ON A FUNCTIONAL EQUATION CONNECTED WITH INFORMATION AND INFORMATION IMPROVEMENT

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The paper deals with unified study of functional equations in two and three variables whose solutions under suitable conditions are used to characterize information theoretic measures associated with two and three distributions of a random discrete variate. The three parametric family of information improvement studied here for the first time includes measures studied earlier as particular cases. So is the case for study in two variables.

1. INTRODUCTION

Kendall (1964) formed a functional equation and in terms of its solution (information function) characterized under suitable conditions Shannon's entropy (1948). This has been generalized by Darcozy (1970) for information of type β, which gives entropy of type β.

Rathie and Kannappan (1971) have unified their study by considering the functional equation,

$$f(x)+g(x)f\left(\frac{y}{1-x}\right)=f(y)+g(y)f\left(\frac{x}{1-x}\right),\ x, y, \in [0,1] \text{ with } x+y \in [0,1] \tag{1.1}$$

where the function g satisfies the functional equation,

$$g(x+y-xy)=g(x)\,g(y) \text{ for } x, y \in [0,1] \tag{1.2}$$

and have found that when $f(0)=f(1)$ and $f(\frac{1}{2})=1$, the non-trivial solutions of (1.2) are $g(x)=(1-x)$ and $g(x)=(1-x)^{\beta}$.

Extending the study of Kendall and Daroczy to functions of two and three variables, Sharma and Autar (1973) and Autar (1975) have studied under suitable conditions the functional equations,

$$f(x,y)+(1-x)^{\beta}(1-y)^{\alpha-\beta}f\left(\frac{u}{1-x},\frac{v}{1-y}\right)$$
$$=f(u,v)+(1-u)^{\beta}(1-v)^{\alpha-\beta}f\left(\frac{x}{1-u},\frac{y}{1-v}\right), \quad (1.3)$$

and

$$f(x,y,z)+(1-x)^{\alpha}(1-y)^{\beta-\alpha}(1-z)^{\alpha-\beta}f\left(\frac{u}{1-x},\frac{v}{1-y},\frac{w}{1+z}\right)$$
$$=f(u,v,w)+(1-u)^{\alpha}(1-v)^{\beta-\alpha}(1-w)^{\alpha-\beta}f\left(\frac{x}{1-u},\frac{y}{1-v},\frac{z}{1-w}\right) \quad (1.4)$$

for $x, y, z, u, v, w \in [0, 1)$ with $x+u, y+v, z+w \in [0, 1]$.

Functional equation (1.3) leads to characterization of non-additive information (directed divergence) of type (α, β) while (1.4) leads to that of non-additive information improvement of type (α, β).

In this paper we shall study functional equations, which are extensions of (1.1) for functions of two and three variables. Such a study gives a unified study of information theoretic measures associated with two and three distributions of a given random variate and in the case of three variables, it gives rise to a three parameter family of solutions, obtained here for the first time. These measures generalize Kullback's (1959) information which is useful in statistical estimation and Theil's (1967) information improvement which has found applications in Economic Analysis.

2. Unified Functional Equation in two Variables

Let us consider a real valued function $f(x, y)$ defined over $[0, 1]^2$ satisfying functional equation

$$f(x,y)+g(x,y)f\left(\frac{u}{1-x},\frac{v}{1-y}\right)=f(u,v)+g(u,v)f\left[\frac{x}{1-u},\frac{y}{1+v}\right] \quad (2.1)$$

for $x, y, u, v, \in [0, 1)$ with $x+u, y+v \in [0, 1]$,

where

$$g(x+u-xu, y+v-yv) = g(x, y)\, g(u, v) \text{ for } x, y, u, v \in [0, 1]. \quad (2.2)$$

It would be noted that (1.3) is included in (2.1) as a particular case. We obtain the solution of (2.1) in theorem 1 under suitable conditions.

