ELECTROSTATIC PROBLEMS OF TWO COPLANAR PARALLEL STRIPS

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We present here solutions of two electrostatic problems involving two charged coplanar parallel strips, each having unit total charge per unit length, when the strips are lying symmetrically either inside a long grounded cylinder in presence of a line source of a suitable line density along the axis of the cylinder or outside a long grounded cylinder. By usual Green's function approach, each problem is first reduced to governing Fredholm integral equation of the first kind which is further solved by the regular perturbation technique to obtain an approximate expression for the charge density of the two strips.

1. Introduction

Recently, some two dimensional electrostatic problems of two charged coplanar parallel strips have been discussed by various authors—Tranter (1960), Srivastava and Gupta (1971), Goel and Jain (1976). Tranter (1960) has determined the charge densities of the two strips when these are charged to potentials ± 1 in a free space. Srivastava and Gupta (1971) have studied the perturbation in the charge densities of the two strips when these are placed symmetrically inside or outside a grounded cylinder by reducing each problem to a Fredholm integral equation of the second kind with the help of triple integral equations and finite Hilbert transform techniques given by Srivastava and Lowengrub (1970). Goel and Jain (1976) observed that analysis of Srivastava and Gupta (1971) can be simplified by applying the usual Green's function approach to reduce each problem to a governing Fredholm integral equation of the first kind which can be easily solved by the perturbation technique of Jain and Kanwal (1972) to obtain approximate expressions for the charge densities of the two strips.

We present here solutions of two electrostatic problems involving two coplanar parallel strips charged to an unknown constant potential A such that each strip has unit total charge per unit length. In the first problem, the two strips are situated symmetrically inside a grounded cylinder and are parallel to a line source of line density —2 along the axis of the cylinder whereas in the second problem, the two strips are situated symmetrically outside a grounded cylinder. In each problem we proceed by the usual Green's function approach to obtain a governing Fredholm integral equation of the first kind which embodies the differential equation as well as the boundary conditions of the problem. The governing Fredholm integral equation in each problem is further solved by the perturbation technique of Jain and Kanwal (1972) to get an approximate expression for the charge density of the two strips in terms of some unknown constants which are finally evaluated by using the condition that each strips has unit total charge per unit length.

2. Electrostatic Problem I

We first take up the electrostatic problem of two coplanar parallel strips, charged to an unknown constant potential A, placed symmetrically inside a grounded cylinder of radius $c >> 1$ and parallel to a line source of line density -2 along the axis of the cylinder when each strip has unit total charge per unit length. We use cylindrical coordinates (r, θ, z) and take the z-axis along the axis of the cylinder. Thus we have to solve the following two dimensional boundary value problem for the electrostatic potential $\phi(r, \theta)$

$$\nabla^2 \phi(r, \theta) = 0 \qquad \text{in } D \tag{2.1}$$

$$\phi(c, \theta) = 0, \qquad 0 \leq \theta \leq 2\pi \tag{2.2}$$

$$\phi(r, \theta) = A, \qquad a < r < 1, \ \theta = 0, \pi \tag{2.3}$$

$$\phi, \ \frac{\partial \phi}{\partial \theta}$$

are continuous across the line segments $0 < r < a$, $1 < r < c$, $\theta = 0, \pi$ where D is the whole region lying inside the circle $r = c$ except the origin and the line segments

$$a < r < 1, \quad \theta = 0, \pi. \tag{2.4}$$

The integral representation formula for ϕ is obtained by the usual Green's function approach due to Stakgold (1968), Kanwal (1971) satisfying (2.1.), (2.2) and (2.4) and it is given by

$$\phi(r, \theta) = \int_a^1 \sigma(r_0, 0)\, g(r, \theta \mid r_0, 0)\, dr_0 + \int_a^1 \sigma(r_0, \pi)\, g(r, \theta \mid r_0, \pi)\, dr_0 - 2g(r, \theta \mid 0, 0) \tag{2.5}$$

where $\sigma(r_0, 0)$ and $\sigma(r_0, \pi)$ are the unknown total charge densities (per unit length) of the two strips and the Green function g is defined by

$$g(r, \theta \mid r_0, \theta_0) = -\frac{1}{4\pi} \log (r^2 + r_0^2 - 2rr_0 \cos(\theta - \theta_0)) +$$

$$+\frac{1}{4\pi} \log \left(c^2 + \frac{r^2 r_0^2}{c^2} - 2rr_0 \cos(\theta - \theta_0) \right). \tag{2.6}$$

