ON THE THERMO-ELASTIC PROBLEM OF A PLATE—PART I

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(Communicated by F. C. Auluck, F. N. A.)

(*Received 16 May 1974*)

In this work the solution to the thermoelastic problem of a plate is obtained when a point-source of heat is applied on one face while the other face is presumed exposed to loss of heat by radiation. The primary aim is to assess the axial deflection (towards the thickness direction) both due to the parabolic nature of the thermal field and the hyperbolic nature of the thermoelastic waves. In this preliminary part of the comprehensive work, while the complete solution for the plate is developed, the numerical data basic to the above comparison between the parabolic and hyperbolic fields is presented for the case when the effects of the plate are negligible. These effeets will, however. be included in a future presentation.

NOMENCLATURE

θ_0 = initial temperature

a = thermal diffusivity

u, w = displacements

θ = temperature

c_1, c_2 = velocities of compressional and shear waves

γ = $(3\lambda+2\mu)\alpha_t$

λ, μ = lame's constants

ρ = density

c = specific heat

h = thickness of plate

α_t = coefficient of linear expansion

$\theta_1, \theta_2, k, m, n$ = parameters occurring in the boundary conditions

$\delta(r)$ = Dirac's delta function

$$\nabla^2 \equiv \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \right)$$

$$f,_z \equiv \frac{\partial f}{\partial z}$$

1. Introduction

The effects of thermoelastic deformation due to sudden appearance of heat sources are an important part of structural aspect studies in various contexts (such as high speed projectiles). The basic equations governing the elastic and thermal fields under various coupling conditions and applications to certain classical problems can be found in standard books (see for instance Nowacki 1962 and Boley and Weiner 1960). In the present work our main interest is to consider a simple problem of a plate of thickness h in which one face is subjected to a time-dependent point source of heat while the remaining face is under a general condition of loss of heat by radiation. For simplicity we consider only one-way coupling of the fields, viz. the thermal field is assumed independent while the elastic field is coupled to the thermal effects. The complete solution to the plate problem is obtained using Intergral Transform techniques. The resulting approximations are further examined for comparing the deflection of the plate caused by the parabolic thermal field against the hyperbolic thermoelastic wave-field. To simplify this part of our analysis we restrict this comparison only to the case when the thickness of the plate is too large ($h >> 1$) to include the effects of the second face. Extension of our numerical work to include these effects as well will be carried out in a future communication.

2. The Heat Conduction Problem

The plate is initially at a temperature $\theta = \theta_0$. At the time $t = 0$ a heat source appears on one face of the plate. Let the faces of the plate be defined by $z = 0$ and $z = h$ in the cylindrical system (r, ϕ, z). Then the conditions that we impose on $z = 0$ and $z = h$ are

$$\left.\begin{aligned} \theta &= \theta_0 + \theta_1 e^{-kt} \frac{\delta(r)}{r}, \quad & z=0 \\ m \frac{\partial \theta}{\partial z} + n\theta &= \theta_2 \quad , & z=h \end{aligned}\right\} \quad t \geqslant 0. \tag{1}$$

The problem is to solve for θ from the heat conduction equation

$$a \nabla^2 \theta = \frac{\partial \theta}{\partial t} \tag{2}$$

satisfying the conditions (1).

Define the Laplace transform by

$$\bar{f}(p) = \int_0^\infty f(t)\, e^{-pt}\, dt \tag{3}$$

and then the Fourier transform by

$$\hat{f}(p, \xi, z) = \int_0^\infty \bar{f}(p, r, z)\, J_0(\xi r)\, r\, dr. \tag{4}$$

Then (2) becomes

$$\hat{\theta}_{,zz} - \left(\xi^2 + \frac{p}{a}\right)\hat{\theta} = -\frac{\theta_0}{a}\, \frac{\delta(\xi)}{\xi} \tag{5}$$

where $\delta(\xi)$ is the Dirac delta function. The condition (1) become

$$\hat{\theta}(p, \xi, 0) = \frac{\theta_0}{p} \cdot \frac{\delta(\xi)}{\xi} + \frac{\theta_1}{p+k}$$

and

$$\left(m\frac{d}{dz} + n\right)\hat{\theta}(p, \xi, z) = \frac{\theta_2}{p}\, \frac{\delta(\xi)}{\xi} \quad (z=h). \tag{6}$$

