PROJECTION OPERATORS ON UNIFORMLY CONVEX SEMI-INNER PRODUCT SPACES

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In this paper we extend the Hilbert space type arguments to the study of projection operators on uniformly convex s.i.p. space. Since every Banach space could be converted into a uniformly convex s.i.p. space, our arguments in fact extend to projection operators on Banach spaces. We study the sum and the product of projection operators. We also consider the concept of partial isometry in this setting.

§1. Koehler (1971) explored some aspects of the operator theory in smooth uniformly convex Banach space with the semi-inner product constructed on it. Lumer (1961) defined a semi-inner product (s.i.p.) on a complex vector space X as a complex function $[x, y]$ on $X \times X$ with the following properties :

$$[x + y, z] = [x, z] + [y, z] \quad \ldots(1)$$

$$[\lambda x, y] = \lambda [x, y] \text{ for all complex } \lambda. \quad \ldots(2)$$

$$[x, x] > 0, \text{ when } x \neq 0 \quad \ldots(3)$$

$$|[x, y]|^2 \leqslant [x, x] [y, y]. \quad \ldots(4)$$

A vector space with an s.i.p. is called a semi-inner product space. An s.i.p. space is a normed linear space with the norm $\| x \| = [x, x]^{1/2}$. Giles (1967) showed that any normed linear space can be represented as an s.i.p. space. Hence it is worthwhile to carry over Hilbert space type arguments to the theory of Banach space. Pethe and Thakare (*in press a*) not only established the projection theorem for the uniformly convex Banach spaces but also proved Riesz representation theorem, Lax-Milgram theorem for uniformly convex s.i.p. spaces.

The purpose of this paper is to study projection operators on uniformly convex s.i.p. space X. By the projection theorem on X (Pethe and Thakare *in press a*) every

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$x \in \mathscr{X}$ can be written in the form $x = y + z$ where $y \in M$ and $z \in M^{\perp}$. The point y is called the projection of x in M and an operator P on $\mathscr{X}$ with the property $Px = y$ will be called projection on M. We shall denote such operators by P_M, and M will be called the subspace of the projection P.

It is well known that Hilbert space is a special type of Banach space, having an additional property which enables us to state when two vectors are orthogonal. James (1947) has introduced the concept of orthogonality in any normed linear space. Recently this concept has also been extended to Frcèhet and semi-normed spaces by Pethe and Thakare (*in press b*). James (1947) showed that if the orthogonality is additive and symmetric then real normed linear space of dimension $\geqslant 3$ is an inner product space. Day (1947) in fact went further and showed that in a real Banach space of dimension $\geqslant 3$ the orthogonality is symmetric if and only if the Banach space is isometric with a Hilbert space. Recently Tapia (1973) showed that if a generalised inner product could be defined on a real normed linear space $\mathscr{X}$ by

$$\langle x, y \rangle = f'_{+}(x)\ (y),\ x,\ y \in \mathscr{X}$$

where $f'_{+}(x)\ (y)$ is a right-hand Gâteaux derivative of a functional f on $\mathscr{X}$ at x in the direction y, then every real normed linear space is an inner product space if either this generalised inner product is symmetric or linear in x.

It is observed that the underlying space which we consider is of any dimension (it can be of dimension < 3); besides the underlying field is that of complex numbers. Giles (1967) gave in his paper examples of non-Euclidean s.i.p. spaces. The authors have not yet come across any extension of the results of James (1947), Day (1947), Tapia (1973) to that of complex case.

It is remarked that for the concepts of orthogonality in Banach and such general spaces refer Pethe and Thakare (*in press b*) or James (1947). In our discussion the orthogonality will be assumed symmetric. By this we shall mean that

$$x \perp y \Rightarrow y \perp x.$$

§2. In what follows let $\mathscr{X}$ be a uniformly convex semi-inner-product space which is complete in its norm. Koehler (1971) defines the generalised adjoint A^* of a bounded linear A on $\mathscr{X}$ by

$$[A(x), y] = [x, A^*(y)].$$

Phadke and Thakare (1974) established that an s.i.p. space which is complete in its norm and satisfying the homogeneity property is a Hilbert space if the s.i.p. is additive in the second variable. Hence in order that A^* turns out to be linear we have the following restrictive definition.

Definition 2.1—If T is bounded linear operator on $\mathscr{X}$, then the adjoint operator T^* of T is defined by the relations:

$$[T^*x, y] = [x, Ty]$$

and

$$[Tx, y] = [x, T^*y].$$

Of course, this does not go to mean that an s.i.p. is symmetric.

