ON CERTAIN q-ORTHOGONAL POLYNOMIALS

by B. K. KARANDE, *Science College, Karad (Maharashtra)*

and

N. K. THAKARE*, *Department of Mathematics, Shivaji University, Kolhapur (Maharashtra)*

(Communicated by F. C. Auluck, F. N. A.)

(*Received 27 December 1973 ; after revision 10 April 1974*)

In this paper we solve the problems posed by Carlitz (1973) of summing the following series :

$$\sum_{m,n,k=0}^{\infty} H_{n+k}(a)\, H_{k+n}(b)\, H_{m+n}(c)\, \frac{x^m y^n z^k}{(q)_m (q)_n (q)_k}$$

and

$$\sum_{m,n,k=0}^{\infty} H_m(a)\, H_n(b)\, H_k(c)\, H_{m+n+k}(d)\, \frac{x^m y^n z^k}{(q)_m (q)_n (q)_k}$$

where

$H_n(x) = H_n(x \; ; \; q)$ are the q-orthogonal polynomials introduced by Szegö (1926). We also give number of generalizations of these results.

§ 1. Szegö (1926) discussed the q-orthogonal polynomials $H_n(x \; ; \; q)$ defined by

$$H_n(x) = H_n(x \; ; \; q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q)_n}{(q)_k \, (q)_{n-k}}, \quad \text{with}$$

$$(q)_n = (1-q)(1-q^2)(1-q^3) \cdots (1-q^n).$$

Carlitz (1955, 1958) showed that these polynomials have analogous properties to those of the Hermite polynomials. In a recent paper Carlitz (1973) posed the problem that it would be of interest to sum the following series :

$$\sum_{m,n,k=0}^{\infty} H_{n+k}(a)\, H_{k+m}(b)\, H_{m+n}(c)\, \frac{x^m y^n z^k}{(q)_m (q)_n (q)_k} \tag{1.1}$$

$$\sum_{m,n,k=0}^{\infty} H_m(a)\, H_n(b)\, H_k(c)\, H_{m+n+k}(d)\, \frac{x^m y^n z^k}{(q)_m (q)_n (q)_k}. \tag{1.2}$$

* *Present address* : Department of Mathematics and Statistics, Marathwada University, Aurangabad 431004.

In this paper an attempt has been made to sum the series (1.1) and (1.2) and also to obtain generalizations of these results.

§ 2. Put $e(x) = \prod_{n=0}^{\infty} (1-q^n x)^{-1} = \sum_{n=0}^{\infty} x^n/(q)_n$.

It can be seen that the series $e(x)$ converges absolutly for $|x|<1$ and $|q|<1$.

Hence the polynomials $H_n(a)$ could be given by

$$e(x)\, e(ax) = \sum_{n=0}^{\infty} H_n(a) \frac{x^n}{(q)_n}.$$

Next define the polynomials $\phi_n(a\,;x) = \phi_n(a\,;x, q)$ by means of

$$\frac{e(z)\, e(xz)}{e(axz)} = \sum_{n=0}^{\infty} \phi_n(a\,;x) \frac{z^n}{(q)_n}$$

so that

$$\phi_n(a\,;x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (a)_k\, x^k$$

where $\begin{bmatrix} n \\ k \end{bmatrix}$ has the same meaning as that in section I, and

$$(a)_k = (1-a)(1-qa) \ldots \ldots (1-q^{k-1}a).$$

Clearly, if we have $a = 0$, $\phi_n(a\,;x)$ reduces to $H_n(x)$.

In the course of our investigation we shall need the following results (see Carlitz 1973) :

$$(x)_r\, e(x) = e(q^r x), \tag{2.1}$$

$$\sum_{n=0}^{\infty} \phi_n(a\,;x)\, \phi_n(b\,;y) \frac{z^n}{(q)_n} = \frac{e(z)\, e(xz)\, e(yz)}{e(axz)\, e(byz)} \sum_{k=0}^{\infty} \frac{(a)_k\, (b)_k\, (c)_k\, (xyz)^k}{(q)_k\, (axz)_k\, (byz)_k}. \tag{2.2}$$

