FINITE AND CONVOLUTION HARDY TRANSFORMATIONS

by H. D. CHAUBEY and R. S. PATHAK, *Department of Mathematics Banaras Hindu University, Varanasi 5*

(Communicated by R. S. Mishra, F. N. A.)

(*Received 14 May 1973 ; after revision 10 April 1974*)

An inversion theorem for finite Hardy transformation is established which is analogous to the finite Hankel transform theorem. A convolution Hardy transformation is defined. It is shown that it can be inverted by means of a suitable differential operater. Many interesting particular cases are deduced.

1. INTRODUCTION

The Hardy transformation (Hardy 1925), which we also call C_ν—transformation, is defined by

$$f(x) = \int_0^\infty F_\nu(tx)\, t\, dt \int_0^\infty C_\nu(vt)\, v\, f(v)\, dv \tag{1.1}$$

where

$$C_\nu(z) = \cos(a\pi)\, J_\nu(z) + \sin(a\pi)\, Y_\nu(z) \tag{1.2}$$

and

$$F_\nu(z) = \sum_{m=0}^{\infty} (-1)^m \frac{(\frac{1}{2}z)^{\nu+2a+2m}}{\Gamma(a+m+1)\,\Gamma(a+m+\nu+1)}$$

$$= 2^{2-\nu-2a}\, s_{\nu+2a-1,\,\nu}(z) \,/\, \{\Gamma(a)\,\Gamma(\nu+a)\} \tag{1.3}$$

$s_{\mu,\nu}(z)$ being Lommel's function (Watson 1966, p. 345), valid under following conditions (Cooke 1925, p. 384)

(*i*) $a > -1.\ a+\nu > -1,\ |\nu+2a| < 3/2.$

(*ii*) $t^\sigma f(t)$ integrable over $(0, \delta)$, $\sigma = \min(1-\nu-2a,\ 1-|\nu|,\ \frac{1}{2})$, $\delta > 0$

(*iii*) $t^{\frac{1}{2}} f(t)$ integrable over (δ, ∞).

The theory of the expansion formula (1.1) has been given by Cooke.

In this paper finite Hardy transformation

$$f(x) = \int_0^\infty F_\nu (tx)\, t dt \int_p^q C_\nu (vt)\, v f(v)\, dv \tag{1.4}$$

has been introduced. An inversion theorem for this transformation has been established and the inversion formula for the classical Hardy transformation (1.1) has been deduced. The results are extensions of the works already done by Hardy and Cooke.

The Hankel transformation, Y-transformation and H-transformation are particular cases of the Hardy transformation. Some more particular cases are given by Cooke. Therefore the inversion and some of the properties of these transformations can be deduced as particular cases of the results contained in this paper.

Fox (1953) has introduced the following Hankel convolution transformation. Let $\phi(v) \in L(o, \infty)$
and let

$$f(x) = \int_0^\infty K(x, v)\, (xv)^{\frac{1}{2}}\, \phi(v)\, dv, \tag{1.5}$$

where

$$K(x, v) = \int_0^\infty \frac{u\, J_\nu (xu)\, J_\nu (uv)\, du}{E(u)} \quad (\nu \geqslant -\tfrac{1}{2})$$

$$E(u) = \prod_{n=1}^\infty (1 + u^2 a_n^{-2}) \tag{1.6}$$

the a_n's being real, $\sum_{n=1}^\infty a_n^{-2}$ and $\phi(v)$ be of bounded variation in the neighbourhood of the point $v = x$, then

$$\prod_{n=1}^\infty \left\{1 - \frac{1}{a_n^2}\left(D^2 - \frac{\nu^2 - \frac{1}{4}}{x^2}\right)\right\} f(x) = \tfrac{1}{2}\{\phi(x+0) + \phi(x-0)\}$$

$$\left(D = \frac{d}{dx}\right) \tag{1.7}$$

In section 4, of the present paper a Hardy convolution transformation

$$f(x) = \int_0^\infty G(x, v)\, v\phi(v)\, dv \tag{1.8}$$

where

$$G(x, v) = \int_0^\infty \frac{u\, C_\nu (ux)\, F_\nu (uv)\, du}{E(u)}$$

has been studied. An inversion theorem has been established. This is an extension of the aforesaid Hankel convolution transformation.