Definition 2.2—If $T = T^*$, then T is called self-adjoint.

If P_L and P_M are projection operators on $\mathcal{X}$, with L and M as the respective subspaces, then it is easy to show that $(P_L P_M)^* = P_M^* P_L^*$.

Definition 2.3—The projection operators P_L and P_M are orthogonal iff

$$P_L P_M = 0.$$

Theorem 2.1—The projection operator P is a bounded operator on $\mathcal{X}$ satisfying $P^2 = P$ and $\| P \| = 1$ if $P \neq 0$.

PROOF: Let $P = P_M$ and $x_i = y_i + z_i$ $(i = 1, 2)$ with $y_i \in M$, $z_i \in M^\perp$. If $x_1, x_2, \in \mathcal{X}$, then by noting that M and $M^\perp$ are subspaces, we can easily show that P is linear. In order to prove continuity of P, assume that $\{x_n\}$ is a sequence in $\mathcal{X}$, converging to x. If $x_n = y_n + z_n$, $x = y + z$ where y_n, $y \in M$ and z_n, $z \in M^\perp$. Then it is clear that $y_n \to y$, $z_n \to z$. Hence $Px_n = P(y_n + z_n) = y_n \to y = Px$ implying the continuity of P.

If $x \in \mathcal{X}$ with $x = y + z$; $y \in M$, $z \in M^\perp$, then

$$P^2 x = P\{P(y + z)\} = Py = P(y + 0) = y = Px \;\forall\; x \in \mathcal{X}.$$

Thus $P^2 = P$.

If $P \neq 0$, $\exists x \neq 0$ in M [suppose not; let $M = \{0\}$. Take $z \in \mathcal{X}$, $z = 0 + z$. Then $P(z) = 0$; $\forall\; z \in \mathcal{X}$ implying that $P = 0$]. Write $x = x + 0$, where $x \in M$, $0 \in M^\perp$.

$$\therefore\; Px = x, \text{ implying } \| Px \| = \| x \|.$$

Now

$$\| P \| = \sup_{\|x\|=1} \{\| Px \|\} = \sup_{\|x\|=1} \{\| x \|\} = 1,$$

proving that

$$\| P \| = 1.$$

It is observed that under the conditions of the theorem, the projection operator is self-adjoint if the underlying space is a Hilbert space.

Theorem 2.2—The projection operators P_M and P_N are orthogonal if $N \perp M$.

PROOF: $\forall\; x \in \mathcal{X}$, $P_N x \in N$ implying that $P_N x \perp M$. Hence $P_M(P_N x) = 0$. $\forall\; x \in \mathcal{X}$, leading to the conclusion that P_M and P_N are orthogonal.

Corollary—If in X, we assume that orthogonality is symmetric, then P_M and P_N are orthogonal if $M \perp N$.

Theorem 2.3—If the projection operators P_M and P_N are self-adjoint and orthogonal then $N \perp M$.

PROOF: On account of the orthogonality and self-adjoint nature of P_M and P_N, we have $P_N P_M = 0$.

If

$$x \in M, \ y \in N,$$

then

$$[x, y] = [P_M x, P_N y] = [P_N P_M x, y] = [0, y] = 0.$$

Thus $[x, y] = 0$. $\forall x \in M$, $\forall y \in N$ implying that $y \perp x$ $\forall x \in M$, $\forall y \in N$, hence $N \perp M$.

Theorem 2.4—The projection operators P_M and P_N are orthogonal, P_M is self-adjoint, then $M \perp N$.

PROOF: If $x \in M$, $y \in N$, then

$$[y, x] = [P_N y, P_M x] = [P_M{}^* P_N y, x] = [P_M P_N y, x] = [0, x] = 0$$

implying $x \perp y$, $\forall x \in M$, $\forall y \in N$. This leads to the desired result.

Theorem 2.5—If the sum of projections, i.e., $(P_M + P_N)$ is a projection, then

$$P_M P_N = 0.$$

PROOF: The proof runs parallel to Hilbert space arguments.

Theorem 2.6—If P is a projection operator on X, then $(I-P)$ is a projection operator on X.

PROOF: By projection theorem (see Pethe and Thakare *in press a*) every $x \in X$ can be written uniquely as $x = y + z$ where $y \in M$, $z \in M^{\perp}$, M being the closed subspace of X. If M is the subspace of P then $P_x = y$. Hence

$$(I - P)(x) = (I - P)(y + z) = I(y + z) - P(y + z) = y + z - y = z.$$

Thus $(I - P)$ is the projection on $M^{\perp}$.