Using bounday conditions (2.3) in (2.5), we obtain two simultaneous Fredholm integral equations of the first kind

$$\int_a^1 \sigma(r_0, 0) g(r, 0 \mid r_0, 0) \, dr_0 + \int_a^1 \sigma(r_0, \pi) g(r, 0 \mid r_0, \pi) \, dr_0 -$$

$$-\frac{1}{\pi} \log \left(\frac{c}{r} \right) = A, \quad a < r < 1 \tag{2.7}$$

$$\int_a^1 \sigma(r_0, 0) g(r, \pi \mid r_0, 0) \, dr_0 + \int_a^1 \sigma(r_0, \pi) g(r, \pi \mid r_0, \pi) \, dr_0 -$$

$$-\frac{1}{\pi} \log \left(\frac{c}{r} \right) = A, \quad a < r < 1 \tag{2.8}$$

where

$$g(r, 0 \mid r_0, 0) = g(r, \pi \mid r_0, \pi) = -\frac{1}{2\pi} \log \left| 1 - \frac{r_0}{r} \right|$$

$$+\frac{1}{2\pi} \log \left(1 - \frac{rr_0}{c^2} \right) + \frac{1}{2\pi} \log \left(\frac{c}{r} \right) \tag{2.9}$$

$$g(r, 0 \mid r_0, \pi) = g(r, \pi \mid r_0, 0) = -\frac{1}{2\pi} \log \left(1 + \frac{r_0}{r} \right) +$$

$$+\frac{1}{2\pi} \log \left(1 + \frac{rr_0}{c^2} \right) + \frac{1}{2\pi} \log \left(\frac{c}{r} \right). \tag{2.10}$$

Since each strip has unit total charge per unit length i. e.

$$\int_a^1 \sigma(r_0, 0) \, dr_0 = \int_a^1 \sigma(r_0, \pi) \, dr_0 = 1 \tag{2.11}$$

it readily follows from equations (2.7) to (2.11) that

$$\sigma(r_0, 0) = \sigma(r_0, \pi) = r_0^{-1} g(r_0^2) \tag{2.12}$$

and the unknown function $g(r_0^2)$ is defined by the Fredholm integral equation of the first kind

$$\int_a^1 r_0^{-1}\, g(r_0^2)\, K(r_0, r)\, dr_0 = -2\pi A, \quad a < r < 1 \tag{2.13}$$

where

$$\begin{aligned} K(r_0, r) &= \log\left|1 - \frac{r_0^2}{r^2}\right| - \log\left(1 - \frac{r_0^2\, r^2}{c^4}\right) \\ &= \log\left|1 - \frac{r_0^2}{r^2}\right| + \frac{r_0^2\, r^2}{c^4} + 0\,(c^{-8}). \end{aligned} \tag{2.14}$$

Now we set

$$g(r_0^2) = g_0(r_0^2) + \frac{1}{c^4}\, g_4(r_0^2) + 0\,(c^{-8})\,; \quad A = A_0 + \frac{1}{c^4}\, A_4 + 0\,(c^{-8}) \tag{2.15}$$

and equations (2.13) to (2.15) lead to the following set of equations

$$\int_a^1 r_0^{-1}\, g_0(r_0^2)\, \log\left|1 - \frac{r_0^2}{r^2}\right| dr_0 = -2\pi A_0, \quad a < r < 1 \tag{2.16}$$

$$\int_a^1 r_0^{-1}\, g_4(r_0^2)\, \log\left|1 - \frac{r_0^2}{r^2}\right| dr_0 = -\, r^2 \int_a^1 r_0\, g_0(r_0^2)\, dr_0 - 2\pi A_4, \quad a < r < 1 \tag{2.17}$$

and so on. The above two integral equations are of the form

$$\int_a^1 r_0^{-1}\, t(r_0^2)\, \log\left|1 - \frac{r_0^2}{r^2}\right| dr_0 = f(r), \quad a < r < 1 \tag{2.18}$$

which has the inversion formula (see Appendix A)

