also

$$|h(z)| \leqslant \frac{1+|z|^n}{1-\left(\frac{\alpha-1}{\alpha}\right)|z|^n} .$$

$$\therefore \left| \frac{zf'(z)}{f(z)} - 1 \right| \leqslant \frac{2n\,|z|^n (1-v)}{(1-|z|^n)\,[1+(1-2v)\,|z|^n]}$$

$$+ \frac{n\,|z|^n}{1-|z|^{2n}} \left\{ 2 - \frac{(1-|z|^n}{\alpha+(\alpha-1)\,|z|^n} \right\} \left\{ \frac{\alpha-(\alpha-1)\,|z|^n}{(\alpha-\alpha\lambda)-(\alpha-1+\alpha\lambda)\,|z|^n} \right\}$$

This last inequality is valid for

$$|z|^n = r < \frac{(1-\lambda)\,\alpha}{\alpha\lambda+\alpha-1} .$$

The right-hand member of the above inequality can be made less than one [after a lengthy but straightforward calculation] provided

$$Ar^4 + Br^3 + Cr^2 + Dr + E > 0 \qquad \text{...(1)}$$

where

$$A = (1 - 2v)\ (\alpha - 1)\ (\alpha - 1 + \alpha\lambda)$$

$$B = 2v\ (\alpha - 1)\ (\alpha - 1 + \alpha\lambda) + (1 - 2v)\ (2\alpha^2 \lambda - \alpha\lambda) + 2n\,(1 - v) \times (\alpha - 1)\ (\alpha - 1 + \alpha\lambda) + n\,(1 - 2v)\ (\alpha - 1)\ (2\alpha - 1)$$

$$C = -(\alpha - 1)\ (\alpha - 1 + \alpha\lambda) - (1 - 2v)\,\alpha^2\,(1 - \lambda) + 2v\ (2\alpha^2 \lambda - \alpha\lambda) + 2n\,(1 - v)\ (2\alpha^2 \lambda - \alpha\lambda) - \alpha n\,(1 - 2v)\,(2\alpha - 1) + n\,(\alpha - 1)\,(2\alpha - 1).$$

$$D = \alpha\lambda - 2\alpha^2 \lambda - \alpha n\,(2\alpha - 1) - \alpha^2\,(1 - \lambda)\ (2v + 2n - 4nv)$$

$$E = \alpha^2\,(1 - \lambda).$$

Denoting the left member of the inequality (1) by $F(r)$ we see that $F(0) = E > 0$;

$$F\left[\frac{(1 - \lambda)\,\alpha}{\alpha\lambda + \alpha - 1}\right] < 0 \text{ if } \alpha\lambda \geqslant \tfrac{1}{2} \text{ and if } \alpha\lambda < \tfrac{1}{2}\ F(1) < 0.$$

There exists a least positive root R of the equation $F(r) = 0$ in $0 < r < 1$. Therefore $f(z)$ is starlike for $|z| < R^{1/n}$.

Special cases :

(1) $\alpha = 1$; $v = 0$, we get Theorem 4 of Shah (1972).

(2) Allowing $\alpha \to \infty$ and $v = 0$, we get Theorem 1 of Shah (1972).

(3) Let $v = 0$; $\lambda = 0$; $n = 1$, we get Theorem 1 of Padmanabhan (1972).

Note : (1) When $\lambda = 0$; $n = 1$; $v = 0$; the result is sharp for all α. This was shown by Padmanabhan (1972).