Following results are well-known (Erdelyi, 1953, pp. 74, 40, 41, 85, 12, 41 ; Babister 1967 p. 83, Watson, 1966, p. 457).
In our analysis a and ν will be assumed to be real. If ν is an integer, we assume that the expression on the right of (1.9) is defined by its limiting value as $\nu \to$ an integer.

$$C_\nu(x) = \frac{\sin(a+\nu)\pi}{\sin \nu\pi} J_\nu(x) - \frac{\sin a\pi}{\sin \nu\pi} J_{-\nu}(x). \tag{1.9}$$

$$s_{\mu,\nu}(z) = S_{\mu,\nu}(z) - 2^{\mu-1}\Gamma\left(\frac{\mu-\nu+1}{2}\right)\Gamma\left(\frac{\mu+\nu+1}{2}\right)\operatorname{cosec}\nu\pi \times$$
$$\times\left[\cos\tfrac{1}{2}\pi(\mu-\nu)J_{-\nu}(z) - \cos\tfrac{1}{2}\pi(\mu+\nu)J_\nu(z)\right] \tag{1.10}$$

$$S_{\mu,\nu}(z) \sim z^{\mu-1}\{1 - [(\mu-1)^2-\nu^2]z^{-2} + [(\mu-1)^2-\nu^2][(\mu-3)^2-\nu^2]z^{-4} - \ldots\} \tag{1.11}$$

$$J_\nu(x) \sim \left(\frac{2}{\pi}\right)^{\frac{1}{2}} x^{-\frac{1}{2}} \cos(x - \tfrac{1}{4}\pi - \tfrac{1}{2}\nu\pi), \text{ as } x \to \infty. \tag{1.12}$$

$$\frac{d}{dx} C_\nu(\beta x) = \frac{\nu}{x} C_\nu(\beta x) - \beta C_{\nu+1}(\beta x). \tag{1.13}$$

$$\frac{d}{dx} s_{\mu,\nu}(\alpha x) = \frac{\nu}{x} s_{\mu,\nu}(\alpha x) + \alpha(\mu-\nu-1)s_{\mu-1,\nu+1}(\alpha x). \tag{1.14}$$

$$\int_0^h t^\mu J_\nu(t)\,dt = (\mu+\nu-1)hJ_\nu(h)s_{\mu-1,\nu-1}(h) - hJ_{\nu-1}(h)s_{\mu,\nu}(h) \tag{1.15}$$

$$\int_0^h t^\mu J_{-\nu}(t)\,dt = (\mu+\nu-1)hJ_\nu(h)s_{\mu-1,\nu-1}(h) + hJ_{1-\nu}(h)s_{\mu,\nu}(h) \tag{1.16}$$

If $\nu \geq -\frac{1}{2}$ and $\int_a^b F(R)\sqrt{R}\,dR$

exists and is absolutely convergent,

$$\int_a^b F(R)J_\nu(\lambda R)R\,dR = o\left(\frac{1}{\lambda^{\frac{1}{2}}}\right), \text{ as } \lambda \to \infty. \tag{1.17}$$

2. A Definite Integral Associated With The Hardy Transformation

Theorem 2.1—If N is positive and finite and $\mu - |\nu| + 1 > 0$, then

$$\int_0^N x F_\nu(\alpha x) C_\nu(\beta x)\,dx$$

$$= \frac{1}{\alpha^2-\beta^2}\Big[\Big(\frac{\alpha}{\beta}\Big)^{\mu+1}\frac{(\beta N)}{\sin \nu\pi}\{(\mu+\nu-1)(\sin(a+\nu)\pi J_\nu(\beta N)$$

$$-\sin a\pi J_\nu(\beta N)\} s_{\mu-1,\nu-1}(\beta N)-\{\sin(a+\nu)\pi J_{\nu-1}(\beta N)$$

$$+\sin a\pi J_{1-\nu}(\beta N) s_{\mu,\nu}(\beta N)\}-AN\{\alpha(\mu-\nu-1) s_{\mu-1,\nu+1}(\alpha N) C_\nu(\beta N)$$

$$+\beta s_{\mu,\nu}(\alpha N) C_{\nu+1}(\beta N)\}\Big] \tag{2.1}$$

where $a>-1$, $\nu+a>-1$,

$$A=\frac{2^{2-\nu-2a}}{\Gamma(a)\Gamma(\nu+a)} \text{ and } \mu=\nu+2a-1. \tag{2.2}$$

Proof: We know that $u=F_\nu(\alpha x)$ and $v = C_\nu(\beta x)$ are solutions of the differential equations

$$\frac{d^2u}{dx^2}+\frac{1}{x}\frac{du}{dx}+\Big(\alpha^2-\frac{\nu^2}{x^2}\Big)u=x^{\mu-1}\alpha^{\mu+1} \tag{2.3}$$

$$\frac{d^2v}{dx^2}+\frac{1}{x}\frac{dv}{dx}+\Big(\beta^2-\frac{\nu^2}{x^2}\Big)v=0 \tag{2.4}$$

respectively.

From these two equations we easily get

$$(\alpha^2-\beta^2)\int_0^N x\,uv\,dx=\int_0^N v\,\alpha^{\mu+1}x^\mu\,dx-\Big[x\Big\{v\frac{du}{dx}-u\frac{dv}{dx}\Big\}\Big]_0^N \tag{2.5}$$

where $a>-1$ and $\nu+a>-1$.