$$t(r_2^0) = m(r_0^2) + C/T \tag{2.19}$$

where

$$m(r_0^2) = \frac{2}{\pi^2}\left(\frac{r_0^2 - a^2}{1 - r_0^2}\right)^{\frac{1}{2}} \int_a^1 \left(\frac{1 - r^2}{r^2 - a^2}\right)^{\frac{1}{2}} \frac{r^2 f'(r)\, dr}{(r^2 - r_0^2)}$$

$$C = \frac{2a}{\pi \log\left(\dfrac{1-a}{1+a}\right)} \left\{ f(r) - \int_a^1 r_0^{-1}\, m(r_0^2)\, \log\left|1 - \frac{r_0^2}{r^2}\right| dr_0 \right\}$$

$$T = \{(1 - r_0^2)\,(r_0^2 - a^2)\}^{\frac{1}{2}}.$$

Using the above inversion formula, we obtain by inverting equation (2.16)

$$g_0(r_0^2) = \frac{-4a\, A_0}{T \log\left(\frac{1-a}{1-a}\right)} . \tag{2.20}$$

Now we substitute this value of g_0 in right side of (2.17), invert the resulting equation by using formula (2.19) and thus obtain

$$g_4(r_0^2) = \frac{-4a}{T\log\left(\frac{1-a}{1+a}\right)}\left[\left\{r_0^4 - \frac{(1+a^2)}{2}\, r_0^2 - \frac{a(1+a^2)}{2\log\left(\frac{1-a}{1+a}\right)}\right\}A_0 + A_4\right] \tag{2.21}$$

We have from eqns. (2.12), (2.15), (2.20), and (2.21)

$$\sigma(r_0, 0) = \sigma(r_0\, \pi) = \frac{-4a}{r_0\, T\log\left(\frac{1-a}{1+a}\right)}\left[\left\{1 + \frac{1}{c^4}\right.\right.$$

$$\left.\left[r_0^4 - \frac{(1+a^2)}{2}\, r_0^2 - \frac{a(1+a^2)}{2\log\left(\frac{1-a}{1+a}\right)}\right]\right\} A_0 + \frac{A_4}{c^4} + 0(c^{-8})\Big] . \tag{2.22}$$

The above expression for the total charge density of the two strips still contains the unknown constants A_0, A_4 which are readily evaluated by substituting (2.22) in (2.11) and we get

$$A_0 = -\frac{1}{2\pi}\log\left(\frac{1-a}{1+a}\right); \; A_4 = \frac{-a}{4\pi}(1+a^2). \tag{2.23}$$

Finally, we put the above values of A_0, A_4 in (2.22) and obtain an approximate expression for the total charge density of the two strips

$$\sigma(r_0, 0) = \sigma(r_0, \pi) = \frac{2a}{\pi\, r_0 T}\left\{1 + \frac{1}{c_4}\left[r_0^2 - \frac{1}{2}(1+a^2)\right] r_0^2 + 0(c^{-8})\right\} . \tag{2.24}$$

Even the limiting result obtained from (2.24) when $c \to \infty$, seems to be new.

3. Electrostatic Problem II

Let us now consider the second electrostatic problem of two coplanar parallel strips, charged to an unknown constant potential b, situated symmetrically outside a grounded cylinder of radius $c<<1$ when each strip has unit total charge per unit length and the cylinder has total charge-2 per unit length. When we formulate this two dimensional electrostatic problem as in the last section, it follows that the total charge densities $\sigma(r_0, 0)$ and $\sigma(r_0\ \pi)$ of the two strips in this problem are governed by the equations

$$\sigma(r_0, 0) = \sigma(r_0, \pi) = r_0^{-1} h(r_0^2) \tag{3.1}$$

$$\int_a^1 r_0^{-1} h(r_0^2) L(r_0, r)\, dr_0 = -2\pi B, \quad a < r < 1 \tag{3.2}$$

where

$$L(r_0, r) = \log\left|1 - \frac{r_0^2}{r^2}\right| - \log\left(1 - \frac{c^4}{r_0^2 r^2}\right) = \log\left|1 - \frac{r_0^2}{r^2}\right| + \frac{c^4}{r^2 r_0^2} + 0(c^8) \tag{3.3}$$

$$B = b + \frac{1}{\pi} \log c - \frac{1}{\pi} \int_a^1 \sigma(r_0, 0) \log r_0\, dr_0 \tag{3.4}$$

$$\int_a^1 \sigma(r_0, 0)\, dr_0 = \int_a^1 \sigma(r_0, \pi)\, dr_0 = 1. \tag{3.5}$$