$$\sum_{k=0}^{\infty} H_{m+k}(a)\, H_{n+k}(b) \frac{z^k}{(q)_k} = \frac{e(z)\, e(az)\, e(bz)\, e(abz)}{e(abz^2)} \times \sum_{s=0}^{m} \sum_{t=0}^{n} \begin{bmatrix} m \\ s \end{bmatrix} \begin{bmatrix} n \\ t \end{bmatrix} \frac{(az)_s\, (bz)_t\, (abz)_{s+t}\, a^{m-s} b^{n-t}}{(abz^2)_{s+t}} \tag{2.3}$$

$$\sum_{m,n=0}^{\infty} H_{m+n}(a)\, H_m(b)\, H_n(c) \frac{x^m y^n}{(q)_m\, (q)_n} = \frac{e(x)\, e(bx)}{e(abcxy)} \times e(abx)\, e(y)\, e(cy)\, e(acy) \sum_{k=0}^{\infty} \frac{(bx)_k\, (cx)_k\, \phi_k^{*}(a\,;b)\, a^k}{(q)_k\, (abcxy)_k} \tag{2.4}$$

where in this case $\phi_k^*(a, b)$ is defined by

$$\frac{e(az)\,e(bz)}{e(abz)} = \sum_{k=0}^{\infty} \phi_k^*(a, b)\frac{z^k}{(q)_k}$$

$$\sum_{n=0}^{\infty} H_{n+k}(x)\frac{z^n}{(q)_n} = e(z)\,e(xz)\,\phi_k(z\,;x). \tag{2.5}$$

The generalisation of this result is also needed.

$$\sum_{n=0}^{\infty} \phi_{n+k}(a\,;x)\quad\frac{z^n}{(q)_n} = \frac{e(z)\,e(xz)}{e(axz)}\sum_{r=0}^{k}\begin{bmatrix}k\\r\end{bmatrix}\frac{(a)_r\,(z)_r x^r}{(axz)_r} \tag{2.6}$$

which reduces to (2.5) when $a = 0$.

§ 3. To begin with we shall sum the series

$$\sum_{m,n,k=0}^{\infty} H_{m+k}(a)\,\phi_{n+k}(c, b)\,\frac{x^m\,y^n\,z^k}{(q)_m\,(q)_n\,(q)_k}.$$

This can be written as

$$\sum_{m,n,k=0}^{\infty} H_{m+k}(a)\;\phi_{n+k}(c,b)\;\frac{x^m\,y^n\,z^k}{(q)_m\,(q)_n\,(q)_k}$$

$$= \sum_{k=0}^{\infty}\left\{\sum_{m=0}^{\infty} H_{m+k}(a)\;\frac{x^m}{(q)_m}\right\}\left\{\sum_{n=0}^{\infty}\phi_{n+k}(c, b)\;\frac{y^n}{(q)_n}\right\}\frac{z^k}{(q)_k}.$$

On account of (2.5) and (2.6) the right-hand side above can be written as

$$= \sum_{k=0}^{\infty}\left\{e(x)\,e(ax)\,\phi_k(x,a)\right\}\left\{\frac{e(y)\,e(by)}{e(bcy)}\sum_{r=0}^{k}\begin{bmatrix}k\\r\end{bmatrix}\frac{(c)_r\,(y)_r\,b^r}{(cby)_r}\right\}\frac{z^k}{(q)_k}$$

$$= \frac{e(x)\,e(ax)\;e(y)\,e(by)}{e(bcy)}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\phi_{k+r}(x, a)\;\frac{(c)_r\,(y)_r\,b^r\,z^{k+r}}{(q)_r\,(q)_k\,(bcy)_r}.$$

Applying (2.6) again we now get

$$L.\,H.\,S. = \frac{e(y)\,e(x)\,e(z)\,e(ax)\,e(by)\,e(az)}{e(bcy)\;e(axz)}$$

$$\sum_{r=0}^{\infty}\left[\sum_{k=0}^{r}\frac{(c)_r\;(y)_r\;(bz)^r}{(q)_r\,(bcy)_r}\begin{bmatrix}r\\k\end{bmatrix}\frac{(x)_k\,(z)_k\,a^k}{(azx)_k}\right].$$