Theorem 1 —The most general continuous non-constant functions $f(x, y)$ satisfying (2.1) while $g(x, y)$ satisfies (2.2) under the boundary conditions,

$$f(0, 0) = f(1, 1) \quad (2.3)$$

and
$$f(1, \tfrac{1}{2}) = 1 \tag{2.4}$$
are given by
$$f(x, y) = \frac{x^{\lambda} y^{\mu} + (1-x)^{\lambda} (1-y)^{\mu} - 1}{2^{-\mu} - 1} = f_{\lambda, \mu}(x, y) \tag{2.5}$$
where λ and μ are arbitrary constants,
and
$$f(x, y) = -x \log y - (1-x) \log (1-y) = f^{K}(x, y) \tag{2.6}$$
the second solution being obtained for $\lambda = 1$ and $\mu \to 0$.

Proof : The proof is on the lines of one given by Rathie and Kannappan (1971) the case of single variable.

Let us define
$$h(x, y) = g(1-x, 1-y) \text{ for } x, y \in [0, 1]. \tag{2.7}$$
Then (2.2) reduces to the Cauchy functional equation
$$h(xu, yv) = h(x, y)\, h(u, v) \text{ for } x, y, u, v \in [0, 1]. \tag{2.8}$$
Setting $x = y = 0$ in (2.2), there arise two possibilities, viz. $g(u, v) = 0$ and $g(0.0) = 1$, for all $u, v \in [0, 1]$. The first of these leads to $f(x, y) = 1$, a constant value We would therefore consider only the second possibility.

Next, putting $x = y = 1$ in (2.2), there are again two possibilities, viz. $g(u, v) = 1$ and $g(1, 1) = 0$ for all $u, v \in [0, 1]$ As can be easily seen here also when $g(u, v) = 1$, we obtain $f(x, y) = 1$, a constant solution.

Thus for non-trivial and non-constant solutions, we shall take
$$g(0, 0) = 1 \tag{2.9}$$
and
$$g(1, 1) = 0 \tag{2.10}$$
Now setting $u = x, v = y$ in (2.2), we have
$$g(2x - x^2, 2y - y^2) = g^2(x, y) \text{ for all } x, y \in [0, 1]$$
showing that $g(x, y) \geqslant 0$ for all $x, y \in [0, 1]$.

Now, it can be shown as by Rathie and Kannappan (1971) that $g(x, y) > 0$ for all $x, y \in [0, 1)$. Taking $x = y = 0$ in (2.1) and uslng (2.9), we get that $f(0,0) = 0$ and hence from (2.3), we have
$$f(0, 0) = f(1, 1) = 0. \tag{2.11}$$
Now replacing u, v in (2.1) by $1-x$ and $1-y$ respectively and using (2.11), we get
$$f(x, y) = f(1-x, 1-y) \text{ for } x, y \in (0,1). \tag{2.12}$$

Consider now four arbitrary numbers p_1, p_2, q_1, q_2 belonging to the open interval (0,1). On setting $p_1=1-x$, $p_2=u(1-x)^{-1}$, $q_1=1-y$, $q_2=v(1-y)^{-1}$ in (2.1), we get.

$$f(p_1q_1)+g(1-p_1,1-q_1)f(p_2,q_2)$$
$$=f(p_1p_2,q_1q_2)+g(p_1p_2,q_1q_2)f\left(\frac{1-p_1}{1-p_1p_2},\frac{1-q_1}{1-q_1q_2}\right) \tag{2.13}$$

for $p_1, q_1 \in [0,1,]$ $p_2, q_2 \in (0,1)$ such that $p_1p_2 \neq 1$ and $q_1q_2 \neq 1$.

Let us consider the function

$$G(p_1,p_2;q_1,q_2)=f(p_1,q_1)+[g(p_1,q_1)+g(1-p_1,1-q_1)]f(p_2,q_2)$$

for all $p_1, p_2, q_1, q_2 \in (0,1)$.

