A NOTE ON ELECTROSTATIC PROBLEMS INVOLVING TWO STRIPS

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We discuss here solutions of two electrostatic problems involving two coplanar parallel charged strips placed symmetrically inside or outside a long grounded cylinder. By applying the usual Green's function approach, such a boundary value problem is first reduced to a governing Fredholm integral equation of the first kind. The governing Fredholm integral equation of the first kind, in each problem, is further solved by the perturbation technique to obtain approximate expression for the charge densities of the two strips.

1. INTRODUCTION

Recently, Tranter (1960) has discussed the electrostatic problem involving two parallel coplanar strips charged to potentials ± 1 in a free space. Srivastava and Gupta (1971) have further studied the perturbation in the charge density of the two strips when these are placed symmetrically inside or outside a grounded cylinder by reducing such a problem to a single Fredholm integral equation of the second kind with the help of triple integral equations and finite Hilbert transform techniques given by Srivastava and Lowengrub (1970). We also discuss here the solutions of two electrostatic boundary value problems involving two coplanar parallel strips charged to potentials ± 1 when the two strips are placed symmetrically inside or outside a grounded long cylinder. Our analysis has become quite simplified by applying the usual Green's function approach to reduce such a two demensional boundary value problem to a governing Fredholm integral equation of the first kind which embodies the differential equation as well as the boundary conditions of the problem. The governing Fredholm integral equation is further solved by the regular perturbation technique as explained by Jain and Kanwal (1972). Approximate expressions for the unknown charge densities of the strips are obtained in both the problems discussed here and these expressions agree with those of Srivastava and Gupta (1971) provided necessary corrections are carried out in their analysis of the second problem.

2. Electrostatic Problem I

We consider here the electrostatic problem of finding the charge densities of two coplanar parallel strips charged to prescribed potentials when the strips are placed symmetrically with respect to the axis and parallel to the generators of the bounding grounded cylinder of radius $c \gg 1$. We use cylindrical co-ordinates (r, θ, z) with the axis of the cylinder along the z-axis and we have a two-dimensional boundary value problem of a circle of radius $c \gg 1$ and two symmetrically placed line segments occupying the regions $o \leqslant r \leqslant c,\ o \leqslant \theta \leqslant 2\pi\ ;\ a < r < 1,\ \theta = o, \pi.$

Thus we have to solve the following two-dimensional boundary value problem for the electrostatic potentiol $\phi(r, \theta)$:

$$\nabla^2 \phi\ (r, \theta) = o, \qquad \text{in } D \tag{2.1}$$

$$\phi\ (c, \theta) = o, \qquad o \leqslant \theta \leqslant 2\pi \tag{2.2}$$

$$\phi\ (r, o) = f_1(r), \quad a < r < 1 \tag{2.3}$$

$$\phi\ (r, \pi) = f_2(r), \quad a < r < 1 \tag{2.4}$$

ϕ, $\dfrac{\partial \phi}{\partial \theta}$ are continuous across the line segments

$$0 \leqslant r < a, \quad 1 < r < c, \quad \theta = 0. \pi, \tag{2.5}$$

where D is the whole region lying within the circle of radius c except the two line segments $a < r < 1$, $\theta = 0, \pi$ and $f_i\ (r)$, $(i = 1, 2)$ are the prescribed potentials of the two strips.

The integral representation formula for $\phi(r, \theta)$ follows from the usual Green's function approach. Indeed, the function ϕ satisfying (2.1), (2.2) and (2.5) is

$$\phi(r, \theta) = \int_a^1 \sigma(r_0, 0)\, g(r, \theta \mid r_0, 0)\, dr_0 + \int_a^1 \sigma(r_0, \pi)\, g(r, \theta) \mid r_0, \pi)\, dr_0 \tag{2.6}$$

where Green's function g, as explained by Stakgold (1968), is given by

$$g(r, \theta \mid r_0, \theta_0) = -\frac{1}{4\pi} \log\,(r^2 + r_0^2 - 2r\, r_0 \cos\,(\theta - \theta_0))$$
$$+ \frac{1}{4\pi} \log\left(c^2 + \frac{r_0^2 r^2}{c^2} - 2r\, r_0 \cos\,(\theta - \theta_0)\right) \tag{2.7}$$

and $\sigma\ (r_0, 0)$ and $\sigma\ (r_0, \pi)$ are the unknown total charge densities of the two strips (per unit length) . Finally, when we use the boundary conditions (2.3) and (2.4) in (2.6), we obtain two simultaneous Fredholm integral equations of the first kind

$$\int_a^1 \sigma\ (r_0, 0)\ g\ (r, 0 \mid r_0, 0)\ dr_0 + \int_a^1 \sigma\ (r_0, \pi)\ g\ (r, 0 \mid r_0, \pi)\ dr_0 = f_1\ (r),\ a<r<1 \tag{2.8}$$

$$\int_a^1 \sigma\ (r_0, 0)\ g\ (r, \pi \mid r_0, 0)\ dr_0 + \int_a^1 \sigma\ (r_0, \pi)\ g\ (r, \pi \mid r_0, \pi)\ dr_0 = f_2\ (r),\ a<r<1, \tag{2.9}$$

where

$$g(r, o \mid r_0, 0) = g(r, \pi \mid r_0, \pi) = -\frac{1}{2\pi} \log | r-r_0 | + \frac{1}{2\pi} \log\left(c - \frac{r_0 r}{c}\right) \quad (2.10)$$

$$g(r, o \mid r_0, \pi) = g(r, \pi \mid r_0, 0) = -\frac{1}{2\pi} \log (r+r_0) + \frac{1}{2\pi} \log\left(c + \frac{r_0 r}{c}\right). \quad (2.11)$$