Hence we have

$$\sum_{m,n,k=0}^{\infty} H_{m+k}(a)\,\phi_{n+k}(c, b)\;\frac{x^m\,y^n\,z^k}{(q)_m\,(q)_n\,(q)_k} = \frac{e(x)\,e(y)\,e(ax)}{e(cby)\,e(axz)}\times$$

$$\times\, e(z)\,e(by)\,e(az)\;\sum_{r,k=0}^{\infty}\frac{(c)_{r+k}\,(y)_{r+k}\,(x)_k\,(z)_k\,(bz)^r\,(abz)^k}{(q)_r\,(q)_k\,(bcy)_{r+k}\,(azx)_k}.$$

Now consider,

$$\sum_{m,n=0}^{\infty}\left\{\sum_{k=0}^{\infty}\frac{z^k}{(q)_k}\sum_{r=0}^{\min(m,n)}\frac{(q)_r}{(q)_m\,(q)_n}\begin{bmatrix}n\\r\end{bmatrix}\begin{bmatrix}m\\r\end{bmatrix}H_{k+m-r}(x)\,H_{k+n-r}(y)\,u^n\,v^m\right\}$$

$$=\sum_{k=0}^{\infty}\frac{z^k}{(q)_k}\sum_{r=0}^{\infty}\frac{(uv)^r}{(q)_r}\sum_{n=0}^{\infty}\frac{H_{k+n}(x)\,u^n}{(q)_n}\sum_{m=0}^{\infty}\frac{H_{k+m}(y)\,v^m}{(q)_m}.$$

Use of the result (2.5) allows us to write this as

$$=\sum_{k=0}^{\infty}\frac{z^k}{(q)_k}\,e(uv)\,e(u)\,e(v)\,e(yv)\,e(xu)\,\phi_k(u,x)\,\phi_k(v,y).$$

If we use (2.2), however we shall get

$$L.\,H.\,S.=e(u)\,e(v)\,e(uv)\,e(xu)\,e(yv)\,e(z)\,e(xz)\,e(yz)\times$$

$$\times\frac{1}{e(xuz)\,e(vyz)}\sum_{r=0}^{\infty}\frac{(u)_r\,(v)_r\,(z)_r\,(xyz)^r}{(q)_r\,(uxz)_r\,(vyz)_r}. \tag{3.1}$$

But it can be seen that

$$\frac{e(u)\,e(xu)\,e(v)\,e(yv)}{e(xuz)\,e(yvz)}\sum_{r=0}^{\infty}\frac{(u)_r\,(v)_r\,(z)_r\,(xyz)^r}{(q)_r\,(uxz)_r\,(vyz)_r}$$

$$=\sum_{m,n=0}^{\infty}u^n\,v^m\sum_{r=0}^{\infty}\frac{(z)_r\,(xyz)^r}{(q)_r}\sum_{s=0}^{n}\frac{(xz)_s\,q^{rs}\,x^{n-s}}{(q)_s\,(q)_{n-s}}\times$$

$$\times\sum_{t=0}^{m}\frac{(uz)_t\,q^{rt}\,y^{m-t}}{(q)_t\,(q)_{m-t}}.$$

This again can be put in the form

$$\sum_{m,n=0}^{\infty}\frac{u^n\,v^m}{(q)_n\,(q)_m}\sum_{r=0}\frac{(z)_r}{(q)_r}(xyz)^r\,x^n\,\phi_n(xz,q^r/x)\,y^m\,\phi_m(yz,q^r/y).$$

Hence the right-hand side of (3.1) now becomes

$$\sum_{m,n=0}^{\infty}u^n\,v^m\sum_{s=0}^{\min(m,n)}\frac{1}{(q)_{n-s}\,(q)_{m-s}\,(q)_s}$$

$$\times\sum_{r=0}^{\infty}\left[\frac{x^{n-s}\,y^{m-s}}{(q)_r}\phi_{m-s}(yz,q^r/y)\,(xyz)^r\,\phi_{n-s}(xz,q^r/x)\right].$$