Theorem 2.7—A subspace M with projection P is invariant under an operator A iff $AP = PAP$.

PROOF: If $AP = PAP$ and if $x \in M$, then

$$Ax = APx = PAPx \in M$$

proving the first part. Conversely let M be invariant under A. If $x \in \mathcal{X}$, then $Px \in M$ implying that $APx \in M$ which, in turn, means that $APx = PAPx$. Hence $AP = PAP$.

Theorem 2.8—If P is a self-adjoint projection with M as the subspace, then M reduces an operator A iff $AP = PA$.

PROOF: The proof runs parallel to Hilbert space arguments.

§3. In the following discussion $\mathcal{X}$ will be a uniformly convex s.i.p. space which is complete in its norm, and in which orthogonality is symmetric, i.e., $x \perp y \Rightarrow y \perp x$ (see Pethe and Thakare *in press b*).

Theorem 3.1—If P is a projection operator on $\mathcal{X}$, then

$$\| P \| \leqslant 1.$$

PROOF: Let $x \in \mathcal{X}$ such that $x = y + z$ where $y \in M$, $z \in M^{\perp}$. If M is the subspace of P, then $Px = y$. On account of orthogonality and its symmetric nature (see Giles 1967, Pethe and Thakare *in press b*) we have $\| y + z \| \geqslant \| y \|$ and since $\| x \| \geqslant \| Px \|$ it follows by the definition that $\| P \| \leqslant 1$.

Theorem 3.2—A self-adjoint operator P on $\mathcal{X}$ with $P^2 = P$ is a projection.

PROOF: Let the range of P be M. It can be shown that M is a closed linear subspace of $\mathcal{X}$. Now if $x, y \in \mathcal{X}$, then

$$Py \perp (x - Px),$$

for

$$[x - Px,\ Py] = [P^* \ (x - Px),\ y] = [Px - P^2 x,\ y] = [0,\ y] = 0.$$

Hence $(x - Px) \perp Py \in M$, on account of the symmetry of orthogonality. Thus $(x - Px) \in M^{\perp}$. Hence $x = Px + (x - Px)$ is the unique decomposition of x as a sum $x_1 + x_2$ with $x_1 \in M$ and $x_2 \in M^{\perp}$. This leads to the desired result.

Theorem 3.3—If P_M, P_N are self-adjoint and orthogonal then $P_M + P_N$ is idempotent.

PROOF: Let $P = P_M + P_N$. Because of orthogonality and self-adjoint nature of P_M and P_N, we have $P_M P_N = 0 = P_N P_M$. Now it follows that $P^2 = P$; and the proof is complete.

Theorem 3.4—If P_M, P_N are self-adjoint projection operators then $P = P_M P_N$ is a projection if P_M and P_N commute. In this case $P_M \cdot P_N = P_{M \cap N}$.

PROOF: It is easy to see from Theorem 3.2 that $P = P_M P_N = P_N P_M$ is a projection. To prove the remaining part, let L be the subspace of P. For any x, $Px \in L$, also $Px \in M \cap N$. Conversely if $x \in M \cap N$, then $x = P_M x, x = P_N x$

$$Px = P_M (P_N x) = P_M x = x.$$

Therefore

$$x = Px \in L.$$

Thus

$$L = M \cap N.$$

Theorem 3.5—The following statements are equivalent :

(1) A projection P_L is a part of projection P_M where P_L and P_M are self-adjoint,

(2) $P_M P_L = P_L$.

(3) $P_L P_M = P_L$,

(4) $\| P_L x \| \leqslant \| P_M x \| \ \forall x \in X$.

PROOF : (*a*) To prove (1) $\Rightarrow$ (2).

We have the definition : "A projection P_L is a part of a projection P_M iff $L \subset M$." Therefore if P_L is a part of P_M then $L \subset M$. Hence $\forall x \in X$, $P_L \ x \in M$ and $P_M P_L x = P_L x$ implying that $P_M P_L = P_L$.

(*b*) To prove (2) $\Rightarrow$ (3).

It is obvious.

(*c*) To prove (3) $\Rightarrow$ (4).

Using Theorem 3.1 it follows that $\| P_L x \| \leqslant \| P_M x \|$.

(*d*) To prove (4) $\Rightarrow$ (1).