Using results (1.9), (1.13)—(1.16) this can easily be shown to be equivalent to

$$(\alpha^2-\beta^2)\int_0^N x\,F_\nu(\alpha x)\,C_\nu(\beta x)\,dx$$

$$=\Big(\frac{\alpha}{\beta}\Big)^{\mu+1}\frac{(\beta N)}{\sin\nu\pi}(\mu+\nu-1)\{\sin(a+\nu)\pi J_\nu(\beta N)$$

$$-\sin a\pi J_{-\nu}(\beta N)\} s_{\mu-1,\nu-1}(\beta N)$$

$$-\{\sin(a+\nu)\pi J_{\nu-1}(\beta N)+\sin a\pi J_{1-\nu}(\beta N)\} s_{\mu,\nu}(\beta N)$$

$$-A[N\{\alpha(\mu-\nu-1) C_\nu(\beta N) s_{\mu-1,\nu+1}(\alpha N)$$

$$+\beta C_{\nu+1}(\beta N) s_{\mu,\nu}(\alpha N)\}] \tag{2.6}$$

where A is the same as given in (2.2), $a>-1$ and $\nu+a>-1$. This completes the proof of the theorem.

3. Inversion of the Hardy Transformation

Now we shall establish an inversion theorem for finite Hardy Transformation.

Theorem 3.1— If for $p<\rho\leqslant q$, whre p and q are real and positive $\phi(\rho)$

is continuous, except for a finite number of finite discontinuities, and has only a finite number of maxima and minima and if (*i*) $a>-1$, $\nu+a>-1$ (*ii*) $-\frac{1}{2}\leq \nu\leq\frac{1}{2}$; (*iii*) $\nu+2a<3/2$ and (*iv*) $\int_p^q r\,\phi(r)\,dr$ is absolutely convergent, then

$$\int_0^\infty x\,dx \int_p^q F_\nu(\rho x)\,C_\nu(rx)\,r\,\phi(r)\,dr$$

$$\begin{aligned}
&=\tfrac{1}{2}\{\phi(\rho+o)+\phi(\rho-o)\}, && \text{if } p<\rho<q\\
&=\tfrac{1}{2}\{\phi(\rho+o)\}, && \text{if } \rho=p\\
&=\tfrac{1}{2}\{\phi(\rho-o)\}, && \text{if } \rho=q\\
&=0 \quad , && \text{if } 0<\rho<p \text{ or } \rho>q.
\end{aligned} \tag{3.1}$$

Proof: From the previous theorem, we have

$$\int_0^h x\,F_\nu(\rho x)\,C_\nu(r\,x)\,dx = I_1-I_2 \tag{3.2}$$

where

$$I_1=\frac{1}{\rho^2-r^2}\left[\left(\frac{\rho}{r}\right)^{\mu+1}\frac{(rh)}{\sin\nu\pi}\left\{(\mu+\nu-1)\,(\sin(a+\nu)\pi\,J_\nu(rh)-\sin a\pi\,J_{-\nu}(rh)\right\}\right.$$
$$\left.\times s_{\mu-1,\nu-1}(rh)-(\sin(a+\nu)\pi\,J_{\nu-1}(rh)+\sin a\pi\,J_{1-\nu}(rh)\,)\,s_{\mu,\nu}(rh)\right]$$

and

$$I_2=\frac{Ah}{\rho^2-r^2}\left\{\rho(\mu-\nu-1)\,s_{\mu-1,\nu+1}(\rho h)\,C_\nu(rh)+r s_{\mu,\nu}(\rho h)\,C_{\nu+1}(rh)\right\}.$$

Multiplying both sides of eqn. (3.2) by $r\,\phi(r)$, integrating from p to q with respect to r and taking the limit as h tends to infinity, we have

$$\int_p^q r\,\phi(r)\int_0^\infty x\,F_\nu(\rho x)\,C_\nu(rx)\,dr dx=\int_p^q I_1\,r\phi(r)\,dr-\int_p^q I_2\,r\phi(r)\,dr. \tag{3.3}$$

Now

$$\int_p^q I_1\,r\phi(r)\,dr$$

$$=\int_p^q\left[(\rho/r)^{\mu+1}\frac{(rh)}{\sin\nu\pi}\left\{(\mu+\nu-1)\,[\sin(a+\nu)\pi\,J_\nu(rh)-\sin a\pi\,J_{-\nu}(rh)]\,s_{\mu-1,\nu-1}(rh)\right.\right.$$
$$\left.\left.-[\sin(a+\nu)\pi\,J_{\nu-1}(rh)+\sin a\pi\,J_{1-\nu}(rh)]\,s_{\mu,\nu}(rh)\right\}\frac{r\,\phi(r)}{\rho^2-r^2}\,dr\right]$$