Hence equating the coefficients of $(u^n\,v^m)$ on both the sides we are led to an interesting identity

$$\sum_{k=0}^{\infty}\frac{z^k}{(q)_k}\sum_{r=0}^{\min(m,n)}\frac{(q)_r}{(q)_m\,(q)_n}\begin{bmatrix}n\\r\end{bmatrix}\begin{bmatrix}m\\r\end{bmatrix}H_{k+n-r}(x)\,H_{k+m-r}(y)$$

$$=\sum_{r=0}^{\infty}\frac{(z)_r\,(xyz)^r}{(q)_r}\sum_{s=0}^{\min(m,n)}\frac{x^{n-s}\,y^{m-s}\phi_{n-s}\left(xz,\frac{q^r}{x}\right)\phi_{m-s}\left(yz,\frac{q^r}{y}\right)}{(q)_{n-s}\,(q)_{m-s}\,(q)_s}.$$

Next we shall try to sum the series (1.1). We have after using (2.3)

$$\sum_{k,m,n=0}^{\infty} H_{m+k}(a)\, H_{n+k}(b)\, H_{m+n}(c)\, \frac{x^m y^n z^k}{(q)_m (q)_n (q)_k}$$

$$= \sum_{m,n=0}^{\infty} H_{m+n}(c)\, \frac{x^m y^n}{(q)_m (q)_n} \sum_{k=0}^{\infty} H_{m+k}(a)\, H_{n+k}(b) \frac{z^k}{(q)_k}$$

$$= \frac{e(z)\, e(az)\, e(bz)\, e(abz)}{e(abz^2)} \sum_{m,n=0}^{\infty} \Big\{ H_{m+n}(c)\, \frac{x^m y^n}{(q)_m (q)_n}$$

$$\sum_{s=0}^{m} \sum_{t=0}^{n} \begin{bmatrix} m \\ s \end{bmatrix} \begin{bmatrix} n \\ t \end{bmatrix} \frac{(az)_s\, (bz)_t\, (abz)_{s+t}\, a^{m-s}\, b^{n-t}}{(abz^2)_{s+t}} \Big\}$$

$$= \frac{e(z)\, e(az)\, e(bz)\, e(abz)}{e(abz^2)} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \Big\{ H_{m+n+s}(c)\, \frac{y^n}{(q)_n (q)_{m+s}}$$

$$\sum_{s=0}^{\infty} \sum_{t=0}^{n} \begin{bmatrix} m+s \\ s \end{bmatrix} \begin{bmatrix} n \\ t \end{bmatrix} \frac{(az)_s\, (bz)_t\, (abz)_{s+t}\, a^m\, b^{n-t}}{(abz^2)_{s+t}} \Big\}$$

$$= \frac{e(z)\, e(bz)\, e(az)\, e(abz)}{e(abz^2)} \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} \sum_{t=0}^{n} \begin{bmatrix} n \\ t \end{bmatrix} \frac{(az)_s\, (bz)_s\, (abz)_{s+t}}{(abz^2)_{s+t}\, (q)_s} \times$$

$$\times\, x^s\, b^{n-t}\, y^n \sum_{m=0}^{\infty} H_{m+n+s}(c)\, \frac{(xa)^m}{(q)_m}\,.$$

On account of (2.5) we get

$$L.H.S. = \frac{e(z)\, e(az)\, e(bz)\, e(abz)}{e(abz^2)} \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} \sum_{t=0}^{n} \begin{bmatrix} n \\ t \end{bmatrix} \frac{(az)_s\, (bz)_t}{(abz^2)_{s+t}} \times$$

$$\times \frac{(abz)_{s+t}\, x^s b^{n-t} y^n}{(q)_s\, (q)_n}\, e(ax)\, e(acx)\, \phi_{n+s}(xa, c)\,.$$

Using the usual summation techniques we now obtain

$$L.H.S. = \frac{e(z)\, e(az)\, e(bz)\, e(abz)\, e(xa)\, e(acz)}{e(abz^2)} \times$$

$$\times \sum_{s,t=0}^{\infty} \frac{(az)_s\, (bz)_t\, (abz)_{s+t}\, x^s y^t}{(abz^2)_{s+t}\, (q)_s\, (q)_t} \sum_{n=0}^{\infty} \phi_{n+s+t}(xa, c)\, \frac{(by)^n}{(q)_n}\,.$$