For the solution of (5) we take

$$\hat{\theta}(p, \xi, z) = Ae^{-\alpha z} + Be^{\alpha z} + \frac{\theta_0}{a} \cdot \frac{\delta(\xi)}{\xi} \cdot \frac{1}{\xi^2 + (p/a)}$$

$$\left[\alpha = \left(\xi^2 + \frac{p}{a}\right)^{1/2} \;;\quad \text{Re. } \alpha \geqslant 0\right]. \tag{7}$$

Substituting this in (6) gives

$$A + B = \theta_0 \frac{\delta(\xi)}{\xi}\left\{\frac{1}{p} - \frac{1}{p + a\xi^2}\right\} + \frac{\theta_1}{p+k}\quad (n - m\alpha)\, Ae^{-h\alpha} +$$

$$(n + m\alpha)\, Be^{h\alpha} = \left\{\frac{\theta_2}{p} - \frac{n\theta_0}{p + a\xi^2}\right\}\frac{\delta(\xi)}{\xi}\,. \tag{8}$$

A and B can be solved from (8) and $\hat{\theta}(p, \xi, z)$ is then determined from (7). The temperature distribution $\theta(t, r, z)$ is recovered by the inversion of the transformations (3) and (4) viz.

$$\bar{\theta}(p, r, z) = \int_0^\infty \hat{\theta}(p, \xi, z)\, J_0(\xi r)\, \xi\, d\xi \tag{9}$$

and

$$\theta(t, r, z) = \frac{1}{2\pi i}\int_{Br} \bar{\theta}(p, r, z)\, e^{pt}\, dp \tag{10}$$

where Br is the Bromwich contour.

3. The Thermo-Elastic Problem

For the axi-symmetric problem considered, the elastic displacements (u,o,w) are given by the equations of motion

$$\frac{\partial^2 u}{\partial t^2}=c_1^2\frac{\partial\Delta}{\partial r}+c_2^2\frac{\partial\Omega}{\partial z}-\frac{\gamma}{\rho}\frac{\partial\theta}{\partial r}$$

$$\frac{\partial^2 w}{\partial t^2}=c_1^2\frac{\partial\Delta}{\partial z}-\frac{c_2^2}{r}\frac{\partial}{\partial r}(r\Omega)-\frac{\gamma}{\rho}\frac{\partial\theta}{\partial z} \tag{11}$$

where Δ and Ω are the dilatation and rotation functions satisfying

$$\left(\nabla^2-\frac{1}{c_1^2}\frac{\partial^2}{\partial t^2}\right)\Delta=\frac{\gamma}{\lambda+2\mu}\nabla^2\theta$$

$$\left(\nabla^2-\frac{1}{r^2}-\frac{1}{c_2^2}\frac{\partial^2}{\partial t^2}\right)\Omega=0. \tag{12}$$

Also it is assumed that the boundaries $z=0,\ h$ are stress-free. In usual notations this is stated by

$$\sigma_{zz}=0=\sigma_{rz} \text{ on } z=0,\ h \tag{13}$$

where the stress displacement relations concerned are

$$\left.\begin{aligned}\sigma_{zz}&=\lambda\Delta+2\mu\frac{\partial w}{\partial z}-\gamma\theta\\ \sigma_{rz}&=\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right).\end{aligned}\right\} \tag{14}$$

Assuming that before $t=0$ there was no elastic deformation the Laplace transforms of (11) and (12) are obtained by merely replacing $\frac{\partial^2}{\partial t^2}$ by p^2 and the quantities $u,\ \bar{w},\ \Delta,\ \Omega$ and θ by $\bar{u},\ \bar{w}$...etc. The solution of these may be taken as

$$\bar{\Delta}(p,r,z)=\int_0^\infty[E(\xi)\,e^{-z\alpha_1(\xi)}+F(\xi)\,e^{z\alpha_1(\xi)}]\,J_0(r\xi)\,d\xi+\bar{\Delta}_{(\theta)}$$