$$=(\mu+\nu-1)\,\rho^{\mu+1}\int_p^q\frac{(rh)}{r^{\mu+1}}\,s_{\mu-1,\nu-1}(rh)\,\text{cosec}\,\nu\pi\,[\sin(a+\nu)\pi\,J_\nu(rh)$$

$$- \sin a\pi\, J_{-\nu}(rh)\,]\frac{r\,\phi(r)}{\rho^2-r^2}\,dr$$

$$-\rho^{\mu+1}\int_p^q \frac{(rh)}{r^{\mu+1}}\, s_{\mu,\nu}(rh)\,\mathrm{cosec}\,\nu\pi\,[\,\sin(a+\nu)\pi\, J_{\nu-1}(rh)$$

$$+\sin a\pi\, J_{1-\nu}(rh)]\,\frac{r\,\phi(r)}{\rho^2-r^2}\,dr.$$

Expressing the Lommel function $s_{\mu,\nu}(x)$ in terms of Bessel functions by means of the relation (1.10) and using the asymptotic orders (1.11) and (1.12) we see that

$$(\mu+\nu-1)\,\rho^{\mu+1}\,h\int_p^q s_{\mu-1,\nu-1}(rh)\,\mathrm{cosec}\,\nu\pi\,[\,\sin(a+\nu)\pi\, J_\nu(rh)$$

$$-\sin a\pi\, J_{-\nu}(rh)\,]\frac{r^{1-\mu}\,\phi(r)}{\rho^2-r^2}\,dr$$

$$=(\mu+\nu-1)\,\rho^{\mu+1}\,h\int_p^q \mathrm{cosec}\,\nu\pi\,[\,\sin(a+\nu)\pi\, J_\nu(rh)-\sin a\pi\, J_{-\nu}(rh)\,]\times$$

$$\times[\,S_{\mu-1,\nu-1}(rh)+2^{\mu-2}\,\Gamma\left(\frac{\mu-\nu+1}{2}\right)\Gamma\left(\frac{\nu+\nu-1}{2}\right)\mathrm{cosec}\,\nu\pi\,\times$$

$$\times\{\cos\tfrac{1}{2}\pi(\mu-\nu)\,J_{-\nu+1}(rh)+\cos\tfrac{3}{2}\pi(\mu+\nu)\,J_{\nu-1}(rh\}\,]\,\frac{r^{1-\mu}\,\phi(r)}{\rho^2-r^2}\,dr$$

$$=(\mu+\nu-1)\,\rho^{\mu+1}\,h\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_p^q (rh)^{-\frac{1}{2}}\,\mathrm{cosec}\,\nu\pi\,[\,\sin(a+\nu)\pi\cos(rh-\tfrac{1}{4}\pi-\tfrac{1}{2}\nu\pi)$$

$$-\sin a\pi\cos(rh-\tfrac{1}{4}\pi+\tfrac{1}{4}\nu\pi)]\,\{\,(rh)^{\mu-2}+2^{\mu-2}$$

$$\times\,\Gamma\left(\frac{\mu-\nu+1}{2}\right)\Gamma\left(\frac{\mu+\nu-1}{2}\right)\mathrm{cosec}\,\nu\pi\left(\frac{2}{\pi}\right)^{\frac{1}{2}}(rh)^{-\frac{1}{2}}\,[\cos\tfrac{1}{2}\pi(\mu-\nu)\times$$

$$\times\cos(rh-\tfrac{3}{4}\pi+\tfrac{1}{2}\nu\pi)+\cos\tfrac{1}{2}\pi(\mu+\nu)\cos(rh+\tfrac{1}{4}\pi-\tfrac{1}{2}\nu\pi]\}\,\frac{r^{1-\mu}\,\varphi(r)}{\rho^2-r^2}\,dr.$$

$$=(\mu+\nu-1)\,\rho^{\mu+1}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}h^{\mu-\frac{3}{2}}\int_p^q \frac{r^{-\frac{3}{2}}\,\phi(r)}{\rho^2-r^2}\,\mathrm{cosec}\,\nu\pi$$

$$\times\,[\sin(a+\nu)\pi\cos(rh-\tfrac{1}{4}\pi-\tfrac{1}{2}\nu\pi)-\sin a\pi\cos(rh-\tfrac{1}{4}\pi+\tfrac{1}{2}\nu\pi)]\,dr$$

$$+(\mu+\nu-1)\,\rho^{\mu+1}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}2^{\mu-2}\,\Gamma\left(\frac{\mu-\nu+1}{2}\right)\Gamma\left(\frac{\mu+\nu-1}{2}\right)h^{\frac{1}{2}}\times$$

$$\times\int_p^q\{\cos\tfrac{1}{2}\pi(\mu-\nu)\cos(rh-\tfrac{3}{4}\pi+\tfrac{1}{2}\nu\pi)+\cos\tfrac{1}{2}\pi(\mu+\nu)\cos(rh+\tfrac{1}{4}\pi-\tfrac{1}{2}\nu\pi)\}$$