$$=f(p_1p_2,q_1q_2)+g(p_1p_2,q_1q_2)f\left(\frac{1-p_1}{1-p_1p_2},\frac{1-q_1}{1-q_1q_2}\right)$$
$$+g(p_1,q_1)f(p_2,q_2) \text{ using (2.13)}$$
$$=f(p_1p_2,q_1q_2)+g(p_1p_2,q_1q_2)\left[f\left(\frac{1-p_1}{1-p_1p_2},\frac{1-q_1}{1-q_1q_2}\right)\right.$$
$$\left.+\frac{g(p_1,q_1)}{g(p_1p_2,q_1q_2)}f(p_2q_2)\right]$$
$$=f(p_1p_2,q_1q_2)+g(p_1p_2,q_1q_2)\left[f\left(\frac{1-p_1}{1-p_1p_2},\frac{1-q_1}{1-q_1q_2}\right)\right.$$
$$\left.+g\left(\frac{p_1(1-p_2)}{1-p_1p_2},\frac{q_1(1-q_2)}{1-q_1q_2}\right)f(p_2,q_2)\right]$$

using (2.2)

$$=f(p_1p_2,q_1q_2)+g(p_1p_2,q_1q_2)\left[f\left\{\frac{p_2(1-p_1)}{1-p_1p_2},\frac{q_2(1-q_1)}{1-q_1q_2}\right\}\right.$$
$$\left.+g\left\{\frac{p_2(1-p_1)}{1-p_1p_2},\frac{q_2(1-q_1)}{1-q_1q_2)}\right\}f(p_1,q_1)\right]$$

using (2.13)

$$=f(p_1p_2,q_1q_2)+g(p_1p_2,q_1q_2)\left[f\left\{\frac{1-p_2}{1-p_1p_2},\frac{1-q_2}{1-q_1q_2}\right\}\right.$$
$$\left.+g\left(\frac{p_2(1-p_1)}{1-p_1p_2},\frac{q_2(1-q_1)}{1-q_1q_2}\right)f(p_1,q_1)\right]$$

using (2.12)

$$=G(p_2,p_1;q_2,q_1). \tag{2.14}$$

Equations (2.14) and (2.4) together yield

$$f(p_1,q_1)=\frac{g(p_1,q_1)+g(1-p_1,1-q_1)-1}{g(1,\frac{1}{2})+g(0,\frac{1}{2})-1}$$

for all $p_1 \cdot q_1 \in [0,1]$ provided $g(1,\frac{1}{2})+g(0,\frac{1}{2}) \neq 1$. (2.15)

The most general continuous solution of (2.8) is given by $h(x,y) = x^{\lambda} y^{\mu}$ where λ, μ are arbitrary constants.

Therefore

$$g(x,y) = (1-x)^{\lambda}(1-y)^{\mu} \text{ for } x,y \in [0,1]. \tag{2.16}$$

The solution obtained from (2.15) and (2,16) is

$$f_{\lambda,\mu}(x,y) = \frac{x^{\lambda} y^{\mu} + (1-x)^{\lambda}(1-y)^{\mu} - 1}{2^{-\mu} - 1}.$$

Lastly we consider the case when

$$g(1,\tfrac{1}{2}) + g(0,\tfrac{1}{2}) = 1 \Rightarrow 2^{-\mu} = 1 \text{ i.e. } \mu = 0.$$

Since an explicit finite value of $f(x,y)$ does not exist in this case, when $\lambda \neq 1$, we take

$$f_{\lambda,0}(x,y) = \lim_{\mu \to 0} f_{\lambda,\mu}(x,y), \qquad \lambda \neq 1,$$

and when $\lambda = 1$, it becomes the limit of an indeterminate form given by

$$f_{1,0}(x,y) = \lim_{\mu \to 0} f_{1,\mu}(x,y) = -x \log y - (1-x)\log(1-y) = f^{K}(x,y).$$

This completes the proof of the theorem.