Theorem 2a—Let

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

and

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k$$

be regular and satisfy

$$\left|\frac{f(z)}{\lambda f(z) + (1 - \lambda)\,g(z)} - \alpha\right| < \alpha$$

where

$$\frac{1}{2} < \alpha \leqslant \frac{1}{1+\lambda}\,; \quad 0 \leqslant \lambda < 1 \text{ for } |z| < 1.$$

Let

$$\operatorname{Re}\left\{\frac{g(z)}{z}\right\} > \frac{1}{2} \text{ for } |z| < 1.$$

Then $f(z)$ is starlike for $|z| < R_1$ where R_1 is the smallest positive root of the equation

$$\begin{aligned} r^4 & [(1-t)^2 + 8(t-\lambda)^2] + r^3 [2(1-t)(1-3t+2\lambda) - 16(1-\lambda) \\ & \times (t-\lambda)] + r^2 [(1-3t+2\lambda)^2 + 4(1-t)(1-\lambda) - 8(t-\lambda)^2 \\ & + 8(1-\lambda)^2] + r[4(1-\lambda)(1-3t+2\lambda) + 16(1-\lambda)(\lambda-t)] \\ & - 4(1-\lambda)^2 = 0 \end{aligned}$$

and the result is sharp.

PROOF: As in Theorem 1 we have,

$$\frac{zf'(z)}{f(z)} = 1 + \frac{zp'(z)}{p(z)} + \frac{zh'(z)}{h(z)\big(1-\lambda h(z)\big)} \quad \text{where } g(z) = z\,p(z).$$

Now

$$\operatorname{Re}\,(p(z)) > \tfrac{1}{2}.$$

$$\therefore \left|\frac{zp'(z)}{p(z)}\right| < \frac{r}{1-r}\,; \quad |z| = r < 1$$

and

$$\left|\frac{zh'(z)}{h(z)\big(1-\lambda h(z)\big)}\right| \leq \frac{(1-t)\,r}{(1-r)\big(1-\lambda-(t-\lambda)\,r\big)} < \frac{r}{1-r}$$

as $r < 1$.

$$\therefore \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqslant 1 - \frac{r}{1-r} - \frac{r}{1-r} \geqslant 0 \text{ for } r \leqslant \frac{1}{3}\,.$$

But to see that f can be starlike for $|z| < R_1$ where $R_1 > 1/3$, we sharpen the estimate for

$$\operatorname{Re}\left\{\frac{zp'(z)}{p(z)}\right\} \text{ for } r \geqslant \frac{1}{3}\,.$$

By Lemma 4

$$\operatorname{Re}\left\{\frac{zp'(z)}{p(z)}\right\} \geqslant -\frac{(\sqrt{2}-\sqrt{1-r^2})^2}{1-r^2}\ ;\ \frac{1}{3} \leqslant r \leqslant \sqrt{8\sqrt{2}-11}\,.$$

$$\therefore\ \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqslant 1 - \frac{(\sqrt{2}-\sqrt{1-r^2})^2}{1-r^2} - \frac{(1-t)\,r}{(1-r)\,[(1-\lambda)-(t-\lambda)\,r]}$$

$$\therefore\ \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqslant 0 \text{ as } |z| = r$$

and hence

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \text{ for } |z| < r$$

provided

$$\begin{aligned} Q_1(r) \equiv {} & r^4\,[(1-t)^2 + 8\,(t-\lambda)^2] + r^3\,[2\,(1-t)\,(1-3t+2\lambda) \\ & - 16\,(1-\lambda)\,(t-\lambda)] + r^2\,[(1-3t+2\lambda)^2 + 4\,(1-\lambda)\,(1-t) \\ & - 8\,(t-\lambda)^2 + 8\,(1-\lambda)^2] + r\,[4\,(1-\lambda)\,(1-3t+2\lambda) + 16\,(1-\lambda) \\ & \times (t-\lambda)] - 4\,(1-\lambda)^2 \leqslant 0. \end{aligned}$$

$$Q_1\left(\frac{1}{3}\right) < 0 \text{ as } \alpha < \frac{1}{2\lambda}\ ;\ Q_1\,(\sqrt{8\sqrt{2}-11}) > 0.$$

Therefore there exists a least positive root R_1 in

$$\frac{1}{3} \leqslant r \leqslant \sqrt{8\sqrt{2}-11}.$$

Therefore $f(z)$ is starlike for $|z| < R_1$.