$$\times\,\mathrm{cosec}\,\nu\pi\,.\,[\sin(a+\nu)\pi\, J_\nu(rh)-\sin a\pi\, J_{-\nu}(rh)]\,r^{-\mu+\frac{1}{2}}\frac{\phi(r)}{\rho^2-r^2}\,dr$$

$$=O(h^{\mu-3/2})\int_p^q\frac{r^{-3/2}\,\phi(r)}{\rho^2-r^2}\,dr+O(h^{\frac{1}{2}})\int_p^q r\,J_{\pm\nu}(rh)\,\frac{r^{-\mu-\frac{1}{2}}}{\rho^2-r^2}\,\phi(r)\,dr \tag{3.4}$$

as $h\to\infty$.

Clearly the first term on the right-hand side of (3.4) tends to zero as $h \to \infty$, if $\mu < 3/2$ and

$$\int_p^q \frac{r^{-3/2}\,\phi(r)}{\rho^2 - r^2}\,dr$$

exists. Applying the result (1.17) to the second integral on the right hand side of (3.4), we see the last term of (3.4) can be replaced by

$$K\, h^{\frac{1}{2}}[o\,(1/h^{\frac{1}{2}})] \to 0, \text{ as } h \to \infty,$$

where K is a certain appropriate constant, provided that

$$\int_p^q \frac{r^{-\mu}\,\phi(r)}{\rho^2 - r^2}\,dr$$

exists and is absolutely convergent.

A similar analysis shows that the integral

$$\rho^{\mu+1}\int_p^q \frac{(rh)}{r^{\mu+1}}\, s_{\mu,\nu}(rh)\,\text{cosec}\,\nu\pi\,[\sin(a+\nu)\pi\, J_{\nu-1}(rh)$$
$$+\sin a\pi\, J_{1-\nu}(rh)]\;\frac{r\,\phi(r)}{\rho^2 - r^2}\,dr$$

tends to zero as h tends to infinity provided that $\nu \geqslant -\frac{1}{2}$, $\mu < \frac{3}{2}$ and the integral

$$\int_p^q r^{\sigma}\,\frac{\phi(r)}{\rho^2 - r^2}\,dr \text{ exists for } \sigma = -\tfrac{3}{2}, -\mu, -\mu-1,$$

Thus

$$\int_p^q r\,\phi(r)\int_0^\infty x\,F_\nu(\rho x)\,C_\nu(rx)\,dr dx = -\int_p^q I_2\, r\,\phi(r)\,dr \tag{3.5}$$

Now $\int_p^q I_2\, r\,\phi(r)\,dr$

$$= A\,h\int_p^q\Big[\;\rho\,(\mu-\nu-1)\Big\{(\rho h)^{\mu-2} + 2^{\mu-2}\,\Gamma\Big(\frac{\mu-\nu-1}{2}\Big)\,\Gamma\Big(\frac{\mu+\nu+1}{2}\Big)$$
$$\text{cosec}\,\nu\pi\;[\cos\tfrac{1}{2}\pi\,(\mu-\nu-2)\,J_{-\nu-1}(\rho h) - \cos\tfrac{1}{2}\pi\,(\mu+\nu)\,J_{\nu+1}(\rho h)\;\Big]\Big\}\times$$
$$\times\Big\{\frac{\sin(a+\nu)\pi}{\sin\nu\pi}\,J_\nu(rh) - \frac{\sin a\pi}{\sin\nu\pi}\,J_{-\nu}(rh)\Big\} +$$
$$+\,r\,\Big\{(\rho h)^{\mu-1} - 2^{\mu-1}\,\Gamma\Big(\frac{\mu-\nu+1}{2}\Big)\Gamma\Big(\frac{\mu+\nu+1}{2}\Big)\text{cosec}\,\nu\pi\quad\times$$
$$\times\;[\cos\tfrac{1}{2}\pi\,(\mu-\nu)\,J_{-\nu}(\rho h) - \cos\tfrac{1}{2}\pi\,(\mu-\nu)\,J_\nu(\rho h)]\Big\}\times$$
$$\times\Big\{\frac{\sin(a+\nu)\pi}{\sin\nu\pi}\,J_{\nu+1}(rh) + \frac{\sin a\pi}{\sin\nu\pi}\,J_{-\nu-1}(rh)\Big\}\Big]\,\frac{r\,\varphi(r)}{\rho^2 - r^2}\,dr. \tag{3.6}$$