$$\bar{\Omega}(p,r,z)=\int_0^\infty[G(\xi)\,e^{-z\alpha_2(\xi)}+H(\xi)\,e^{z\alpha_2(\xi)}]\,J_1(r\xi)\,d\xi \tag{15}$$

with

$$\alpha_j=\left(\xi^2+\frac{p^2}{c_j^2}\right)^{\frac{1}{2}}\qquad j=1,2,\ \mathrm{Re}\,\alpha_j\geqslant 0,$$

the functions E, F, G, H being so far arbitrary, and

$$\bar{\Delta}_{(\theta)}=\int_0^\infty\hat{\Delta}_{(\theta)}=J_0(r\xi)\,\xi\,d\xi \tag{16a}$$

where

$$\widehat{\Delta}_{(\theta)} = \frac{\gamma}{\lambda+2\mu}\left[\{Ae^{-\kappa z}+Be^{\kappa z}\}\frac{p}{p-\frac{ap^2}{c_1^2}} + \frac{\theta_0}{a}\frac{\delta(\xi)\,\xi}{\left(\xi^2+\frac{p^2}{c_1^2}\right)\left(\xi^2+\frac{p}{a}\right)}\right]. \tag{16b}$$

That is, $\bar{\Delta}_{(\theta)}$ is the particular integral due to the term in $\nabla^2\theta$ occuring in (12) with θ defined by (7) to (10).

Next applying the boundary conditions (14) after Laplace transformation, we obtain the following equations for determining $E(\xi)$, $F(\xi)$, $G(\xi)$ and $H(\xi)$:

$$\left(\lambda + \frac{2\mu c_1^2}{p^2}\alpha_1^2\right)(E+F) + \frac{2\mu c_2^2}{p^2}\alpha_2\xi\,\{G(\xi)-H(\xi)\}=f_1^0(\xi)$$

$$\frac{2c_1^2\xi}{p^2}\alpha_1\{E(\xi)-F(\xi)\} + \frac{c_2^2}{p^2}(\xi^2+\alpha_2^2)\{G(\xi)+H(\xi)\}=f_2^0(\xi) \tag{17a}$$

due to the conditions on $z=0$; and

$$\left(\lambda+\frac{2\mu c_1^2}{p^2}\right)\{E(\xi)\,e^{-h\alpha_1}+F(\xi)\,e^{h\alpha_1}\}+\frac{2\mu c_2^2}{p^2}\alpha_2\xi$$
$$\times\{G(\xi)\,e^{-h\alpha_2}-H(\xi)\,e^{h\alpha_2}\}=f_1^h(\xi) \tag{17b}$$

$$\frac{2c_1^2\xi}{p^2}\alpha_1\{E(\xi)\,e^{-h\alpha_1}-F(\xi)\,e^{h\alpha_1}\}+\frac{c_2^2}{p^2}(\xi^2+\alpha_2^2)$$
$$\times\{G(\xi)\,e^{-h\alpha_2}+H(\xi)\,e^{h\alpha_2}\}=f_2^h(\xi)$$

due to the conditions on $z=h$. Here, $f_1^0(\xi)$, $f_2^0(\xi)$ and $f_1^h(\xi)\,f_2^h(\xi)$ are obtained from the following definition of $f_j^z(\xi)$ by taking $z=0$ and h respectively :

$$f_1^z(\xi)=\gamma\xi\,\widehat{\theta}(p,\xi,z)+\frac{2\mu\gamma}{p^2\rho}\xi\,\widehat{\theta}_{,zz}$$
$$-\lambda\xi\,\widehat{\Delta}_{(\theta)}-\frac{2\mu c_1^2\xi}{p^2}\widehat{\Delta}_{(\theta)zz} \tag{18}$$

$$f_2^z(\xi)=-\frac{2\gamma\xi^2}{\rho p^2}\widehat{\theta}_{,z}+\frac{2c_1^2\xi^2}{p^2}\widehat{\Delta}_{(\theta),z}$$

where $\widehat{\Delta}_{(\theta)}$ is defined in (16b) while $\widehat{\theta}$ is given by (7) and (8).