Remarks : For $g(x,y) = (1-x)^{\lambda}(1-y)^{\mu}$ the functional equation (2.1) becomes

$$f(x,y) + (1-x)^{\lambda}(1-y)^{\mu} f\left(\frac{u}{1-x}, \frac{v}{1-y}\right)$$
$$= f(u,v) + (1-u)^{\lambda}(1-v)^{\mu} f\left(\frac{x}{1-u}, \frac{y}{1-v}\right) \tag{2.17}$$

while when $\mu = 0$, this takes the form

$$f(x,y) + (1-x)^{\lambda} f\left\{\frac{u}{1-x}, \frac{v}{1-y}\right\}$$
$$= f(u,v) + (1-u)^{\lambda} f\left\{\frac{x}{1-u}, \frac{y}{1-v}\right\}, (\lambda \neq 1) \tag{2.18}$$

The functional equation (2.18) appears to be untenable for study under the present boundary conditions. However, when $\lambda = 1$, the functional equation reduces to

$$f(x,y) + (1-x) f\left\{\frac{u}{1-x}, \frac{v}{1-y}\right\}$$
$$= f(u,v) + (1-u) f\left\{\frac{x}{1-u}, \frac{y}{1-v}\right\} \tag{2.19}$$

which characterizes the function $f^{K}(x,y)$ called the Kerridge's (1961) inaccuracy function studied by Sharma and Autar. (1973).

Note 1: The function $f_{\lambda,\mu}(x,y)$ obtained above would be called as information function of type (λ, μ).

Particular Cases

(1) If we take $\lambda=\beta$ and $\mu=\alpha-\beta$ in (2.5), we get

$$f(x,y)=\frac{x^{\beta}y^{\alpha-\beta}+(1-x)^{\beta}(1-y)^{\alpha-\beta}-1}{2^{\beta-\alpha}-1} \text{ for } x,y\in[0,1], \alpha>\beta\geqslant 0. \quad (2.20)$$

which has been obtained otherwise by Sharma and Autar (1974). The function given by (2.20) is called as relative information function of type (α,β).

(2) Further taking $\alpha=1$ in (2.20), we get

$$f(x,y)=\frac{x^{\beta}y^{1-\beta}+(1-x)^{\beta}(1-y)^{1-\beta}-1}{2^{\ 1}-1}, \text{ for } x,y\in[0,1],\ \beta>0\ (\beta\neq 1) \quad (2.21)$$

which have been studied by Rathie and Kannappan (1972*a*)

(3) Next taking $\beta=1$ in (2.20). we get

$$f(x,y)=\frac{xy^{\alpha-1}+(1-x)(1-y)^{\alpha-1}-1}{2^{1-\alpha}-1} \text{ for } x,y\in[0,1],$$

$$\alpha>0\ (\alpha\neq 1) \quad (2.22)$$

which is the inaccuracy of type $(\alpha, 1)$ (Sharma and Autar 1973, 1976),

Thus $f_{\lambda,\mu}(x,y)$ unifies the study made by Rathie and Kannappan (1972*a*) and Sharma and Autar (1973, 1974, 1976).

3. Unified Functional Equation in three Variables

Let us consider a real-valued function $f(x,y,z)$ defined on $[0,1]^3$, and satisfying the functional equation,

$$f(x,y,z)+g(x,y,z)\,f\left(\frac{u}{1-x},\frac{v}{1-y},\frac{w}{1-z}\right)$$

$$=f(u,v,w)+g(u,v,w)\,f\left(\frac{x}{1-u},\frac{y}{1-v},\frac{z}{1-w}\right) \quad (3.1)$$

for $x,y,z,u,v,w\in[0,1]$ with $x+u,\ y+v,\ z+w\in[0,1]$,
where

$$g(x+u-xu,\ y+v-yv,\ z+w-zw)=g(x,y,z)\,g(u,v,w)$$

$$\text{for} \quad x,y,z,u,v,w\in[0,1]. \quad (3.2)$$

It would be noted that (1.4) is included in (3.1) as a particular case. It is straight to obtain explicit solutions of the functional equation (3.1) on lines identical to Theorem 1. We state the results in the following theorem.

Theorem 2—The most general continuous and non-constant functions $f(x,y,z)$ satisfying (3.1) while $g(x,y,z)$ satisfies (3.2) under the boundary conditions,

$$f(0,0,0)=f(1,1,1) \quad (3.3)$$

and

$$f(1, 1, \tfrac{1}{2}) = 1 \tag{3.4}$$

are given by

$$f(x, y, z) = \frac{x^{\lambda} y^{\mu} z^{\nu} + (1-x)^{\lambda} (1-y)^{\mu} (1-z)^{\nu} - 1}{2^{-\nu} - 1} = f_{\lambda,\mu,\nu}(x, y, z) \tag{3.5}$$

and

$$f(x, y, z) = x \log \frac{y}{z} + (1-x) \log \frac{(1-y)}{(1-z)} = f^{T}(x, y, z) \tag{3.6}$$

the second solution being obtained for $\lambda = 1$ and $\mu = -\nu \to 0$.