We process eqn. (3.2) as (2.13) by setting

$$h(r_0^2) = h_0(r_0^2) + c^4 h_4(r_0^2) + 0(c^8) \; ; \quad B = B_0 + c^4 B_4 + 0(c^8) \tag{3.6}$$

and consequently obtain

$$h(r_0^2) = \frac{-4a}{T \log\left\{\frac{1-a}{1+a}\right\}} \left[\left\{1 + \frac{c^4(1+a^2)}{2a^2}\right.\right.$$

$$\left.\left.\left[r_0^{-2} - \frac{1 + \frac{1+a^2}{2a} \log\left(\frac{1-a}{1+a}\right)}{a \log\left(\frac{1-a}{1+a}\right)}\right]\right\} B_0 + c^4 B_4 + 0(c^8)\right]. \tag{3.7}$$

When we substitute (3.7) in (3.1), we readily obtain from (3.5) the values of the unknown constants B_0, B_4 and these are given by

$$B_0 = -\frac{1}{2\pi} \log\left(\frac{1-a}{1+a}\right), \quad \text{i.e. } B_4 = \left\{-\frac{(1+a^2)}{4\pi a^3}\right\}. \tag{3.8}$$

Finaly, we put these values of B_0, B_4 in (3.7) and obtain by using relation (3.1)

$$\sigma(r_0, 0) = \sigma(r_0, \pi) = \frac{2a}{\pi r_0 T}\left\{1 - \frac{c^4(1+a^2)^2}{4a^4}\left[1 - \frac{2a^2}{(1+a^2) r_0^2}\right] + (c^8)\right\} \tag{3.9}$$

which also seems to be a new result.

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APPENDIX A

In order to obtain inversion formula of integral equation of the first kind (2.18), we first differentiate both sides of this equation with respect to r and get

$$\int_a^1 \frac{r_0\, t(r_0^2)}{r^2-r_0^2}\, dr_0 = \frac{r}{2} f'(r), \quad a < r < 1 \tag{A.1}$$

The solution of this integral equation is known and is given by Srivastava and Lowengrub (1970) as

$$t(r_0^2) = m(r_0^2) + C/T \tag{A.2}$$

where

$$m(r_0^2) = \frac{2}{\pi^2}\left\{\frac{r_0^2-a^2}{1-r_0^2}\right\}^{\frac{1}{2}} \int_a^1 \left\{\frac{1-r^2}{r^2-a^2}\right\}^{\frac{1}{2}} \frac{r^2 f'(r)}{(r^2-r_0^2)}\, dr$$

and C is an unknown constant. The unknown constant C is easily obtained by substituting (A.2) in (2.18) in the form

$$C = \frac{2a}{\pi \log\left\{\dfrac{1-a}{1+a}\right\}} \left\{ f(r) - \int_a^1 r_0^{-1}\, m(r_0^2) \log\left|1-\frac{r_0^2}{r^2}\right| dr_0 \right\} \tag{A.3}$$

Appendix B

We give here values of some definite integrals used in our analysis.

$$\int_a^1 (r_0\, T)^{-1} \log \left| 1 - \frac{r_0^2}{r^2} \right| dr_0 = \frac{\pi}{2a} \log \left\{ \frac{1-a}{1+a} \right\}, \quad a < r < 1 \tag{B.1}$$

$$\int_a^1 r_0 T^{-1} \log \left| 1 - \frac{r_0^2}{r^2} \right| dr_0 = \pi \left[\tfrac{1}{2} \log (1-a^2) - \log 2r \right] \quad a < r < 1 \tag{B.2}$$

$$\int_a^1 r_0^3\, T^{-1} \log \left| 1 - \frac{r_0^2}{r^2} \right| dr_0 = \frac{\pi}{4} \left[(1+a^2) \{ (1+\log (1-a^2)) - 2 \log 2r\} - 2r^2 \right], \quad a < r < 1 \tag{B.3}$$

$$\int_a^1 r_0^{-3}\, T^{-1} \log \left| 1 - \frac{r_0^2}{r^2} \right| dr_0 = \frac{\pi}{2a^2} \left[1 + \left\{ \frac{1+a^2}{2a} \right\} \log \left\{ \frac{1-a}{1+a} \right\} - \frac{a}{r^2} \right], \quad a < r < 1 \tag{B.4}$$