If we solve (17) and substitute in (15) the elastic functions $\bar{\Delta}$ and $\bar{\Omega}$ are also completely determined. The displacements can be obtained from (11) (after Laplace inversion of $\bar{\Delta}$ and $\bar{\Omega}$).

In this preliminary work we are interested only in the axial deflection of the plate which means we need to find the shape of any plane $z=$constant (between the faces $z=0$ and $z=h$) at any time after the heat source is introduced. In order to establish the necessary background analysis we therefore present in the next section a discussion of the component w of the axial displacement. To simplify the presentation we have ignored the effects of the boundary $z=h$ of the plate, i. e. we consider the auxiliary problem of an elastic half-space $z \geq 0$ where on $z=0$ the condition

$$\theta=\theta_1\, e^{-kt}.\frac{\delta(r)}{r} \quad (t>0) \tag{19}$$

is applied after $t=0$. This also implies that the plate temperature (θ_0) before $t=0$ is reckoned as the zero initial value for θ. The extension of this discussion to the case when the effects of $z=h$ are also included will present no special difficulty, except in details and therefore this will be carried out in a sequel to this paper.

4 Axial Deflection : Discussion of Axial Displacement $w\,(t, r, z)$

As just mentioned we take (19) as the condition for θ on $z=0$ so that $\theta_0=0$ in (1), and also that $h \to \infty$ so that the effects of the second face are not important for points at finite distances from $z=0$. This will give some insight into the nature of results to be expected in general.

When $h \to \infty$ we drop the terms in $F(\xi)$ and $H(\xi)$ from eqns. (15) and also omit the conditions (17*b*). Then (17*a*) leads to

$$\left.\begin{aligned} E(\xi) &= \frac{(2\xi^2+(p^2/c_2^2))\, f_1^0(\xi)-2\mu\xi\alpha_2\, f_2^0(\xi)}{\mu\,(c_1^2/p^2)\,\triangle(\xi)} \\ G(\xi) &= \frac{\mu\,(2\xi^2+(p^2/c_2^2))\, f_2^0(\xi)-2\xi\alpha_1\, f_1^0(\xi)}{\mu\,(c_2^2/p^2)\,\triangle(\xi)} \end{aligned}\right\} \tag{20}$$

where

$$\triangle(\xi) = \left(2\xi^2+\frac{p^2}{c_2^2}\right)^2-4\xi^2\alpha_1\alpha_2 \tag{21}$$

while the functions $f_1^0(\xi)$, $f_2^0(\xi)$ defined in (18) (with $z=0$) are evaluated (for $\theta_0=0$) to be

$$\left.\begin{aligned} f_1^0(\xi) &= -\frac{\theta_1}{p+k}\cdot\frac{\mu\gamma\xi\,(2\xi^2+p^2/c_2^2)}{pc_1^2\left(\dfrac{p}{a}-\dfrac{p^2}{c_1^2}\right)} \\ f_2^0(\xi) &= -\frac{\theta_1}{p+k}\cdot\frac{2\gamma\xi^2(\xi^2+p/a)^{1/2}}{pc_1^2\left(\dfrac{p}{a}-\dfrac{p^2}{c_1^2}\right)} \end{aligned}\right\} \tag{22}$$

This is derived easily noting that for $\theta_0=0$ and $h\to\infty$ (8) is solved by

$$A=\frac{\theta_1}{p+k}, \qquad B=0$$

so that

$$\widehat{\theta}\,(p, \xi, z) = \frac{\theta_1}{p+k}\, e^{-\alpha z} \tag{23}$$

from (7), and

$$\hat{\Delta}_{(\theta)} = \frac{\gamma}{\lambda+2\mu} \cdot \frac{(p/a)}{\left(\frac{p}{a} - \frac{p^2}{c_1^2}\right)} \cdot \hat{\theta}$$

$$= \frac{\gamma}{(\gamma+2\mu)} \cdot \frac{(p/a)}{\left(\frac{p}{a} - \frac{p^2}{c_1^2}\right)} , \frac{\theta_1}{(p+k)} e^{-\alpha z} \tag{24}$$