The result is sharp. Let

$$\frac{g(z)}{z} = p(z) = \frac{1+b_1 z}{1+2b_1 z+z^2}$$

where

$$b_1 = \frac{p-r^2}{r\,(1-p)} \quad p = \sqrt{2(1-r^2)} - 1.$$

$$h(z) = \frac{1-z^n}{1-tz^n}\ ;\quad t = \frac{1-\alpha}{\alpha}\,.$$

If

$$f(z) = \frac{(1-\lambda)\,g(z)\,h(z)}{[1-\lambda\,h(z)]}\,.$$

then $f(z)$ satisfies the hypothesis of the theorem and

$$\frac{zf'(z)}{f(z)} = 0 \text{ at } z = R_{\cdot}.$$

This completes the proof.

For the use in Theorem 2*b* we introduce the following notations for the constants involved :

$$A = (\alpha-1)^2 \ [(2\alpha-1)^2 + 8(\alpha+\alpha\lambda-1)^2]$$

$$B = (2\alpha-1)(\alpha-1) \ \{(\alpha+\alpha\lambda-1)(16\alpha\lambda+4\alpha-4) - 2(2\alpha-1)\}$$

$$C = -4(\alpha-1)^2 \ (\alpha+\alpha\lambda-1)^2 + (2\alpha-1)^2 [1-2\alpha \ (1-2\lambda) \ (\alpha-1) + 8\alpha^2\lambda^2] - (\alpha-1) \ (\alpha+\alpha\lambda-1) \ [4(2\alpha-1) + 16\alpha^2(1-\lambda)]$$

$$D = 2\alpha(1-2\lambda)(2\alpha-1)^2 - 4\alpha(2\alpha-1)(\alpha-1)(\alpha+\alpha\lambda-1)(1+2\lambda) - 4\alpha^2(2\alpha-1) \ (1-\lambda) \ (\alpha-1+\alpha\lambda)$$

$$E = (2\alpha-1)^2\alpha^2 \ (1-4\lambda-4\lambda^2) + 4\alpha^2(2\alpha-1) \ (1-\lambda) + 8\alpha^4(1-\lambda)^2 + 8\alpha^2 \ (1-\lambda) \ (\alpha-1) \ (\alpha+\alpha\lambda-1)$$

$$F = 4\alpha^2 \ (1-\lambda) \ (2\alpha-1) \ [\alpha \ (1-2\lambda) + 4\alpha\lambda(2\alpha-1)]$$

$$G = -4\alpha^4 \ (1-\lambda)^2.$$

Theorem 2*b*—Let

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

and

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k$$

be analytic and satisfy

$$\left| \frac{f(z)}{\lambda f(z) + (1-\lambda) g(z)} - \alpha \right| < \alpha ; \ \alpha \geqslant 1 ; \ 0 \leqslant \lambda < 1 ;$$

for $|z| < 1$. Let

$$\operatorname{Re}\left\{\frac{g(z)}{z}\right\} > \frac{1}{2} \text{ for } |z| < 1.$$

Then

(i) If $4\alpha\lambda(1-4\alpha) + (1+2\alpha) \leqslant 0$ then $f(z)$ is starlike for $|z| < x$ where x is the least positive root of the equation

$$r^3(\alpha-1) \ (3\alpha-2+\alpha\lambda) + r^2 \ [-\alpha^2(1-\lambda)] - r\,[\alpha(3\alpha-1-\lambda + \alpha\lambda)] \ + \alpha^2 \ (1-\lambda) = 0.$$

(ii) If $4\alpha\lambda(1-4\alpha)+(1+2\alpha)>0$ then $f(z)$ is starlike for $|z|<R_2$ where R_2 is the least positive root of the equation

$$Ar^6+Br^5+Cr^4+Dr^3+Er^2+Fr+G=0,$$

the symbols being explained before the start of the theorem.