We solve the above simultaneous equations for the simple case when $f_1(r)=1$ and $f_2(r)=-1$, i.e. the two strips are charged to potentials ± 1. In this particular case, we have from eqns. (2.8) to (2.11)

$$\sigma(r_0, 0) = -\sigma(r_0, \pi) = h(r_0^2) \quad (2.12)$$

where the unknown function $h(r_0^2)$ is given by

$$\int_a^1 h(r_0^2)\, K(r_0, r)\, dr_0 = 2\pi, \qquad a < r < 1, \quad (2.13)$$

$$\begin{aligned} K(r_0, r) &= \log\left|\frac{r+r_0}{r-r_0}\right| + \log\left(\frac{1-r_0 r/c^2}{1+r_0 r/c^2}\right) \\ &= \log\left|\frac{r+r_0}{r-r_0}\right| - \frac{2r_0 r}{c^2} + O(c^{-6}). \end{aligned} \quad (2.14)$$

This suggests that the Fredholm integral eqn. (2.13) can be solved by setting

$$h(r_0^2) = h_0(r_0^2) + \frac{1}{c^2} h_2(r_0^2) + O(c^{-4}) \quad (2.15)$$

and we have from (2.13) to (2.15)

$$\int_a^1 h_0(r_0^2) \log\left|\frac{r+r_0}{r-r_0}\right| dr_0 = 2\pi, \qquad a < r < 1 \quad (2.16)$$

$$\int_0^1 h_2(r_0^2) \log\left|\frac{r+r_0}{r-r_0}\right| dr_0 = 2r \int_a^1 r_0 h_0(r_0^2)\, dr_0, \qquad a < r < 1. \quad (2.17)$$

Both the above equations are of the form

$$\int_a^1 t(r_0^2) \log\left|\frac{r+r_0}{r-r_0}\right| dr_0 = P(r), \qquad a < r < 1 \quad (2.18)$$

which has the inversion formula [see Appendix A]

$$t(r_0^2) = l\,(r_0^2) + C/T, \quad (2.19)$$

where

$$l\,(r_0^2) = -\frac{2}{\pi^2}\left(\frac{r_0^2-a^2}{1-r_0^2}\right)^{\frac{1}{2}} \int_a^1 \left(\frac{1-r^2}{r^2-a^2}\right)^{\frac{1}{2}} \frac{r\, P'(r)\, dr}{(r^2-r_0^2)},$$

$$C = \frac{1}{\pi K(a)}\left\{ P(r) - \int_a^1 l(r_0^2) \log\left|\frac{r+r_0}{r-r_0}\right| dr_0 \right\},$$

$T = \{ (r_0^2 - a^2)(1 - r_0^2) \}^{1/2}$ and $K(a) = F(\pi/2, a)$ is the complete elliptic integral of the first kind.

Using the inversion formula (2.19), we obtain from (2.16)

$$h_0(r_0^2) = \frac{2}{K(a)\, T} \tag{2.20}$$

and when we substitute this value in (2.17) and similarly invert the resulting equation

$$\int_a^1 h_2(r_0^2) \log \left| \frac{r + r_0}{r - r_0} \right| dr_0 = 2\pi r / k(a), \quad a < r < 1, \tag{2.21}$$

we obtain by using formula (B.2) given in Appendix B

$$h_2(r_0^2) = \frac{2}{K(a)\, T} \left\{ r_0^2 + (E(a)/k(a) - 1) \right\} \tag{2.22}$$

where $E(a) = E\left(\frac{\pi}{2}, a\right)$ is the complete elliptic integral of the second kind.