from (16*b*). Using (22) in (20) we finally get

$$\left.\begin{aligned} E(\xi) &= - \frac{\theta_1}{p+k} \cdot \frac{\gamma p^2}{\rho c_1^4} \cdot \frac{\xi \Delta_1(\xi)}{\left(\frac{p}{a} - \frac{p^2}{c_1^2}\right) \Delta(\xi)} \\ G(\xi) &= - \frac{\theta_1}{p+k} \cdot \frac{2\gamma p^2}{\rho c_1^2 c_2^2} \cdot \frac{\xi^2(2\xi^2+(p^2/c_2^2))(\alpha-\alpha_1)}{\left(\frac{p}{a} - \frac{p^2}{c_1^2}\right) \Delta(\xi)} \end{aligned}\right| \tag{25}$$

where

$$\Delta_1(\xi) = \left(2\xi^2 + \frac{p^2}{c_2^2}\right)^2 - 4\xi^2 \alpha\alpha_1. \tag{26}$$

The Laplace transform of the axial displacement, obtained from (11) is

$$p^2 \bar{w}(p, r, z) = c_1^2 \frac{\partial \bar{\Delta}}{\partial z} - \frac{c_2^2}{r} \frac{\partial}{\partial r} (r \bar{\Omega}) - \frac{\gamma}{\rho} \frac{\partial \bar{\theta}}{\partial z} \tag{27}$$

where for the present special case ($h \to \infty$)

$$\left.\begin{aligned} \bar{\Delta} &= \int_0^\infty E(\xi) e^{-z\alpha_1} J_0(r\xi)\, d\xi + \bar{\Delta}_{(\theta)} \\ \bar{\Omega} &= \int_0^\infty G(\xi) e^{-z\alpha_2} J_1(r\xi)\, d\xi \end{aligned}\right| \tag{28}$$

with E and G is given by (25) and $\bar{\Delta}_{(\theta)}$ fram (16*a*) and (24), by

$$\bar{\Delta}_{(\theta)} = \int_0^\infty e^{-\alpha z} J_0(\xi r)\, \xi\, d\xi \times \left\{ \frac{\gamma}{\lambda+2\mu} \cdot \frac{(p/a)\,\theta_1}{\left(\frac{p}{a} - \frac{p^2}{c_1^2}\right)(p+k)} \right\} \tag{29}$$

Using (28) and (29) in (27) and making some simplifications, we have

$$\bar{w}(p, r, z) = \frac{\gamma\theta_1}{\rho c_1^2} \cdot \frac{1}{(p+k)\left(\frac{p}{a} - \frac{p^2}{c_1^2}\right)} \left[-\int_0^\infty e^{-z\alpha(\xi)} .\, \xi \alpha(\xi) J_0(\xi r)\, d\xi \right.$$

$$+ \int_0^\infty e^{-z\alpha_1(\xi)} .\, \xi \alpha_1(\xi) . \frac{\Delta_1(\xi)}{\Delta(\xi)} . J_0(\xi r)\, d\xi$$

$$\left. + \int_0^\infty e^{-z\alpha_2(\xi)} . \frac{\xi^3 (\alpha-\alpha_1)(2\xi^2+(p^2/c_2^2))}{\Delta(\xi)} . J_0(\xi r)\, d\xi \right] . \tag{30}$$

The three terms inside the brackets [...] here respectively denote the contributions due to the parabolic thermal field, and the hyperbolic thermo-elastic compressional and shear wavefields. By their well-known properties the first term exists for all times from $t=0$ everywhere, the second and third terms will start existing only when the time of transmission of the waves from the point source is exceeded. The shear wave being slower than the compressional waves we need consider here only the first two terms for comparison. However, for the sake of interest we have evaluated a saddlepoint approximation to all the terms in (30) (see De Bruijn, 1958) leading to the result

$$\bar{w}(p, r, z)=\bar{w}_\theta+\bar{w}_c+\bar{w}_s \tag{31}$$

where

$$\bar{w}_\theta \approx \frac{\gamma\,\theta_1}{\rho\, c_1^2}\cdot\frac{p}{(p+k)\left(\frac{p}{a}-\frac{p^2}{c_1^2}\right)}\left\{\frac{z^2\, e^{-\sqrt{(p/a)}\,R}}{a\,R^3}\right\}$$