PROOF : As before

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}\geqslant 1+\operatorname{Re}\frac{zp'(z)}{p(z)}-\left|\frac{zh'(z)}{h(z)\,(1-\lambda h(z))}\right|$$

where

$$zp(z)=g(z)$$

and

$$h(z)=\frac{f(z)}{\lambda f(z)+(1-\lambda)\,g(z)};$$

$$\operatorname{Re}\,(p(z))>\tfrac{1}{2}\text{ for }|z|<1.$$

$$\therefore\ \operatorname{Re}\left\{\frac{zp'(z)}{p(z)}\right\}>\frac{-r}{1+r};\quad r=|z|<\frac{1}{3}.$$

$$\therefore\ \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}\geqslant 1-\frac{r}{1+r}-\frac{r}{1-r^2}\left\{2-\frac{1-r}{\alpha+(\alpha-1)r}\right\}$$

$$\times\left\{\frac{\alpha-(\alpha-1)r}{(\alpha-\alpha\lambda)-r(\alpha+\alpha\lambda-1)}\right\}\geqslant 0\text{ as }|z|=r$$

provided

$$Q(r)\equiv r^3\,[(\alpha-1)(3\alpha-2+\alpha\lambda)]+r^2[-\alpha^2(1-\lambda)]-r[\alpha\,(3\alpha-1-\lambda+\alpha\lambda)]+\alpha^2(1-\lambda)\geqslant 0.$$

$$Q(0)>0;\quad Q(1/3)\leqslant 0\text{ if }4\alpha\lambda\,(1-4\alpha)+(1+2\alpha)\leqslant 0.$$

Under this condition $Q(r)=0$ has a least positive root x in $0<r\leqslant 1/3$.

This gives a value x, such that $f(z)$ is starlike for $|z|<x$ in the case

$$4\alpha\lambda(1-4\alpha)+(1+2\alpha)\leqslant 0.$$

However, we know by Lemma 4

$$\operatorname{Re}\left\{\frac{zp'(z)}{p(z)}\right\}\geqslant-\frac{(\sqrt{2}-\sqrt{1-r^2})^2}{1-r^2};\quad |z|=r;\quad \frac{1}{3}\leqslant r\leqslant\sqrt{8\sqrt{2}-11}.$$

Applying this estimate we get

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqslant 1 - \frac{(\sqrt{2}-\sqrt{1-r^2})^2}{1-r^2} - \frac{r}{1-r^2}\left\{2 - \frac{1-r}{\alpha+(\alpha-1)r}\right\}$$
$$\times \left\{\frac{\alpha-(\alpha-1)r}{(\alpha-\alpha\lambda)-r(\alpha+\alpha\lambda-1)}\right\}$$

valid for $\frac{1}{3} \leqslant r = |z| \leqslant \sqrt{8\sqrt{2}-11}$.

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqslant 0 \text{ on } |z| = r \text{ if}$$

$$Q_1(r) \equiv Ar^6 + Br^5 + Cr^4 + Dr^3 + Er^2 + Fr + G \leqslant 0$$

the notations being those given before the statement of the theorem.

$$Q_1(1/3) < 0 \text{ by the condition in (ii)}$$

$$Q_1(\sqrt{8\sqrt{2}-11}) > 0.$$

Therefore there exists a least positive root R_2 of the equation

$$Q_1(r) = 0 \text{ in } \tfrac{1}{3} \leqslant r \leqslant \sqrt{8\sqrt{2}-11}.$$

Therefore $f(z)$ is starlike for $|z| < R_2$.

Corollary—1. Allowing $\alpha \to \infty$ we get Theorem 2 of Shah (1972).
2. $\lambda = 0$; we get Theorem 2 of Padmanabhan (1972).

Theorem 3*a*—Let

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k$$

and

$$g(z) = z + \sum_{k=n+1}^{\infty} b_k z^k$$

be regular and satisfy

$$\left|\frac{f(z)}{\lambda f(z) + (1-\lambda) g(z)} - \alpha\right| < \alpha;\ \frac{1}{2} < \alpha \leqslant \frac{1}{1+\lambda},\ 0 \leqslant \lambda < 1,$$

for $|z| < 1$. Let $g(z)$ be univalent and starlike of order β.