Note 2 : Some solutions can obtained in two variables instead of three. These have not been considered in the above theorem. However these are discussed later as particular cases of the above solutions.

Remarks : For $g(x, y, z) = (1-x)^{\lambda} (1-y)^{\mu} (1-z)^{\nu}$, the functional equation (3.1) becomes

$$f(x, y, z) + (1-x)^{\lambda} (1-y)^{\mu} (1-z)^{\nu} f\left(\frac{u}{1-x}, \frac{v}{1-y}, \frac{w}{1-z}\right)$$

$$= f(u, v, w) + (1-u)^{\lambda} (1-v)^{\mu} (1-z)^{\nu} f\left(\frac{x}{1-u}, \frac{y}{1-x}, \frac{z}{1-w}\right) \tag{3.7}$$

while when $\nu = 0$, this takes the form

$$f(x, y, z) + (1-x)^{\lambda} (1-y)^{\mu} f\left(\frac{u}{1-x}, \frac{v}{1-y}, \frac{w}{1-z}\right.$$

$$= f(u, v, w) + (1-u)^{\lambda} (1-v)^{\mu} f\left[\frac{x}{1-u}, \frac{y}{1-v}, \frac{z}{1-w}\right] \tag{3.8}$$

This appears to be untenable for study under the present boundary conditions. However, when $\lambda = 1$ and $\mu = \nu \to 0$ the functional equation reduces to

$$f(x, y, z) + (1-x) f\left[\frac{n}{1-x}, \frac{v}{1-y}, \frac{w}{1-z}\right]$$

$$= f(u, v, w) + (1-u) f\left[\frac{x}{1-u}, \frac{y}{1-v}, \frac{z}{1-w}\right] \tag{3.9}$$

which characterizes the function $f^{T}(x, y, z)$ called the Theil's (1967) informa-improvement.

Note 3 : The function $f_{\lambda,\mu,\nu}(x, y, z)$ obtained above would be called as information improvement function of type (λ, μ, ν).

Particular Cases

(1) If we take $\lambda = \alpha$, $\mu = \alpha - \beta$ and $u = \beta - \alpha$ in (3.5) we get

$$f(x, y, z) = \frac{x^{\alpha} y^{\alpha-\beta} z^{\beta-\alpha} + (1-x)^{\alpha} (1-y)^{\alpha-\beta} (1-z)^{\beta-\alpha} - 1}{2^{\alpha-\beta} - 1} \tag{3.10}$$

which has been earlier obtained differently by Autar (1975). The function given by (3.10) is called the generalized information function of type (α, β).

(2) Further, taking $\alpha = 1$ in (3.10), we get

$$f(x, y, z) = \frac{xy^{1-\beta} z^{\beta-1} + (1-x)(1-y)^{1-\beta}(1-z)^{\beta-1} - 1}{2^{1-\beta}-1}, \tag{3.11}$$

$$\text{for } x, y, z \in [0, 1], \beta > 0 \ (\beta \neq 1)$$

which has been obtained by Rathie and Kannappan (1972 *b*)

Thus $f_{\lambda,\mu,\nu}(x, y, z)$ unifies the study made by Autar (1975) and Rathie and Kannapan (1972 *b*).

