ESCAPE OF NEUTRINOS FROM WITHIN CYLINDRICALLY SYMMETRIC CLUSTER OF PARTICLES

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The stationary cylindrically symmetric cluster of many gravitating masses moving in circles perpendicular to the axis of symmetry described by Raychaudhuri and Som (1962) is considered. The emission of neutrinos from within the cluster has been discussed. It has been shown that neutrinos emitted radially in any plane $z =$ constant of the cluster escape to infinity.

1. INTRODUCTION

The gravitational field of an isolated massive body with spherical symmetry is given by the Schwarzschild exterior solution,

$$ds^2 = -\left(1 + \frac{m}{2r}\right)^4 \{dx^2 + dy^2 + dz^2\} + \left\{\frac{\left(1 - \frac{m}{2r}\right)}{\left(1 + \frac{m}{2r}\right)}\right\}^2 dt^2 \qquad \ldots(1.1)$$

where m is the gravitational mass of the body. The field is singular at the surface $r = m/2$ which is known as the surface of the Schwarzschild singularity. In order to study the existence of this singularity for the fields of physical reality Einstein (1939) constructed a spherically symmetric cluster of particles of equal masses moving in concentric circular orbits around the centre of symmetry under the influence of the gravitational field produced by all of them together. It was found that the matter cannot be concentrated arbitrarily and for a cluster of a given gravitational mass m the particles of the cluster were constrained to move beyond a certain critical distance $r > m/2\ (2 + \sqrt{3})$. This fact led to the conclusion that the Schwarzschild type singularities do not exist in physical reality. Raychaudhuri and Som (1962) studied a stationary cylindrically symmetric cluster of particles of equal masses moving in circles perpendicular to the axis of symmetry under the influence of the gravitational field produced by all of them together. A number of specific solutions for the interior of the cluster were deduced and it was emphasized that the investigation provides an additional class of solutions for a bounded distribution of matter in general relativity.

In this paper the escape of neutrinos from within the stationary cylindrically symmetric cluster in outwards direction has been discussed. It has been shown that neutrinos emitted radially from within the cluster in any plane $z =$ constant escape to infinity. The escape of neutrinos from within Einstein's stationary spherically symmetric cluster of many gravitating masses has already been studied by Hogan (1973).* It has been shown that neutrinos emitted outwards from within Einstein's spherical cluster, making any angle in the range $(0, \pi/2)$ with the radial direction escape to infinity.

2. Stationary Cylindrically Symmetric Cluster of Many Gravitating Masses

The line-element inside the cluster is taken in Weyl's canonical form,

$$ds^2 = e^{2\alpha}\, dt^2 - e^{2\beta - 2\alpha}(dr^2 + dz^2) - r^2 e^{-2\alpha}\, d\varphi^2 \qquad ...(2.1)$$

where α and β are functions of r alone. The circular motion of a particle of the cluster in $r - \varphi$ plane will be given by the equations of the geodesics,

$$\frac{d^2 x^i}{ds^2} + \Gamma^i_{jk} \frac{dx^j}{ds} \cdot \frac{dx^k}{ds} = 0 \qquad ...(2.2.1)$$

together with

$$\frac{dr}{ds} = \frac{dz}{ds} = 0. \qquad ...(2.2.2)$$

For the metric (2.1), eqns. (2.2.1) are reduced to,

$$\frac{d^2 \varphi}{ds^2} = \frac{d^2 t}{ds^2} = 0 \qquad ...(2.3.1)$$

and

$$r(r\alpha' - 1)\left(\frac{d\varphi}{dt}\right)^2 + \alpha' e^{4\alpha} = 0 \qquad ...(2.3.2)$$

where a dash denotes differentiation with respect to r. Using (2.3.2) in (2.1) we get,

$$\left(\frac{ds}{dt}\right)^2 = e^{2\alpha}\left(\frac{1 - 2r\alpha'}{1 - r\alpha'}\right). \qquad ...(2.4)$$

Since ds^2 is to be positive for the world line of a particle in motion we have

either

$$r\alpha' < \tfrac{1}{2} \qquad ...(2.5.1)$$

or

$$r\alpha' > 1. \qquad ...(2.5.2)$$

For (2.5.2), $d\phi/dt$ is imaginary so that the condition (2.5.1) should hold. The exterior static field of the cluster is given by the Marder (1958) solution for the exterior static field of a cylinder

$$ds^2 = r^{2C}\,dt^2 - A^2 r^{-2C\,(1-C)}\,(dr^2 + dz^2) - r^{2\,(1-C)}\,d\phi^2 \qquad ...(2.6)$$

where A and C are arbitrary constants. It may be noted that $C/2$ is the gravitational mass per unit length of the cylinder. Applying this solution for the particles at the boundary of the cluster we get from (2.5.1),

$$C < \tfrac{1}{2}. \qquad ...(2.7)$$

Thus we see that the condition for the velocity of the particles of the cluster to be less than the velocity of light restricts the gravitational mass of the cluster. Contrary to the case of spherical symmetry there is no limit to the radial concentration of the particles and hence the radius of the orbits may shrink arbitrarily. From the Einstein field equations

$$R^i_j - \tfrac{1}{2}\,Rg^i_j = -\,8\pi T^i_j \qquad ...(2.8)$$

the equations for the system are finally obtained as,

$$\frac{\beta'}{r} - \alpha'^2 = 0 \qquad ...(2.9.1)$$

and

$$4\pi mn\,e^{\alpha} = (r\alpha'' + \alpha')\,(1 - r\alpha')^{1/2}\,(1 - 2r\alpha')^{1\,2} \qquad ...(2.9.2)$$

where m is the mass of the particles and n is their number density per unit coordinate volume.