Because of (2.6) we arrive at

$$\sum_{k,m,n=0}^{\infty} H_{m+k}(a)\, H_{n+k}(b)\, H_{m+n}(c)\, \frac{x^m y^n z^k}{(q)_m (q)_n (q)_k}$$

$$= \frac{e(z)\, e(az)\, e(bz)\, e(abz)\, e(xa)\, e(acx)\, e(by)\, e(bcy)}{e(abz^2)\, e(abxy)} \times$$

$$\times \sum_{s,t=0}^{\infty} \frac{(az)_s\, (bz)_t\, (abz)_{s+t}\, x^s y^t}{(abz^2)_{s+t}\, (q_s)\, (q)_t} \sum_{r=0}^{s+t} \begin{bmatrix} s+t \\ r \end{bmatrix} \frac{(xa)_r\, (by)_r\, c^r}{(abcxy)_r}\,.$$

§ 4. Let us now consider a much more general expression of the type

$$\sum_{m,n,k,r=0}^{\infty} H_{m+k}(a) H_{n+r}(b) H_m(c) H_n(d) H_k(g) H_r(f) \frac{x^m y^n z^k u^r}{(q)_m (q)_n (q)_k (q)_r}.$$

Because of (2.4) this can be put in the form

$$\sum_{r,k=0}^{\infty} H_k(g) H_r(f) \frac{z^k u^r}{(q)_k (q)_r} \sum_{m=0}^{\infty} H_{m+k}(a) H_m(c) \frac{x^m}{(q)_m} \times$$
$$\times \sum_{n=0}^{\infty} H_{n+r}(b) H_n(d) \frac{y^n}{(q)_n}.$$

If we use the identity (2.3) when $n = 0$ we can simplify this to the form

$$\sum_{r,k=0}^{\infty} \Big[H_k(g) H_r(f) \frac{z^k u^r}{(q)_k (q)_r} \frac{e(x) e(ax) e(cx) e(acx) e(y) e(by)}{e(acx^2) e(bdy^2)} \times$$
$$\times e(dy) e(bdy) \sum_{s=0}^{k} \sum_{t=0}^{r} \begin{bmatrix} k \\ s \end{bmatrix} \begin{bmatrix} r \\ t \end{bmatrix} \frac{(ax)_s (acx)_s a^{k-s}}{(acx^2)_s} \times$$
$$\times \frac{(by)_t (bdy)_t b^{r-t}}{(bdy^2)_t} \Big],$$

which simplifies further to

$$\frac{e(x) e(y) e(ax) e(by) e(cx) e(dy) e(acx) e(bdy)}{e(acx^2) e(bdy^2)} \times$$
$$\times \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \frac{z^s u^t (ax)_s (acx)_s (by)_t (bdy)_t}{(q)_s (q)_t (acx^2)_s (bdy^2)_t} \times$$
$$\times \sum_{k=0}^{\infty} H_{k+s}(g) \frac{z^k}{(q)_k} \sum_{r=0}^{\infty} H_{r+t}(f) \frac{u^r}{(q)_r}.$$

The use of (2.5) gives us

$$\frac{e(x) e(y) e(ax) e(by) e(cx) e(dy) e(acx) e(bdy) e(z) e(zg) e(u)}{e(acx^2) e(bdy^2)} \times$$
$$\times e(fu) \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \frac{(ax)_s (acx)_s (by)_t (bdy)_t z^s u^t}{(q)_s (q)_t (acx^2)_s (bdy^2)_t} \phi_s(z, g) \phi_t(u, f).$$

Hence we now get

$$\sum_{m,n,k,r=0}^{\infty} H_{m+k}(a) H_{n+r}(b) H_m(c) H_n(d) H_k(g) H_r(f) \frac{x^m y^n z^k u^r}{(q)_m (q)_n (q)_k (q)_r}$$
$$= e(x) e(y) e(z) e(u) e(cx) e(zg) e(fu) \times$$
$$\times \sum_{s=0}^{\infty} \frac{e(q^s ax) e(q^s acx)}{e(q^s acx^2) (q)_s} \phi_s(z, g) z^s \sum_{t=0}^{\infty} \frac{e(q^t by) e(q^t bdy) \phi_k(u, f) u^t}{e(q^t bdy^2) (q)_t}.$$

Here we have made use of the relation (2.1).