Further, using asymptotic order (1.12) it can easily be shown that

$$Ah\left[\int_p^q (\mu-\nu-1)\,\rho\left(\frac{\sin(a+\nu)\pi}{\sin\nu\pi}J_\nu(rh)-\frac{\sin a\pi}{\sin\nu\pi}J_{-\nu}(rh)\right)(\rho h)^{\mu-2}+\right.$$

$$\left.+\,r\left(\frac{\sin(a+\nu)\pi}{\sin\nu\pi}J_{\nu+1}(rh)+\frac{\sin a\pi}{\sin\nu\pi}J_{-\nu-1}(rh)\right)(\rho h)^{\mu-1}\right]\frac{r\,\phi(r)}{\rho^2-r^2}\,dr$$

$\to 0$ as $h\to\infty$, if $\mu<\frac{1}{2}$ and the integrals

$$\int_p^q \frac{r^{\frac{1}{2}}\phi(r)}{\rho^2-r^2}\,dr \text{ and} \int_p^q \frac{r^{3/2}\phi(r)}{\rho^2-r^2}\,dr \text{ exist.}$$

The non-zero term on the right-hand side of (3.6) is easily seen to be equivalent to

$$\frac{2}{\pi}\int_p^q \frac{1}{\sqrt{r\rho}}\,\{\rho\cos(a\pi-rh+\tfrac{1}{4}\pi+\tfrac{1}{2}\nu\pi)\cos(\rho h-\tfrac{1}{4}\pi-\tfrac{1}{2}\mu\pi)$$

$$-r\sin(a\pi-rh+\tfrac{1}{4}\pi+\tfrac{1}{2}\nu\pi)\sin(\rho h-\tfrac{1}{4}\pi-\tfrac{1}{2}\mu\pi)\}\,\frac{r\,\phi(r)}{\rho^2-r^2}\,dr+\frac{P}{h},$$

where P is finite for all the value of h,

$$=\frac{1}{\pi}\int_p^q\left[\frac{(\rho+r)}{\rho^2-r^2}\cos(a\pi+\tfrac{1}{2}\nu\pi-\tfrac{1}{2}\mu\pi+h(\rho-r)\right]+\frac{(\rho-r)}{\rho^2-r^2}\sin(h(\rho+r)-$$

$$-(a\pi+\tfrac{1}{2}\nu\pi+\tfrac{1}{2}\mu\pi)\quad\frac{r\phi(r)}{\sqrt{r\rho}}\,dr$$

$$=\frac{1}{\pi}\int_p^q\left[-\frac{\sin h(\rho-r)}{(\rho-r)}+\cos\frac{\{h(\rho+r)-(2a+\nu)\pi\}}{(r+\rho)}\right]\frac{r\,\phi(r)}{\sqrt{r\rho}}\,dr$$

Using the fact that

$$\lim_{h\to 8}\int_p^q \phi(x)\frac{\sin\{h(x-r)\}}{x-r}\,dx=\tfrac{1}{2}\pi\,\{\phi(r+0)+\phi(r-0)\}\quad\text{for } p<r<q.$$

and that the second part of integral on the right converges to zero as $h\to\infty$,

$$-\int_p^q I_2\,r\,\phi(r)\,dr=\tfrac{1}{2}\,\{\phi(\rho+0)+\phi(\rho-0)\},\ \text{if } p<\rho<q.$$

Thus

$$\int_0^\infty x\,dx\int_p^\rho F_\nu(\rho x)\,C_\nu(rx)\,r\,\phi(r)\,dr$$

$$=\tfrac{1}{2}\begin{cases}\phi(\rho+0)+\phi(\rho-0)\ ; & \text{if } p<\rho<q\\ \phi(\rho+0)\ , & \text{if } p=\rho\\ \phi(\rho-0)\ ; & \text{if } q=\rho\\ 0\ ; & \text{if } 0<\rho<p \text{ or } \rho>q.\end{cases}$$

This completes the proof of the theorem 3.1.

Corollary 3.2.—Let

$$\phi(\rho) = O(\rho^{\xi}) \quad \text{as } \rho \to 0$$
$$= O(\rho^{\eta}) \quad \text{as } \rho \to \infty$$

and let it satisfy Dirichlet's conditions in $0 < \rho < \infty$. Then for $-\frac{1}{2} \leqslant \nu \leqslant \frac{1}{2}$, $\nu + 2a < 3/2$, $\mu = \nu + 2a - 1$, $\xi > \max(\mu, \frac{1}{2})$ and $\eta < \min(-5/2, \mu - \frac{3}{2})$

$$\int_0^\infty x\,dx \int_0^\infty r\phi(r)\, F_\nu(\rho x)\, C_\nu(rx)\, dr = \tfrac{1}{2}\{\phi(\rho+0) + \phi(\rho-0)\}.$$

Special cases : Taking $a = 0$ and $a = \frac{1}{2}$ in (3.1) we deduce the following useful results after certain modifications.