4. Information Theoretic Measures

In this section we shall examine the information theoretic measures that can be defined in terms of the solutions obtained in theorem 1 and 2. Consider a discrete random variate x taking finite number of values x_1, $x_2, \cdots\cdots, x_n$ associated with two probability distributions

$$P = (p_1, p_2, \ldots, p_n) \quad \text{and} \quad Q = (q_1, q_2, \ldots, q_n).$$

Then we have the following :

Definition 1. If $f(x, y)$ is an information function of type (λ, μ), then

$$I_n^{(\lambda,\mu)}\begin{pmatrix} p_1, p_2, \cdots, p_n \\ b_1, q_2, \ldots, q_n \end{pmatrix} = \text{def} \sum_{i=2}^{n} P_i^{\lambda} Q_i^{\mu} f\left[\frac{p_i}{P_i}, \frac{q_i}{Q_i}\right] \tag{4.1}$$

where $P_i = p_1+p_2+\cdots+p_i$, $Q_i = q_1+q_2+\cdots+q_i$, for $i = 1, 2, \cdots, n$ with $P_n = Q_n = 1$.

Theorem 3—The information of type (λ, μ) for distribution P and Q is given by

$$I_n^{(\lambda,\mu)}\begin{pmatrix} p_1, p_2, \ldots, p_n \\ q_1, q_2, \cdots, q_n \end{pmatrix} = \left(\sum_{i=1}^{n} p_i^{\lambda} q_i^{\mu} - 1\right)(2^{-\mu}-1)^{-1}. \tag{4.2}$$

Proof . Substituting the expression for $f(x, y)$ from (2.5) in (4.1) we have

$$I_n^{(\lambda,\mu)}\begin{pmatrix} p_1, p_2, \cdots, p_n \\ q_1, q_2, \cdots, q_n \end{pmatrix} = (2^{-\mu}-1)^{-1} \sum_{i=2}^{n} \left[p_i^{\lambda} q_i^{\mu} + P_{i-1}^{\lambda} Q_{i-1}^{\mu} - P_i^{\lambda} Q_i^{\mu}\right]$$

$$= (2^{-\mu}-1)^{-1}\left[\sum_{i=2}^{n} p_i^{\lambda} q_i^{\mu} + P_1^{\lambda} Q_1^{\mu} - P_n^{\lambda} Q_n^{\mu}\right]$$

$$= (2^{-\mu}-1)^{-1}\left[\sum_{i=1}^{n} p_i^{\lambda} q_i^{\mu} - 1\right], \text{ which is (4.2).}$$

Several properties like null-information, symmetry, expansibility etc. can be easily derived for $I_n^{(\lambda,\mu)}$. We mention below two properties for $I_n^{(\lambda,\mu)}$.

Recursive-property for $I_n^{(\lambda,\mu)}$.

$$I_n^{(\lambda,\mu)}\begin{pmatrix} p_1, p_2, \ldots, p_n \\ q_1, q_2, \ldots, q_n \end{pmatrix} - I_{n-1}^{(\lambda,\mu)}\begin{pmatrix} p_1+p_2, p_3, \ldots, p_n \\ q_1+q_2, q_3, \ldots, q_n \end{pmatrix}$$

$$= (p_1+p_2)^\lambda (q_1+q_2)^\mu I_2^{(\lambda,\mu)}\begin{pmatrix} \frac{p_1}{p_1+p_2}, & \frac{p_2}{p_1+p_2} \\ \frac{q_1}{q_1+q_2}, & \frac{q_2}{q_1+q_2} \end{pmatrix}; \; (n \geqslant 3) \tag{4.3}$$

with $p_1+p_2, q_1+q_2 > 0$.

Strong non-additivity-property for $I_n^{(\lambda,\mu)}$.

$$I_{mn}^{(\lambda,\mu)}\begin{pmatrix} P^* P' \\ Q^* Q' \end{pmatrix} = I_n^{(\lambda,\mu)}\begin{pmatrix} P \\ Q \end{pmatrix} + \sum_{j=1}^{n} p_i^\lambda q_i^\mu \; I_m^{(\lambda,\mu)}\begin{pmatrix} p_{1i}, p_{2i}, \ldots, p_{mi} \\ q_{1i}, q_{2i}, \ldots, q_{mi} \end{pmatrix} \tag{4.4}$$

where $P = (p_1, p_2, \ldots, p_n)$, $P' = (p_{11}, p_{21}, \ldots, p_{m1}, \ldots, p_{1n}, p_{2n}, \ldots, p_{mn})$