We need to show that $L \subset M$. Suppose the contrary, then $\exists \, x_0 \in L$ s.t. $x_0 \notin M$. Let $x_0 = y_0 + z_0$ where $y_0 \in M$, $z_0 \perp M$, $z_0 \neq 0$. (This representation is valid, see Pethe and Thakare 1974*b*).

$$\| P_L x_0 \| = \| x_0 \| = \| y_0 + z_0 \| \geqslant \| y_0 \| = \| P_M x_0 \|,$$

(because of symmetry of orthogonality)

which contradicts (4). Hence $L \subset M$.

Theorem 3.6—If $P_M P_L$ are self-adjoint projection operators on X and $P = P_M - P_L$ is a projection, then P_L is a part of P_M.

PROOF : Let $P = P_M - P_L$ be a projection. Then by Theorem 2.6, $(I - P)$ is also a projection. But $I - P = I - (P_M - P_L) = (I - P_M) + P_L$, implying that $(I - P_M) + P_L$ is a projection. By Theorem 2.5 it means that $(I - P_M) P_L = 0$ implying that $P_L = P_M P_L$. We conclude that P_L is a part of P_M in view of Theorem 3.5.

Theorem 3.7—If P and Q are projection operators on $\mathcal{X}$ with M and N as the respective subspaces, then the following statements are equivalent:

(1) $M \perp N$

(2) $PQ = 0$

(3) $QP = 0$

(4) $PN = 0$

(5) $QM = 0$.

PROOF: Before starting the proof, note that $M \perp N$ iff $N \subset M^{\perp}$.

(*a*) To prove (1) $\Rightarrow$ (2).

Since $M \perp N$, $N \subset M^{\perp}$. Now $Qx \in N$ $\forall x \in \mathcal{X}$, hence $Qx \in M^{\perp}$ implying that $PQx = 0$ $\forall x \in \mathcal{X}$. Hence $PQ = 0$.

(*b*) (2) $\Rightarrow$ (3) is obvious.

(*c*) (3) $\Rightarrow$ (4) is obvious.

(*d*) To prove (4) $\Rightarrow$ (5).

If $x \in N$, then $PN = 0$ implies $Px = 0$. But if $x \in N$, then $Qx = x$. Hence $PQx = 0$ $\forall x \in N$. If $x \perp N$, then $Qx = 0$ implying $PQx = 0$. Thus $PQx = 0$ $\forall x \in \mathcal{X}$. Hence $PQ = 0$. This implies that $QP = 0$. If $x \in M$, then $Px = x$. Hence $Qx = QPx = 0$ implying that $QM = 0$.

(*e*) To prove (5) $\Rightarrow$ (1).

From the fact that $QM = 0$, we can prove that $PQ = 0$. Now if $x \in N$, then $Qx = x$. We have $Px = PQx = 0$ implying that $x \in M^{\perp}$. Thus $N \subset M^{\perp}$ which means that $M \perp N$.

Theorem 3.8—Let P be a self-adjoint operator on $\mathcal{X}$ such that $P = \sum_{j=1}^{n} P$ where P_j's are self-adjoint projection operators on $\mathcal{X}$ with M_j's as the subspaces. Then a necessary and sufficient condition for P to be a projection operator is that P_j's are pairwise orthogonal.

PROOF: Sufficiency is obvious. Let $P = \sum_{j=1}^{n} P_j$ be a projection operator. If $x \in M_k$ for some k, $1 \leqslant k \leqslant n$, then $P_k x = x$. Since $\| P \| \leqslant 1$ we have

$$\| x \|^2 \geqslant \| Px \|^2 = [Px, Px] = [Px, x] \ (\because P^* = P \text{ and } P^2 = P)$$

$$= \sum_{j=1}^{n} [P_j x, x]$$

$$= \sum_{j=1}^{n} [P_j^2 x, x]$$

$$= \sum_{j=1}^{n} [P_j x, P_j x]$$

$$= \sum_{j=1}^{n} \| P_j x \|^2$$

$$\geqslant \| P_k x \|^2$$

$$= \| x \|^2.$$

It is obvious that equality must hold all along the line. Hence

$$\sum_{j=1}^{n} \| P_j x \|^2 = \| P_k x \|^2$$

implies that $P_j x = 0$ if $j \neq k$. This means that $P_j M_k = 0$ if $j \neq k$.

By Theorem 3.7, we conclude that if $j \neq k$, then P_j is orthogonal to P_k and $M_j \perp M_k$.