Then $f(z)$ is starlike for $|z| < R^{1/n}$ where R is the least positive root of the following equation:

$$r^3\ (t-\lambda)\ (2\beta-1) + r^2\ [-(2\beta-1)\ (1-\lambda) - (2\beta-2)\ (t-\lambda)$$
$$-n(1-t)] \ - r\,[(t-\lambda) - (2\beta-2)\ (1-\lambda) + n(1-t)]$$
$$+ 1 - \lambda = 0$$

and the result is sharp.

PROOF : As before we have

$$\frac{zf'(z)}{f(z)} = \frac{zg'(z)}{g(z)} + \frac{zh'(z)}{h(z)[1-\lambda h(z)]}$$

$$g(z) = z + b_{n+1} z^{n+1} + \ldots$$

satisfies

$$\text{Re}\left\{\frac{zg'(z)}{g(z)}\right\} > \beta \text{ for } |z| < 1.$$

$$\therefore \text{Re}\left\{\frac{zg'(z)}{g(z)}\right\} \geqslant \frac{1+(2\beta-1)r}{1+r} \; ; \quad r = |z|^n \text{ (Shah 1972)}$$

$$\text{Re}\left\{\frac{zh'(z)}{h(z)\,[1-\lambda h(z)]}\right\} \leqslant \left|\frac{zh'(z)}{h(z)[1-\lambda h(z)]}\right|$$

$$\leqslant \frac{n(1-t)r}{(1-r)\left(1-\lambda-(t-\lambda)r\right)} \; ; r = |z|^n < 1.$$

$$\therefore \text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqslant \frac{1+(2\beta-1)r}{1+r} - \frac{n(1-t)r}{(1-r)[1-\lambda-(t-\lambda)r]} \geqslant 0$$

provided

$$Q_2(r) \equiv r^3 (t-\lambda)(2\beta-1) + r^2 [-(2\beta-1)(1-\lambda) - (2\beta-2) \times (t-\lambda) - n(1-t)] + r[-(t-\lambda) + (2\beta-2)(1-\lambda) - n(1-t)] + (1-\lambda) \geqslant 0.$$

$$Q_2(0) = 1-\lambda > 0; \quad Q_2(1) = -2n\,(1-t) < 0.$$

Therefore there exists a least positive root R of the equation $Q_2(r) = 0$ in $0 < r < 1$. Therefore $f(z)$ is starlike for $|z| < R^{1/n}$.

The examples

$$g(z) = \frac{z}{(1+z^n)^{2(1-\beta)/n}} \; ; \; h(z) = \frac{1-z^n}{1-tz^n}$$

and

$$f(z) = \frac{(1-\lambda)\, g(z)\, h(z)}{[1-\lambda h(z)]}$$

show that

$$\frac{zf'(z)}{f(z)}$$

can be equal to zero at $z^n = R$ and therefore the result is sharp.

Theorem 3b—Let

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k$$

and

$$g(z) = z + \sum_{k=n+1}^{\infty} b_k z^k$$

be analytic and satisfy

$$\left| \frac{f(z)}{\lambda f(z) + (1-\lambda) g(z)} - \alpha \right| < \alpha; \; 0 \leqslant \lambda < 1 \; ; \; \alpha \geqslant 1 \quad \text{for } |z| < 1.$$