$$\bar{w}_c \approx \frac{\gamma\theta_1}{\rho c_1^2}\cdot\frac{1}{(p+k)\left(p-\frac{c_1^2}{a}\right)}\left\{\frac{z^2\, e^{-(pR/c_1)}}{R^3 e_4}\left(pe_1+e_3\sqrt{\frac{p}{a}-p^2 e_2}\right)\right\}$$

$$w_s \approx \frac{\gamma\theta_1}{\rho c_1^2}\cdot\frac{1}{(p+k)\left(\frac{p}{a}-\frac{p^2}{c_1^2}\right)}\cdot\left\{\frac{2z\,r^2\, e^{-pR/c_2}\left(1-\frac{2r^2}{R^2}\right)}{c_2^5\, R^4\, e_6(p)}\right.$$

$$\left.\times\left(\sqrt{\frac{p}{a}-\frac{p^2\,r^2}{c_2^2\,R^2}}-p\,e_5(p)\right)\right\} \tag{32}$$

Here,

$$R=\sqrt{r^2+z^2}$$

$$e_1=\left(\frac{1}{c_2^2}-\frac{2r^2}{c_1^2\,R^2}\right)^2$$

$$e_2=\frac{r^2}{c_1^2\,R^2}$$

$$e_3=\frac{4r^2}{c_1^2\,R^2}\sqrt{\frac{1}{c_2^2}-\frac{r^2}{c_1^2\,R^2}}$$

$$e_4=\left[\frac{1}{c_2^2}-\frac{2r^2}{c_1^2\,R^2}\right]^2+\frac{4r^2\,z}{c_1^3\,R^3}\sqrt{\frac{1}{c_2^2}-\frac{r^2}{c_1^2\,R^2}}$$

$$e_5(p)=\begin{cases}\sqrt{\dfrac{1}{c_1^2}-\dfrac{r^2}{c_2^2\,R^2}}, & \text{if } r<\dfrac{c_2}{c_1}R;\\[2ex] -i\,\operatorname{sign}(\operatorname{Im}.p)\sqrt{\dfrac{r^2}{c_2^2\,R^2}-\dfrac{1}{c_1^2}}, & \text{if } r>\dfrac{c_2}{c_1}R\end{cases}$$

$$e_6(p)=\left(\frac{1}{c_2^2}-\frac{2r^2}{c_2^2\,R^2}\right)^2+\frac{4r^2\,z}{c_2^3\,R^3}e_5(p). \tag{33}$$

As mentioned earlier, the Laplace inversion of (31) is carried out only for the first two terms. While inverting it is noted that there is also a non-vanishing contribution from the pole at $p=-k$. [The other pole at $p=c_1^2/a$ is excluded here as it lies in the part $\operatorname{Re} p > 0$ which will not be an admissible singularity for a physical system.] Thus we get

$$\bar{w}_{\theta}(t,r,z) \approx \frac{\gamma\theta_1}{\rho c_1^2} \cdot \frac{z^2}{aR^3}\left[\frac{2a}{\left(1+\frac{ak}{c_1^2}\right)} e^{-kt}\cos\left(R\sqrt{\frac{k}{a}}\right)+\right.$$

$$\left.+\sqrt{\frac{a}{\pi}}\;\frac{R\,e^{-\left(\frac{R^2}{4at}\right)}}{2t^{\frac{3}{2}}\left(k+\frac{R^2}{4at^2}\right)\left(1-\frac{R^2}{4c_1^2t^2}\right)}\right] \tag{34}$$

$$\left\{t \geqslant 0,\ t \neq \frac{R}{2c_1}\right\}.$$

When $t=\dfrac{R}{2c_1}$ the second term in (34) has the form

$$\frac{1}{2c_1^2}\;\frac{e^{-(Rc_1/2a)}}{\left[k+\frac{c_1^2}{a}\right]}$$

within the bracket. The first term in the bracket is due to the pole at $p=-k$. It is important to note that this term is very sensitive to the value of k since accordingly the sign of the factor $\cos\left[R\sqrt{\frac{k}{a}}\right]$ will also fluctuate. In fact we have, for illustration later, taken an example where this term is actually negative vertically below the source. The remaining term in (34) is a gradually varying term.