Remark : Under the conditions stated in Theorem 3.8 the range of P is $\sum_{j=1}^{n} M_j$.

§4. *Theorem* 4.1—In an s.i.p. space $\mathscr{X}$, $[Tx, y] = 0$ $\forall x, y \in \mathscr{X}$ iff $T = 0$.

We leave the proof of Theorem 4.1 to the reader.

Theorem 4.2—For any bounded operator U on an s.i.p. space $\mathscr{X}$, Ker $U^*U =$ Ker U.

PROOF : If $f \in$ Ker U^*U, then $U^*Uf = 0$.

Now

$$\| Uf \|^2 = [Uf, Uf] = [U^*Uf, f] = [0, f] = 0$$

implying that $Uf = 0$. Hence $f \in$ Ker U. Thus

$$\text{Ker } U^*U \subset \text{Ker } U \qquad \ldots(1)$$

Conversely if $f \in$ Ker U, then $Uf = 0$.

Now $U^* Uf = U^*(0) = 0$ leading to the fact that $f \in$ Ker U^*U. Thus Ker $U \subset$ Ker U^*U. ...(2)

From (1) and (2), we have, Ker $U =$ Ker U^*U.

Theorem 4.3—If U is a bounded operator on an s.i.p. space $\mathscr{X}$, then Ker U is closed.

Using the usual techniques, one can easily prove Theorem 4.3.

We shall have following definitions :

Definition 4.1—An isometry is a linear transformation from $\mathscr{X}$ into K or from $\mathscr{X}$ into $\mathscr{X}$ satisfying $\|Uf\| = \|f\| \quad \forall f \in \mathscr{X}$ where $\mathscr{X}$ and K are uniformly convex s.i.p. spaces which are complete in their norms.

Definition 4.2—A partial isometry U is a linear transformation that is isometric on the orthogonal complement of its kernel.

Definition 4.3—The orthogonal complement of the kernel of the partial isometry is called as the initial space of U.

Observations from Pethe and Thakare (in press a)

Let $\mathscr{X}$ be a uniformly convex s.i.p. space which is complete in its norm. Then we have

(1) $\mathscr{X}^{\perp} = \{0\}$

(2) If M_1, M_2 are subsets of $\mathscr{X}$ with $M_1 \subseteq M_2$ then $M_1^{\perp} \supseteq M_2^{\perp}$.

(3) If $M \subseteq \mathscr{X}$, then $M \subseteq M^{\perp\perp}$.

(4) If M is a closed vector subspace of $\mathscr{X}$ and if orthogonality is symmetric in $\mathscr{X}$, then $M = M^{\perp\perp}$.

As a consequence of the definition 4.3, we have the following result :

If U is a partial isometry on a uniformly convex s.i.p. space which is complete in its norm, then the initial space of U is the set $\{f: \|Uf\| = \|f\|\}$, i.e.,

$$(\text{Ker } U)^{\perp} = \{f: \|Uf\| = \|f\|\}.$$

Theorem 4.4—Let $\mathscr{X}$ be a uniformly convex s.i.p. space which is complete in its norm and in which orthogonality is symmetric. Then a bounded linear operator U on $\mathscr{X}$ is a partial isometry if U^*U is a projection.

Proof : Assume that $U^*U = P$ is a projection with M as the range. Now

$$\|Uf\|^2 = [Uf, Uf] = [U^*Uf, f] = [Pf, f] \ \forall f \in \mathscr{X}.$$

If $f \in M$, then $Pf = f$. Hence $\|Uf\| = \|f\|$. If $f \perp M$, then $Pf = 0$. Hence $\|Uf\| = 0$ implying $Uf = 0$. Hence $f \in \text{Ker } U$.

Thus $M^{\perp} \subset \text{Ker } U$. But M, being the range of the projection, is closed and hence

$$M \supset (\text{Ker } U)^{\perp}.$$

Since U is isometry on M, it is isometry on $(\text{Ker } U)^{\perp}$ and hence U is a partial isometry.

Corollary—A bonded linear transformation U on $\mathscr{X}$ ($\mathscr{X}$ as specified above) is a partial isometry if $U = U U^*U$.

PROOF : If $UU^*U = U$, then we have

$$U^* (UU^*U) = U^*U,$$

i.e.,

$$(U^*U)^2 = U^*U.$$

Thus U^*U is idempotent. It is obviously self-adjoint. Hence by Theorem 3.2, U^*U is a projection. Then by Theorem 4.4, we conclude that U is a partial isometry.

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