Theorem 3.3 If for $p \leqslant \rho \leqslant q$, where p and q are real and positive, $\phi(\rho)$ is continuous, except for a finite number of finite discontinuities, and has only a finite number of maxima and minima and if

(*i*) $-1 \leqslant \nu$ and (*ii*) $\int_p^q r\,\phi(r)\,dr$ is absolutely convergent, then

$$\int_0^\infty x dx \int_p^q J_\nu(\rho x)\, J_\nu(rx)\, r\phi(r)\, dr$$
$$\begin{aligned} &= \tfrac{1}{2}\{\phi(\rho+0) + \phi(\rho-0)\}, && \text{if } p < \rho < q \\ &= \tfrac{1}{2}\{\phi(\rho+0)\}, && \text{if } \rho = p \\ &= \tfrac{1}{2}\{\phi(\rho-0)\} && \text{if } \rho = q \\ &= 0 && \text{if } 0 < \rho < p \text{ or } \rho > q. \end{aligned}$$

This theorem is given in the treatise of Bessel functions (Gray *et al.* 1952, p. 97).

Theorem 3.4 If for $p \leqslant \rho \leqslant q$, where p and q are real and positive, $\phi(\rho)$ is continuous, except for a finite number of finite discontinuities and has only a finite number of maxima and minima and if

(*i*) $-\frac{1}{2} \leqslant \nu \leqslant \frac{1}{2}$ (*ii*) $\int_p^q r\,\phi(r)\,dr$ is absolutely convergent, then

$$\int_0^\infty x dx \int_p^q H_\nu(\rho x)\, Y_\nu(rx)\, r\phi(r)\, dr$$
$$\begin{aligned} &= \tfrac{1}{2}\{\phi(\rho+0) + \phi(\rho-0)\}, && \text{if } p < \rho < q \\ &= \tfrac{1}{2}\{\phi(p+0)\}, && \text{if } \rho = p \\ &= \tfrac{1}{2}\{\phi(\rho-0)\}, && \text{if } \rho = q \\ &= 0, && \text{if } 0 < \rho < p \text{ or } \rho > q. \end{aligned}$$

This theorem in the finite form is not given elsewhere as far as known to the authors.

4. A Convolution Hardy Transformation

To establish inversion formula for the convolution Hardy transformation defined by

$$f(x) = \int_0^\infty G(x, t)\, \phi(t)\, dt \tag{4.1}$$

where

$$G(x, t) = \int_0^\infty \frac{u\, C_\nu(ux)\, F_\nu(ut)\, du}{E(u)} \tag{4.2}$$

$$E(u) = \prod_{n=1}^\infty \left(1 + \frac{u^2}{a_n^2}\right) \tag{4.3}$$

a_n's are real and $\sum a_n^{-2}$ is convergent, (4.4)

we prove at first, formally that it can be inverted by means of the differential operator

$$\Delta x = D^2 + \frac{1}{x} D - \frac{\nu^2}{x^2}. \tag{4.5}$$

We note that $C_\nu(ux)$ satisfies the homogeneous equation

$$(\Delta x + u^2)\, C_\nu(ux) = 0. \tag{4.6}$$

Let us consider the integral equation

$$f(x) = \int_0^\infty \frac{u\, C_\nu(ux)}{E(u)} \left\{ \int_0^\infty v\, F_\nu(uv)\, \phi(v)\, dv \right\} du. \tag{4.7}$$

In view of (4.6) we have

$$\left(1 - \frac{\Delta x}{a_n^2}\right) C_\nu(ux) = \left(1 + \frac{u^2}{a_n^2}\right) C_\nu(ux). \tag{4.8}$$

Therefore, formally (4.7) yields

$$\begin{aligned} \prod_{n=1}^\infty \left(1 - \frac{\Delta x}{a_n^2}\right) f(x) &= \prod_{n=1}^\infty \left(1 + \frac{u^2}{a_n^2}\right) \left\{ \int_0^\infty \frac{u\, C_\nu(ux)}{E(u)} \int_0^\infty v\, F_\nu(uv)\, \phi(u)\, dv \right\} du \\ &= \int_0^\infty u\, C_\nu(ux)\, du \int_0^\infty v\, F_\nu(uv)\, \phi(v)\, dv \\ &= \phi(x) \end{aligned} \tag{4.9}$$

on using Hardy inversion theorem. Changing the order of integration (4.7) can be written as

$$f(x) = \int_0^\infty G(x, v)\, v\phi(v)\, dv \tag{4.10}$$

where

$$G(x, v) = \int_0^\infty \frac{u\, C_\nu(ux)\, F_\nu(\mu v)\, du}{E(u)}. \tag{4.11}$$

Now we shall establish an inversion theorem which gives conditions under which the above manipulation is valid.