In the case of inversion of $\bar{w}_c$ we derive the result

$$w_c(t,r,z) \approx \frac{\gamma\theta_1}{\rho c_1^2}\;\frac{z^2}{R^3 e_4}\left[\frac{ke_1}{\left(k+\frac{c_1^2}{a}\right)}\;e^{-k(t-R/c_1)}\right.$$

$$\left.+\frac{2e_3}{\sqrt{a\pi}}\left(t-\frac{R}{c_1}\right)^{\frac{1}{2}}\right]\cdot\left(t \geq \frac{R}{c_1}\right)\ \left(\text{vaild for } t \approx \frac{R}{c_1}\right) \tag{35}$$

where the first term is due to contribution from the pole at $p=-k$, while in the second term there is no effective contribution from this pole and therefore only the branch-point at $p=0$ has been utilised.

Summarizing our approximations, we have assumed here

$$w(t, r, z) \approx w_\theta + w_c \quad \left(\text{with } t < \frac{R}{c_2}\right)$$

where w_θ is given in (34) and w_c in (35) each as a sum of two terms. Basically, the contribution (35) arises much later than (34) since it is valid only for $t > \frac{R}{c_1}$. At such times the solution (34) would have become unimportant depending on the value of k selected. In fact the second term of (34) remains small right from $t=0$. Moreover between the two terms in (35) the first is significant for $t \approx \frac{R}{c_1}$ while the second will become equally prominent soon after.

For our numerical illustration we have selected the material to be approximating steel with the following parameters :

$$\begin{aligned}
\rho &= 7.65 \\
c &= \text{specific heat} = 0.113 \text{ cal/gm/°C} \\
a &= (0.161/7.65 \times 0.113) \text{ cm}^2\text{/sec} = 0.1862 \text{ cm}^2\text{/sec.} \\
c_1 &= \sqrt{3}\, c_2 \\
c_2 &= \sqrt{\frac{6 \times 10^{11}}{7.65}} \text{ cm/sec.} \\
\gamma &= (3\lambda + 2\mu)\, \alpha_t = (3c_1^2 - 4c_2^2)\, \rho\, \alpha_t \\
\alpha_t &= 12 \times 10^{-6}/°\text{C}
\end{aligned} \qquad (36)$$

and, most important,

$$k = \frac{c^2}{4a} \text{ /sec}$$

The first term in (34) came out to be negative right below the source as stated earlier. This term was evaluated for $t = 0$ and $t = \frac{1}{k}$ at $z = 1$ cm for $r = 0, 0.25, 0.5, 0.75, 1, \ldots, 2$ expressed in cms. For the two values of time selected, these are plotted in Fig 1. At such times the second term in (34) is very insignificant. Similarly the first term in (35) was calculated at the same location for $t = \frac{R}{c_1}$ and plotted in Fig 2. For $t = \frac{R}{c_1} + \frac{1}{k}$ both the terms in (35) become evenly important and their combined contribution is plotted in Fig 3. All the curves are non-dimensionalised w. r. t. the quantity $\frac{\gamma\theta_1}{\rho c_1^2}$.