Finally, we substitute the values of $h_0(r_0^2)$, $h_2(r_0^2)$ from eqns. (2.20) and (2.22) in (2.15) to obtain an approximate expression for the charge densities of the charge densities of the two strips

$$\sigma(r_0, 0) = -\sigma(r_0, \pi) = \left(\frac{2}{K(a)\, T} \right) \left[1 + \frac{1}{c^2} \left\{ r_0^2 + [E(a)/K(a) - 1] \right\} + O(c^{-4}) \right] \tag{2.23}$$

This result agrees with that of Srivastava and Gupta (1971) when we use the well known Legendre's relation

$$E(a)\, K(a') + E(a')\, K(a) - K(a)\, K(a') = \frac{\pi}{2}, \; a' = (1 - a^2)^{\frac{1}{2}}. \tag{2.24}$$

3. Electrostatic Problem II

We now discuss the second electrostatic problem of finding the charge densities of two coplanar parallel strips charged to prescribed potentials when the strips are lying symmetrically outside parallel to the axis of the grounded long cylinder of radius $c \ll 1$. When we formulate this problem as discussed in the last section, it follows that the unknown total charge densities $\sigma(r_0, 0)$ and $\sigma(r_0, \pi)$ of the two strips of this problem when the strips are charged to potentials ± 1, are governed by

$$\sigma(r_0, 0) = -\sigma(r_0, \pi) = g(r_0^2) \tag{3.1}$$

where

$$\int_a^1 g(r_0^2)\, L(r_0, r)\, dr_0 = 2\pi,\ a < r < 1, \tag{3.2}$$

$$L(r_0, r) = \log \left| \frac{r+r_0}{r-r_0} \right| + \log \left(\frac{1-c^2/r_0 r}{1+c^2/r_0 r} \right)$$

$$= \log \left| \frac{r+r_0}{r-r_0} \right| - 2c^2/r_0 r + O(c^6). \tag{3.3}$$

Therefore, we set in (3.2)

$$g(r_0^2) = g_0(r_0^2) + c^2 g_2(r_0^2) + O(c^4) \tag{3.4}$$

and we get from (3.2) to (3.4)

$$\int_a^1 g_0(r_0^2) \log \left| \frac{r+r_0}{r-r_0} \right| dr_0 = 2\pi,\ a<r<1 \tag{3.5}$$

$$\int_a^1 g_2(r_0^2) \log \left| \frac{r+r_0}{r-r_0} \right| dr_0 = \frac{2}{r} \int_a^1 r_0^{-1} g_0(r_0^2)\, dr_0,\quad a < r < 1. \tag{3.6}$$

The above equations are inverted by using the inversion formula of eqn. (2.18) as explained in section 2 and we have

$$g_0(r_0^2) = \frac{2}{(K(a)\,T)},\ g_2(r_0^2) = \frac{2}{(a^2 K(a)\, T)} \left[\frac{a^2}{r_0^2} + \left\{ \frac{E(a)}{K(a)} - 1 \right\} \right] \tag{3.7}$$

where we have used some formulae given in Appendix B.

Finally relations (3.1), (3.4) (3.7) yield

$$\sigma(r_0, 0) = -\sigma(r_0, \pi) = \left(\frac{2}{K(a)\, T} \right) \left[1 + c^2 \left(\frac{1}{r_0^2} + \frac{1}{a^2} \left\{ \frac{E(a)}{K(a)} - 1 \right\} \right) \right] + O(c^4) \tag{3.8}$$

The above expressions for the charged densities of the two strips agree with the known result of Srivastava and Gupta (1971), when necessary corrections are carried out in their analysis.

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APPENDIX A.

In order to find the inversion formula of the integral equation (2.18), we first differentiate both sides of this equation with respect to r and obtain

$$\int_a^1 \frac{r_0\, t(r_0^2)\, dr_0}{(r^2-r_0^2)} = -\tfrac{1}{2}(P'(r)), \quad a < r < 1. \tag{A.1}$$

The solution of integral equation (A.1), as given by Srivastava and Lowengrub (1970), is

$$t(r_0^2) = l(r_0^2) + C/T, \tag{A.2}$$

where

$$l(r_0^2) = -\frac{2}{\pi^2}\left(\frac{r_0^2-a^2}{1-r_0^2}\right)^{\frac{1}{2}} \int_a^1 \left(\frac{1-r^2}{r^2-a^2}\right)^{\frac{1}{2}} \frac{r\, P'(r)\, dr}{(r^2-r_0^2)}, \tag{A.3}$$

and C is an unknown constant. This unknown constant C is easily evaluated by substituting the value of $t(r_0^2)$ from (A.2) in (2.1.8) and we readily obtain.

$$C = \frac{1}{\pi K(a)}\left\{ P(r) - \int_a^1 l(r_0^2) \log\left|\frac{r+r_0}{r-r_0}\right| dr_0 \right\}. \tag{A.4}$$

APPENDIX B

We give here some standard formulae which has been used in our analysis

$$\int_a^1 (T^{-1}) \log\left|\frac{r+r_0}{r-r_0}\right| dr_0 = \pi K(a), \quad a < r < 1 \tag{B.1}$$

$$\int_a^1 \frac{r_0^2}{T} \log\left|\frac{r+r_0}{r-r_0}\right| dr_0 = \pi \{r + K(a) - E(a)\}, \; a < r < 1 \tag{B.2}$$

$$\int_a^1 \frac{r_0^{-2}}{T} \log\left|\frac{r+r_0}{r-r_0}\right| dr_0 = \frac{\pi}{a^2}\left\{\frac{a}{r} - E(a) + K(a)\right\}, \quad a<r<1. \tag{B.3}$$