Theorem 4.1 If (*i*) $-\frac{1}{2} \leqslant \nu \leqslant \frac{1}{2}$, (*ii*) $\nu+2a < 3/2$, (*iii*) a_n is real for all positive integral value of n and $\sum_{n=1}^{\infty} a_n^{-2}$ is convergent, (*iv*) $\phi(v) \in L(0, \infty)$ and (*v*) $\phi(v)$ is of bounded variation in the neighbourhood of $v = x$, then the integral transform (4.10) is inverted by the differential operator

$$\prod_{n=1}^{\infty}\left\{1-\frac{1}{a_n^2}\left(D^2+\frac{1}{x}D-\frac{\nu^2}{x^2}\right)\right\} f(x) = \phi(x). \tag{4.12}$$

Proof : The following convergence properties of $E_m(u)$ given by Fox (1953), will be used in the proof. For any given positive integer p there exist positive constants M and u_0 such that

$$0 < \frac{1}{E(u)} < \frac{M}{u^p}, \quad 0 < \frac{1}{E_m(u)} < \frac{M}{u^p} \tag{4.13}$$

where

$$u > u_0 \text{ and } E_m(u) = \prod_{n=m+1}^{\infty}\left(1+\frac{u^2}{a_n^2}\right). \tag{4.14}$$

Also $\lim_{m\to\infty} E_m(u) = 1.$ (4.15)

If we differentiate (4.7) with respect to x, it follows from (*iii*), (1.9), (1.12) and (4.13) that the resultant integral is uniformly convergent with respect to x in the interval $0 \leq x \leq X$, for any positive X.

This is also true if we apply the operator $(1-a_n^{-2}\,\Delta x)$ a finite number of times. Thus we have

$$\prod_{n=1}^{m}\left(1-\frac{\Delta x}{a_n^2}\right) f(x) = \int_0^{\infty} \frac{u\,C_\nu(ux)}{E_m(u)} \int_0^{\infty} v\,F_\nu(uv)\,\phi(v)\,dv\,du$$

$$= \int_0^{\infty} p_m(x,u)\,du$$

where

$$P_m(x,u) = \frac{u\,C_\nu(ux)}{E_m(u)} \int_0^{\infty} v\,F_\nu(uv)\,\phi(v)\,dv.$$

Let us write

$$\phi(x) = \int_0^{\infty} P(x,u)\,du = \int_0^{\infty} u\,C_\nu(ux) \int_0^{\infty} v\,F_\nu(uv)\,\phi(v)\,dv\,du.$$

Now

$$\prod_{n=1}^{m}\left(1-\frac{\Delta x}{a_n^2}\right) f(x)-\phi(x) = \int_0^X P_m(x, u)-P(x, u)\} du+$$

$$+\int_X^\infty P_m(x, u)\, du - \int_X^\infty P(x, u)\, du.$$

By the same technique as employed by Fox, it can be proved that for $\epsilon > 0$, we can choose X at first, independent of m, and then choose m_0 so that the absolute value of each of the above integrals on the right is less than $\epsilon/3$ for $m > m_0$. Therefore we have

$$\lim_{m\to\infty} \prod_{n=1}^{m}\left(1-\frac{\Delta x}{a_n^2}\right) f(x) = \phi(x) \tag{4.16}$$

To justify the change of order of integration in (4.7), we note, in view of (iii) and (1.9) – (1.12) that the integral with respect to v in (4.7) is uniformly convergent in the interval $0 \leq u \leq U$, for any positive U.

Hence if $Q(x, u, v)$ stands for the integral in (4.7) then

$$\int_0^U\left\{\int_0^\infty Q(x, u, v)\, dv\right\}du=\int_0^\infty\left\{\int_0^U Q(x, u, v)\, du\right\} dv$$

$$=\int_0^\infty\left\{\int_0^\infty Q(x, u, v)\, du\right\} dv- \int_0^\infty\left\{\int_U^\infty Q(x, u, v)\, du\right\}dv.$$

Now it can be proved as in the case of Fox that the integral tends to zero as $U\to\infty$. So that the change of order of integration in (4.7) is justified.

Thus the universion of (4.10) by differential operator (4.5) is established.

Acknowledgement

Authors express their gratitude to the referee for his valuable comments.

References

Babister, A. W, (1967). Transcendental Functions Satisfying Non-homogeneous Linear Differential Equations. The MacMillan Company, New York.

Cooke, R. G. (1925). The Inversion Formula of Hardy and Titchmarsh. *Proc. Lond. math. Soc.*, **24**, 381-420.

Erde'lyi, A. (1953). Higher Transcendental Functions. Vol. II. Mc-Graw Hill Book Co., Inc., New York,

Fox, C, (1953), The inversion of convolution transform by differential operators. *Proc. Am. math Soc.*, 4. 880-87.

Gray A., Mathews, G. B. and MacRobert, T. M. (1952). A treatise on Bessel functions and their applications to physics. Mac-Millan and Co. London.

Hardy, G. H. (1924). Some formulae in the theory of Bessel functions. *Proc. Lond. math. Soc.*, 23, ixi-ixiii

Watson, G. N- (1966). A Treatise on the Theory of Bessel Functions. Cambridge University Press, London.