§ 5. In order to sum the series (1.2) we shall use the operator δ defined by

$$\delta f(x) = \frac{f(x)-f(qx)}{x}. \tag{5.1}$$

It follows from this definition of the operator δ that

$$\delta^r x^n = \begin{cases} \dfrac{(q)_n\, x^{n-r}}{(q)_{n-r}}, & (0 \leqslant r \leqslant n) \\ 0, & (r > n). \end{cases}$$

On account of this, Carlitz (1973) states that

$$e(\delta)\, x^n = \sum_{r=0}^{\infty} \frac{\delta^r x^n}{(q)_r} = H_n(x) \tag{5.2}$$

and also

$$\delta^r(f(x)\, g(x)) = \sum_{s=0}^{r} \begin{bmatrix} r \\ s \end{bmatrix} \delta^{r-s} f(x)\Big|_{q^r x}\, \delta^s g(x) \tag{5.3}$$

where $\delta^{r-s} f(x)\big|_{q^r x}$ is understood to be the result of replacing x by $q^r x$ in $\delta^{r-s} f(x)$.

Because of (5.2) we can write

$$\sum_{m,n,k=0}^{\infty} H_m(a)\, H_n(b)\, H_k(c)\, H_{m+n+k}(x)\, \frac{u^m v^n z^k}{(q)_m (q)_n (q)_k}$$
$$= e(\delta)\,\{e(xu)\, e(axu)\, e(xv)\, e(bxv)\, e(xz)\, e(xcz)\}.$$

Because of (5.3) the right-hand side of this can be written as

$$e(\delta)\,\{e(xu)\, e(axu)\, e(xv)\, e(bxv)\, e(xz)\, e(cxz)\}$$

$$= \sum_{r,s=0}^{\infty} \frac{\delta^r\, e(xu)\, e(axu)\, e(xz)}{(q)_r}\Big|_{q^s x} \frac{\delta^s\, e(xv)\, e(bxv)\, e(cxz)}{(q)_s}$$

$$= \sum_{r,s=0}^{\infty} \frac{1}{(q)_r\, (q)_s} \left\{ \sum_{j=0}^{r} \begin{bmatrix} r \\ j \end{bmatrix} \delta^{r-j}\, (e(ux)\, e(axu))\Big|_{q^j x}\, \delta^j\, e(xz) \right\}\Big|_{q^s x} \times$$

$$\sum_{t=0}^{\infty} \begin{bmatrix} s \\ t \end{bmatrix} \delta^{s-t}\, e(xv)\, e(bxv)\Big|_{q^t x}\, \delta^t\, e(cxz)$$

$$= \sum_{r,s,t,j=0}^{\infty} \frac{\delta^r\, e(xu)\, e(axu)}{(q)_r}\Big|_{q^{j+s+t} x} \frac{z^j\, e(q^{s+t} xz)}{(q)_j}$$

$$\frac{\delta^s\, e(xv)\, e(bxv)}{(q)_s}\Big|_{q^t x} \frac{\delta^t\, e(cxz)}{(q)_t}. \tag{5.4}$$

Again

$$\delta^r\, e(xu)\, e(axu)\Big|_{q^{j+s+t}x}$$
$$= e(q^{j+s+t}aux)\; e(q^{j+s+t}ux) \sum_{l=0}^{r} \begin{bmatrix} r \\ l \end{bmatrix} (au)^{r-l}\, (q^{j+s+t}aux)_l \tag{5.5}$$

$$\delta^s\, e(xv)\, e(bxv)\Big|_{q^t x}$$
$$= e(q^t bxv)\, e(q^t vx) \sum_{p=0}^{s} \begin{bmatrix} s \\ p \end{bmatrix} (bv)^{s-p}\, (q^t bxv)_p\, v^p\, . \tag{5.6}$$