Let $g(z)$ be starlike of order β $(0 \leqslant \beta < 1)$ Then $f(z)$ is starlike for $|z| < R_1^{1/n}$ where R_1 is the least positive root of the equation

$$\begin{aligned} & r^4 [-(2\beta-1)(\alpha-1)(\alpha\lambda+\alpha-1) + r^3 [(\alpha-1)(2\beta-1)(2\alpha-1) \\ & -n(\alpha-1) - 2n(\alpha-1)^2 - (2\alpha\beta-1)(\alpha\lambda+\alpha-1)] + r^2 [(2\alpha\beta-1) \\ & \times (2\alpha-1) - (\alpha-1)(2\beta-1)\alpha(1-\lambda) - \alpha(\alpha\lambda+\alpha-1) \\ & + n(2\alpha-1)] + r[(\alpha n+\alpha)(2\alpha-1) - \alpha(1-\lambda)(2\beta a-1)] - \alpha^2 \\ & \times (1-\lambda) = 0. \end{aligned}$$

PROOF :

$$\frac{zf'(z)}{f(z)} = \frac{zg'(z)}{g(z)} + \frac{zh'(z)}{h(z)(1-\lambda h(z))}$$

$g(z)$ is of starlike of order $\beta \Rightarrow$ as before

$$\operatorname{Re}\left\{\frac{zg'(z)}{g(z)}\right\} \geqslant \frac{1+(2\beta-1)|z|^n}{1+|z|^n}.$$

$$\begin{aligned} \therefore \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqslant & \frac{1+(2\beta-1)r}{1+r} - \frac{nr}{1-r^2}\left\{2 - \frac{1-r}{[\alpha+(\alpha-1)r]}\right\} \\ & \times \left\{\frac{[\alpha-(\alpha-1)r]}{[(\alpha-\alpha\lambda)-r(\alpha+\alpha\lambda-1)]}\right\} \end{aligned}$$

simplification gives

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqslant 0 \text{ on } |z| = r$$

if

$$\begin{aligned} P(r) \equiv & [-(2\beta-1)(\alpha-1)(\alpha\lambda+\alpha-1)] r^4 + r^3 [(\alpha-1)(2\beta-1) \\ & \times (2\alpha-1) - n(\alpha-1) - 2n(\alpha-1)^2 - (2\alpha\beta-1)(\alpha\lambda+\alpha-1)] \end{aligned}$$

$$+ r^2 [(2\alpha\beta - 1)(2\alpha - 1) - \alpha(\alpha - 1)(2\beta - 1)(1 - \lambda) - \alpha(\alpha\lambda + \alpha - 1) + n(2\alpha - 1)] + r [\alpha(n+1)(2\alpha - 1) - \alpha(1 - \lambda) \times (2\alpha\beta - 1)] - \alpha^2(1 - \lambda) \leqslant 0.$$

$P(0) < 0;\ P(1) > 0.$

Therefore there exists a least positive root R in $0 < r < 1$.

Therefore $f(z)$ is starlike for $|z| < R^{1/n}$.

Corollary—(1) Let $\alpha = 1$, we get Theorem 5 of Shah (1972). (2) Allowing $\alpha \to \infty$ we get Theorem 3 of Shah (1972).

Theorem 4*a*—Let $g(z) = z + b_2z^2 + \ldots$ be analytic and univalent for $|z| < 1$.

Let $f(z) = z + a_2z^2 + \ldots$ be analytic and satisfy

$$\left|\frac{f(z)}{\lambda f(z) + (1 - \lambda) g(z)} - \alpha\right| < \alpha;\quad \frac{1}{2} < \alpha \leqslant \frac{1}{1 + \lambda};\quad 0 \leqslant \lambda < 1;$$

for $|z| < 1$.

Then $f(z)$ is uninvalent for $|z| < R$ where R is the smallest positive root of

$$r^3(t - \lambda) + r^2 [3(\lambda - t)] + r [3(1 - \lambda)] - (1 - \lambda) = 0$$

and the result is sharp.