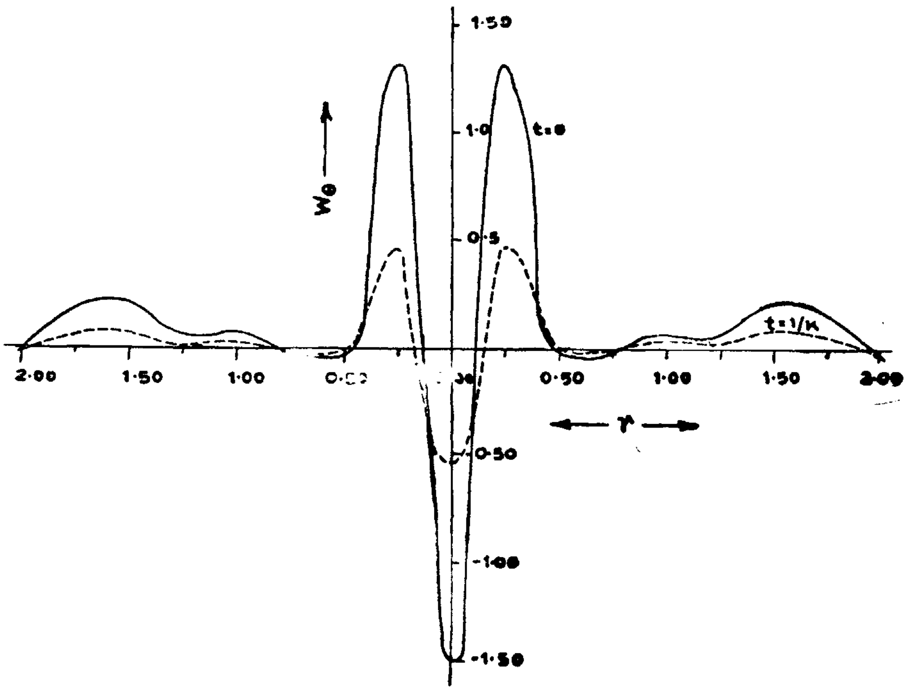

Fig. 1.

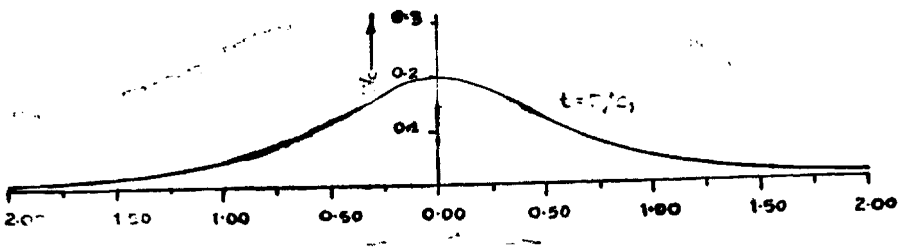

Fig. 2.

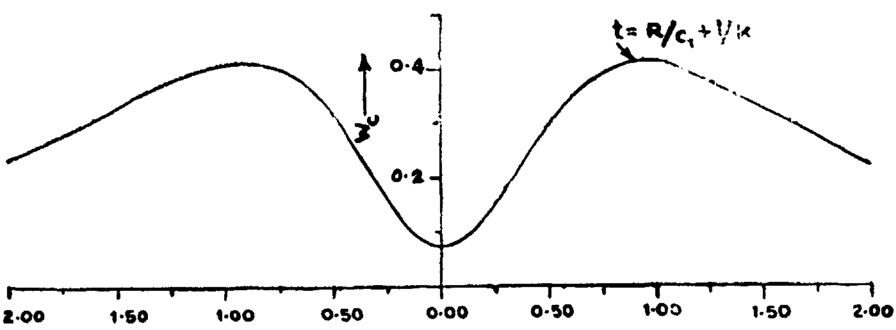

Fig. 3.

Note that θ_1 in (1) must have the dimension of {temperature × (distance)^2}. We have not plotted the second term of (34) since its value is too small compared to those plotted.

A comparison of the magnitudes of Fig. 1 against Figs. 2 and 3 shows that the maximum deflection due to the parabolic thermal field (at $t = 0$) is large by a factor of about 7 near $r = 0$ (on $z = 1$ cm) as compared with that in the thermoelastic wave for $t = \frac{R}{c_1}$. But away from the axis of $r = 0$ the relative importance is transferred to the thermoelastic wave upto some range of r atleast this is particularly evident when we compare Fig. 1 with Fig. 3 the later being the situation when $t = \frac{R}{c_1} + \frac{1}{k}$.

5. Conclusion

The numerical example of the special case of a half-space shows that near the regions vertically below the source of heat, the axial deflection can be initially downwards or upwards depending on the parameters of the heat source such as the value of k. In these regions the thermoelastic field is less important than the parabolic thermal field effects. Away, on the lateral direction, this behaviour seems to be reversed at least upto some distance. The effect of increased depth is not looked into here. So also is the role of the second face $z = h$ of the plate. This is expected to be carried out in our next investigation.

Acknowledgements

The authors are thankful to Dr. R. R. Aggarwal and the Director, Defence Science Laboratory, Delhi, for encouragement and for permission to publish this work.

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