3. Emission of Neutrinos

Let us first consider a thick cylindrical shell of radii r_0 and r_1 $(0 < r_0 < r_1)$. Denote the interior of the shell $(r < r_0)$ by I, the shell $(r_0 < r < r_1)$ by S and the exterior of the shell $(r > r_1)$ by E. Inside S the metric is given by (2.1) which is to be joined continuously at $r = r_1$ to the Marder metric (2.6) in E. We shall study the emission of neutrinos from within the shell, i.e., at a position r for which $r_0 < r < r_1$ considering only neutrinos emitted outwards radially in any plane $z =$ constant.

The equation of motion of neutrinos is obtained from the equations of the null geodesics,

$$\frac{d^2x^i}{du^2} + \Gamma^i_{jk}\frac{dx^j}{du}\cdot\frac{dx^k}{du} = 0 \qquad ...(3.1)$$

where u is an affine parameter along the path. For the metric (2.1), equations (3.1) are as follows :

$$\frac{d^2 r}{du^2} + \{\beta' - \alpha'\}\left(\frac{dr}{du}\right)^2 + re^{-2\beta}\{r\alpha' - 1\}\left(\frac{d\phi}{du}\right)^2 + \{\alpha' - \beta'\}\left(\frac{dz}{du}\right)^2$$

$$+ \alpha' e^{\{4\alpha - 2\beta\}}\left(\frac{dt}{du}\right)^2 = 0 \qquad \text{...(3.1.1)}$$

$$\frac{d^2 \phi}{du^2} + 2\left\{\frac{1 - r\alpha'}{r}\right\}\frac{dr}{du}\cdot\frac{d\phi}{du} = 0 \qquad \text{...(3.1.2)}$$

$$\frac{d^2 z}{du^2} + 2\{\beta' - \alpha'\}\frac{dr}{du}\cdot\frac{dz}{du} = 0 \qquad \text{...(3.1.3)}$$

$$\frac{d^2 t}{du^2} + 2\{\alpha'\}\frac{dr}{du}\cdot\frac{dt}{du} = 0. \qquad \text{...(3.1.4)}$$

For the neutrino emitted in the plane $z = \lambda$ (constant)

$$\left(\frac{dz}{du}\right) \text{initially} = 0.$$

From eqn. (3.1.3) we get

$$\frac{d^2 z}{du^2} = 0$$

and hence it follows that the neutrino emitted in the plane $z = \lambda$ continues to move in the same plane. Integrating eqns. (3.1.2) and (3.1.4) we get

$$\frac{d\phi}{du} = \frac{Ae^{2\alpha}}{r^2} \qquad \text{...(3.2)}$$

and

$$\frac{dt}{du} = ABe^{-2\alpha} \qquad \text{...(3.3)}$$

respectively where A and B are constants of integration. Equation (3.1.1) is now written as

$$\frac{d^2 r}{du^2} + \{\beta' - \alpha'\}\left(\frac{dr}{du}\right)^2$$

$$= -\left[\frac{B^2}{r^3} e^{(4\alpha - 2\beta)}\{r\alpha' - 1\} + \{A^2 \alpha' e^{-2\beta}\}\right]. \qquad \text{...(3.4)}$$

Instead of integrating (3.4) we shall make use of the line element (2.1). For neutrinos we have $ds^2 = 0$. This gives,

$$e^{2\alpha}\left(\frac{dt}{du}\right)^2 - e^{2\beta - 2\alpha}\left(\frac{dr}{du}\right)^2 - r^2 e^{-2\alpha}\left(\frac{d\phi}{du}\right)^2 = 0. \qquad \text{...(3.5)}$$

Making use of eqns. (3.2) and (3.3) we get for the path of the neutrino the differential equation,

$$r^{-4}\, e^{2\beta} \left(\frac{dr}{d\phi}\right)^2 = B^2\, e^{-4\alpha} - r^{-2}. \qquad ...(3.6)$$

For the apses of the path we have,

$$B^2 = \frac{e^{4\alpha}}{r^2}. \qquad ...(3.7)$$

We cannot draw an exact graph of eqn. (3.7) as we do not know the function $\alpha(r)$. It is given by the field eqns. (2.9.1) and (2.9.2) which are only two in three unknowns α, β and n. However, we shall discuss the result qualitatively. For the maxima of B^2 we have from (3.7)

$$r\alpha' = \tfrac{1}{2}, \qquad ...(3.8)$$

which does not hold for any $r\,(r_0 < r < r_1)$ in the light of the condition (2.5.1) and hence B^2 has no maximum in S. Since B^2 has no maximum in S, every neutrino emitted outwards from S fails to encounter an apse on its orbit in S and hence escapes through the boundary $r = r_1$ of the shell S. At the boundary $r = r_1$, eqn. (3.8) leads to $C = \frac{1}{2}$ which further ensures, in the light of the condition (2.7), that the neutrino on leaving the outer surface of the shell will not subsequently have an apse in E and hence will escape to infinity.

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