PROOF: We show that $f(z)$ is univalent on $|z| = r < \mathrm{R}$. Then we can conclude that $f(z)$ is univalent for $|z| \leqslant r$.

$$f(z_1) = f(z_2) \Leftrightarrow$$

$$\frac{g(z_2) - g(z_1)}{g(z_1)(z_2 - z_1)} = \frac{h(z_1) - h(z_2)}{(1 - \lambda h(z_1))(z_2 - z_1) h(z_2)} \quad \ldots(2)$$

where

$$h(z) = \frac{f(z)}{\lambda f(z) + (1 - \lambda) g(z)};$$

$h(z)$ satisfies $|h(z) - \alpha| < \alpha$ for $|z| < 1$.

$$\therefore h(z) = \frac{1 + z\phi(z)}{1 + tz\phi(z)};\quad t = \frac{1 - \alpha}{\alpha}$$

$$\therefore h(z) = \frac{1 + \psi(z)}{1 + t\psi(z)};\quad \psi(z) = z\phi(z).$$

$\psi(z)$ is regular and satisfies $\psi(0) = 0$; $|\psi(z)| < 1$ for $|z| < 1$.

Therefore (2) holds only if

$$\left|\frac{g(z_2)-g(z_1)}{(z_2-z_1)\,g(z_1)}\right| = \left|\frac{(1-t)\,[\psi(z_1)-\psi(z_2)]}{[(1-\lambda)+(t-\lambda)\,\psi(z_1)]\,[(1+\psi(z_2))]}\right| \frac{1}{|z_2-z_1|} \quad \ldots(3)$$

But

$$\psi(z_2)-\psi(z_1) = \int_{z_1}^{z_2} (\psi(z))'\,dz$$

where the path of integration is the line segment joining z_1 and z_2. This yields

$$|\psi(z_2)-\psi(z_1)| \leqslant |z_2-z_1| \text{ Max } |\psi'(z)| \text{ on } |z|=r \quad \ldots(4)$$

by maximum principle applied to $\psi'(z)$ since the line segment joining z_1 and z_2 lies inside the circle $|z|=r$.

$$\psi(0)=0;\ |\psi(z)|<1 \text{ for } |z|<1 \Rightarrow$$

$$|\psi'(z)|<1 \text{ for } |z|<(\sqrt{2}-1)$$ (Caratheodory 1954) and

$g(z)$ is univalent. Therefore by Lemma 3

$$\left|\frac{g(z_1)-g(z_1)}{g(z_1)(z_2-z_1)}\right| \geqslant \frac{|g(z_2)|(1-r^2)}{r^2} \text{ for } |z_1| = |z_2| = r.$$

A distortion theorem for $g(z)$ gives

$$|g(z_2)| \geqslant \frac{r}{(1+r)^2};\quad r=|z_2|$$

$$\therefore \left|\frac{g(z_2)-g(z_1)}{g(z_1)\,(z_2-z_1)}\right| \geqslant \frac{1-r}{r(1+r)}.$$

We see that the equality in (3) will be broken if the right member of (3) does not exceed

$$\frac{1-r}{r(1+r)}.$$

But using (4)

$$\left|\frac{(1-t)\,[\psi(z_1)-\psi(z_2)]}{(z_1-z_2)\,[(1-\lambda)+(t-\lambda)\,\psi(z_1)]\,[1+\psi(z_2)]}\right|$$

$$\leqslant \frac{(1-t)}{[(1-\lambda)-(t-\lambda)\,r]\,[1-r]}.$$

Therefore $f(z)$ will be univalent for $|z|=r$ if

$$\frac{(1-t)}{[1-\lambda-(t-\lambda)\,r]\,[1-r]} < \frac{1-r}{r(1+r)}$$

i.e.,

$$Q_3(r) \equiv r^3(t-\lambda) + r^2\,[3(\lambda - t)] + 3r(1-\lambda) - (1-\lambda) < 0.$$

$$Q_3(0) < 0; \quad Q_3\ (\sqrt{2}-1) \geqslant 0.$$

Therefore there exists a least positive root of this equation $Q_3(r) = 0$ in $0 < r \leqslant \sqrt{2} - 1$. If this is denoted by R we see that $f(z)$ is univalent for $|z| < R$.