$P^*P' = (p_1 p_{11}, p_1 p_{21}, \ldots, p_1 p_{m1}, \ldots\ldots, p_n p_{2n}, \ldots, p_n p_{mn})$

etc. and $\sum_{j=1}^{m} p_{ji} = 1$, $\sum_{j=1}^{m} q_{ji} = 1$.

$$I_{mn}^{(\lambda,\mu)}\begin{pmatrix} P^* P'' \\ Q^* Q'' \end{pmatrix} = I_n^{(\lambda,\mu)}\begin{pmatrix} P \\ Q \end{pmatrix} + I_m^{(\lambda,\mu)}\begin{pmatrix} P'' \\ Q'' \end{pmatrix}$$

$$+(2^{-\mu}-1)\, I_n^{(\lambda,\mu)}\begin{pmatrix} P \\ Q \end{pmatrix} I_m^{(\lambda,\mu)}\begin{pmatrix} P'' \\ Q'' \end{pmatrix} \tag{4.5}$$

where $P'' = (P_1, P_2, \ldots, P_m)$, $P_* P'' = (p_1 P_1, p_1 P_2, \ldots, p_1 P_m, \ldots, p_n P_1, p_n P_2, \ldots, p_n P_m)$

etc. and $\sum_{j=1}^{m} P_j = 1$ and $\sum_{j=1}^{m} Q_j = 1$.

Next let there be three probability distributions

$P = (p_1, p_2, \ldots, p_n)$, $Q = (q_1, q_2, \ldots, q_n)$ and $R = (r_1, r_2, \ldots, r_n)$

associated with random variate x, then we have

Defination 2—If $f(x, y, z)$ is an information improvement function of type (λ, μ, ν), then the information-improvement of type (λ, μ, ν) is defined by the relation

$$I_n^{(\lambda,\mu,\nu)}\begin{bmatrix} p_1, p_2, \ldots, p_n \\ q_1, q_2, \ldots, q_n \\ r_1, r_2, \ldots, r_n \end{bmatrix} \text{ def } \sum_{i=2}^{n} P_i^\lambda Q_i^\mu R_i^\nu f\left[\frac{p_i}{P_i}, \frac{q_j}{Q_j}, \frac{r_i}{R_i}\right] \tag{4.6}$$

where $P_i = p_1 + p_2 + \ldots + p_i$, $Q_i = q_1 + q_2 + \ldots + q_i$, $R_i = r_1 + r_2 + \ldots + r_i$, for $i = 1, 2, \ldots, n$ with $P_n = Q_n = R_n = 1$.

Theorem 4—The information improvement of type (λ, μ, ν) is given by

$$I_n^{(\lambda,\mu,\nu)} \begin{pmatrix} p_1, p_2 \ldots, p_n \\ q_1, q_2 \ldots, q_m \\ r_1, r_2, \ldots, r_1 \end{pmatrix} = \left\{ \sum_{i=1}^{n} p_i^\lambda q_i^\mu r_i^\nu - 1 \right\} (2^{-\nu} - 1)^{-1} . \tag{4.7}$$

Proof: Substituting the expression for $f(x, y, z)$ from (3.5) in (4.6) we have

$$I_n^{(\lambda,\mu\nu)} \begin{pmatrix} p_1, p_2, \ldots, p_n \\ q_1, q_2 \ldots, q_n \\ r_1, r_1, \ldots, r_1 \end{pmatrix} = (2^{-\nu} - 1) \sum_{i=2}^{n} \left[p_i^\lambda q_j^\mu r_j^\nu + P_{i-1}^\lambda Q_{i-1}^\mu R_{i-1}^\nu - P_i^\lambda Q_i^\mu R_i^\nu \right]$$

$$= (2^{-\nu} - 1 \left[\sum_{i=2}^{n} p_i^\lambda q_i^\mu r_i^\nu + P_1^\lambda Q_1^\mu R_n^\nu - P_n^\lambda Q_n^\mu R_n^\nu \right]$$

$$= (2^{-\nu} - 1)^{-1} \left\{ \sum_{i=1}^{n} p_i^\lambda q_i^\mu r_i^\nu - 1 \right\} \quad \text{which is (4.7)}$$

Several properties like null-information improvement, symmetry expansibility etc. can be easily derived for $I_n^{(\lambda,\mu,\nu)}$ also. We mention below two properties for $I_n^{(\lambda,\mu,\nu)}$.