Using (5.5) and (5.6) in (5.4) we obtain

$$\sum_{m,n,k=0}^{\infty} H_m(a)\, H_n(b)\, H_k(c)\, H_{m+n+k}(x)\; \frac{u^m\, v^n\, z^k}{(q)_m\, (q)_n\, (q)_k}$$
$$= \sum_{r,s,t,j=0}^{\infty} \frac{e(q^{j+s+t}aux)\, e(q^{j+s+t}ux)}{(q)_r} \left\{ \sum_{i=0}^{\infty} \begin{bmatrix} r \\ i \end{bmatrix} (q^{s+t+j}aux)_i\, (1/a)^i \right\} \times$$
$$\times \frac{(au)^r\, e(q^{s+t}xz)\, e(q^t bxv)\, e(q^t vx)\, z^j}{(q)_j\, (q)_s} \left\{ \sum_{p=0}^{s} \begin{bmatrix} s \\ p \end{bmatrix} (q^t bvx)_p\, \times \right.$$
$$\left. \times\, (1/b)^p \right\} \frac{(bv)^s\, (cz)^t\, e(czx)}{(q)_t}\, .$$

Further the right hand side can also be put in the form

$$\sum_{r,s,t,j=0}^{\infty} \frac{e(q^{j+s+t}aux)\, e(q^{j+s+t}ux)}{(q)_r}\; \phi_r\,(q^{s+t+j}aux,\, 1/a)\,(au)^r\, z^j \quad \times$$
$$\times \frac{e(q^{s+t}xz)\, e(q^t bxv)\, e(q^t vx)}{(q)_j\, (q)_s}\; \phi_s \frac{(q^t bvx,\, 1/b)\,(bv)^s\,(cz)^t\, e(cx)}{(q)_t}\, .$$

This in turn is equal to

$$\sum_{s,t,j=0}^{\infty} e(q^{j+s+t}aux)\, e(q^{j+s+t}ux)\, z^j\, e(q^{s+t}xz)\, e(q^t bxv)\, e(q^t vx)$$
$$\frac{e(cxz)\, \phi_s\,(q^t bvx,\, 1/b)\,(bv)^s\,(cz)^t}{(q)_j\,(q)_s\,(q)_t} \sum_{r=0}^{\infty} \frac{\phi_r\,(q^{s+t+j}aux,\, 1/a)\,(au)^r}{(q)_r}\, .$$

This further gets reduced to

$$\sum_{s,t,j=0}^{\infty} e(q^{j+s+t}aux)\, e(q^{j+s+t}ux)\, z^j\, e(q^{s+t}xz)\, e(q^t bxv)\, \phi_s\,(q^t bvx,\, 1/b)$$
$$\times \frac{(bv)^s\,(cz)^t\, e(czx)\, e(au)\, e(u)}{(q)_j\,(q)_s\,(q)_t\, e(q^{s+t+j}au^2x)}\, .$$

Hence we now get

$$\sum_{m,n,k=0}^{\infty} H_m(a)\, H_n(b)\, H_k(c)\, H_{m+n+k}(x)\, \frac{u^m\, v^n\, z^k}{(q)_m (q)_n (q)_k}$$

$$= \frac{e(cxz)\, e(au)\, e(u)\, e(aux)\, e(ux)\, e(xz)\, e(bxv)\, e(xv)}{e(axu^2)} \times$$

$$\times \sum_{s,t,j=0}^{\infty} \frac{(aux)_{j+s+t}\, (ux)_{j+s+t}\, (xz)_{s+j}\, (bxv)_t\, (vx)_t}{(q)_j\, (q)_s\, (q)_t\, (axu^2)_{s+j+t}} \times$$

$$\times\ \phi_s(q^t\, bxv,\ 1/b)\, (bv)^s\, c^t\, z^{s+t}.$$

ACKNOWLEDGEMENTS

Authors are grateful to Prof. L. Carlitz of Duke University, USA for encouragement and appreciation of this work. They also thank the referee for his useful suggestions.

REFERENCES

Carlitz, L. (1955). Some polynomials related to theta functions. *Ann. Mat. Pure Appl.* **41**. 359-73.

— — (1958). Note on orthogonal polynomials related to theta functions, *Publ. Math.*, 5, 222-28.

— — (1972). Generating functions for certain *q*-orthogonal polynomials, *Collectanea Math.*, 23, 97-104.

Szegö, G. (1926). Ein Beitrag zur Theoric der Theta funktionen, *Sitzungsberichte Akad. Berlin*, 242-52