The result is sharp, For consider

$$h(z) = \frac{1+z}{1+tz};$$

$$g(z) = \frac{z}{(1-z)^2};$$

$$f(z) = \frac{(1-\lambda)\,g(z)\,h(z)}{1-\lambda h(z)}$$

$f(z)$ satisfies the hypothesis of the theorem and $f'(z) = 0$ at $z = -R$.

Theorem 4*b*—Let

$$f(z) = z + a_{n+1}\,z^{n+1} + \ldots$$

and

$$g(z) = z + b_{n+1}\,z^{n+1} + \ldots$$

be analytic and satisfy

$$\left|\frac{f(z)}{\lambda f(z) + (1-\lambda)\,g(z)} - \alpha\right| < \alpha;\ 0 \leqslant \lambda < 1;\ \alpha \geqslant 1;\ \text{for } |z| < 1.$$

Let $g(z)$ be univalent. Then $f(z)$ is univalent for $|z| < R_1$ where R_1 is the least positive root of the equation

$$\begin{aligned} T(r) \equiv (\alpha + \alpha\lambda - 1)\,r^{2n+1} - (\alpha + \alpha\lambda - 1)\,r^{2n} + r^{n+1}\,[n(2\alpha - 1) \\ - \alpha(1-\lambda) - (\alpha + \alpha\lambda - 1)] + r^n\,[n(2\alpha - 1) + \alpha(1-\lambda) \\ + (\alpha + \alpha\lambda - 1)] + \alpha(1-\lambda)\,r - \alpha(1-\lambda) = 0. \end{aligned}$$

PROOF: As before we show the univalence of $f(z)$ on $|z| = r$

$$f(z_1) = f(z_2) \Leftrightarrow \frac{g(z_2) - g(z_1)}{g(z_1)(z_2 - z_1)} = \frac{h(z_1) - h(z_2)}{(1 - \lambda h(z_1))\,h(z_2)\,(z_2 - z_1)} \qquad \ldots(5)$$

where

$$h(z) = \frac{f(z)}{\lambda f(z) + (1-\lambda)\,g(z)};$$

$h(z)$ satisfies $|h(z) - \alpha| < \alpha$ for $|z| < 1$.

$$\therefore h(z) = \frac{1 + \phi(z)}{1 + t\phi(z)}\,; \; t = \frac{1-\alpha}{\alpha}$$

and where $\phi(z)$ is regular and satisfies $|\phi(z)| \leqslant |z|^n$ for $|z| < 1$.
Substituting for $h(z)$ and estimating the absolute value of the right member of the equality (5) as before we set

$$\left|\frac{h(z_1) - h(z_2)}{(1 - \lambda h(z_1))\, h(z_2)(z_2 - z_1)}\right| \leqslant \frac{nr^{n-1}}{1 - r^{2n}}$$
$$\times \left\{\frac{(2\alpha - 1)\,(1 + r^n)}{[\alpha\,(1 - \lambda) - (\alpha + \alpha\lambda - 1)\, r^n]}\right\}.$$

But

$$\left|\frac{g(z_2) - g(z_1)}{(z_2 - z_1)\, g(z_1)}\right| \geqslant \frac{1 - r}{r(1 + r)} \text{ on } |z| = r < 1 \text{ as before.}$$

Therefore if

$$\frac{1 - r}{r(1 + r)} > \frac{nr^{n-1}}{1 - r^{2n}} \left\{\frac{(2\alpha - 1)(1 + r^n)}{\alpha\,(1 - \lambda) - (\alpha + \alpha\lambda - 1)\, r^n}\right\}$$

then

$$f(z_1) \neq f(z_2) \text{ for } |z_1| = |z_2| = r.$$

This condition reduces to $T(r) \leqslant 0$ where $T(r)$ is given in the theorem.

$$T(0) < 0;\; T(\sqrt{2} - 1) \geqslant 0.$$

If the least positive root of this equation is denoted by R then $f(z)$ is univalent for $|z| < R$.

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