Recursive-property for $I_{n-1}^{(\lambda,\mu,\nu)}$

$$I_n^{(\lambda,\mu,\nu)} \begin{pmatrix} p_1, p_2, \ldots, p_n \\ q_n, q_2, \ldots, q_n \\ r_1, r_2, \ldots, r_n \end{pmatrix} - I_{n-1}^{(\lambda,\mu,\nu)} \begin{pmatrix} p_1 + p_2, p_3, \ldots, p_n \\ q_1, + q_2, q_3, \ldots, q_n \\ r_1 + r_2, r_3, \ldots, r_n \end{pmatrix}$$

$$= (p_1 + p_2)^\lambda (q_1 + q_2)^\mu (r_1 + r_2)^\nu I_2^{(\lambda,\mu,\nu)} \begin{pmatrix} \frac{p_1}{p_1 + p_2}, & \frac{p_2}{p_1 + p_2} \\ \frac{q_1}{q_1 + q^2}, & \frac{q_2}{q_1 + q_2} \\ \frac{r_1}{r_1 + r_2}, & \frac{r_2}{r_1 + r_2} \end{pmatrix} ; (n \geqslant 3) \tag{4.8}$$

Strong non-additivity property for $I_n^{(\lambda,\mu,\nu)}$

$$I_{mn}^{(\lambda,\mu\nu)} \begin{bmatrix} P^* \; P' \\ Q^* \; Q' \\ R^* \; R' \end{bmatrix} = I^{(\lambda,\mu,\nu)} \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$$

$$+ \sum_{i=1}^{n} p_i^\lambda q_i^\mu r_i^\nu I_m^{(\lambda,\mu,\nu)} \begin{bmatrix} p_{1i}, p_{2i}, \ldots, p_{mi} \\ q_{1i}, q_{2i}, \ldots, q_{mj} \\ r_{1i}, r_{2i}, \ldots, r_{mi} \end{bmatrix} \tag{4.9}$$

where

$$P=(p_1, p_2, \ldots, p_n), \quad P'=(p_{11}, p_{21}, \ldots, p_{m1}, \ldots, p_{1n}, p_{2n}; \ldots, p_{mn})$$

$$P*P'=(p_1 p_{11}; p_1 p_{21}, \ldots, p_1 p_{m1}, \ldots, p_n p_{1n}, p_n p_{2n}, \ldots, p_n p_{mn}) \text{ etc.}$$

and

$$\sum_{j=1}^{m} p_{ji}=1, \quad \sum_{j01}^{m} q_{ji}=1.$$

An interesting special case of (4,9) is given below :

$$I_{mn}^{(\lambda,\mu,\nu)}\begin{bmatrix} P*P' \\ Q*Q'' \\ R*R'' \end{bmatrix}=I^{(\lambda,\mu,\nu)}\begin{bmatrix} P \\ Q \\ R \end{bmatrix}+I_m^{(\lambda,\mu,\nu)}\begin{bmatrix} P'' \\ Q'' \\ R'' \end{bmatrix}$$

$$+(2^{-\nu}-1_n^{(\lambda,\mu,\nu)}\begin{bmatrix} P \\ Q \\ R \end{bmatrix} I_m^{(\lambda,\mu,\nu)}\begin{bmatrix} P'' \\ Q'' \\ R'' \end{bmatrix} \qquad (4.10)$$

where

$$P''=(P_1, P_2, \ldots, P_m), \quad P^* P''=(p_1 P_1, p_1 P_2, \ldots, p_1 P_m, \ldots,$$
$$\ldots, p^n p_1, \ldots, p_n p_m^n) \quad \text{etc.}$$

and $\sum_{j=1}^{m} P_j=1, \sum_{j=1}^{m} Q_j=1$ and $\sum_{j=1}^{m} R_j=1.$

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