ON THE THERMO–ELASTIC PROBLEM OF A PLATE—PART II

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The second-order effects of the thermoelastic deflection of a plate acted by a heat-source on one face are discussed using the solution obtained in an earlier paper. Numerical results are added to compare the thermal and elastic contributions individually.

1. INTRODUCTION

In a previous part of this work (*see* Viswanathan and Sharma 1976) we obtained the solution to the problem of a plate under thermoelastic deformation due to prescribed heat source on the boundaries. Further, the solution was numerically computed and, for simplicity of purpose, only direct effects of heat conduction and elastic waves arising from one of these boundaries were considered. The neglecting of the remaining boundary means that these numerical results would correspond to a half space problem. In the current part of our study we extend our numerical work also to include the effects of the second boundary accounting for the reflection of the elastic waves and the modified thermal field.

2. THE FORM OF THE SOLUTION FOR THE PLATE

The solution for the plate was developed by Viswanathan and Sharma (1976) in terms of the functions θ, $\triangle$ and Ω for the temperature and elastic dilatation and rotation respectively.

From equation (7) of Viswanathan and Sharma (1976), the Fourier transform of θ (with respect to r) is given by

$$\hat{\theta}(p, \xi, z) = A\, e^{-\alpha z} + B\, e^{\alpha z} \tag{1}$$

for the case when $\theta_0=0$ in eqn. (1) of Viswanathan and Sharma (1976), A and B which can be solved from eqn. (8) of Viswanathan and Sharma (1976), are now given by

$$A=\left\{(n+m\alpha)\cdot\frac{\theta_1}{p+k}\cdot e^{h\alpha}-\frac{\theta_2}{p}\cdot\frac{\delta(\xi)}{\xi}\right\}/P(\xi),$$

$$B=\left\{(n-m\alpha).\frac{\theta_1}{p+k}\cdot e^{-h\alpha}+\frac{\theta_2}{p}\cdot\frac{\delta(\xi)}{\xi}\right\}/P(\xi) \tag{2}$$

$P(\xi)=(n+m\alpha)\,e^{h\alpha}-(n-m\alpha)\,e^{-h\alpha}$ and p is the Laplace transform parameter for time t.

We shall be dealing with only first order reflection effects between the boundaries $z=0$, h and hence we approximate (2) by

$$A\approx\frac{\theta_1}{p+k}-\frac{\theta_2}{p}\cdot\frac{e^{-h\alpha}}{n+m\alpha}\cdot\frac{\delta(\xi)}{\xi}+O(e^{-2h\alpha})$$

$$B\approx-\frac{n-m\alpha}{n+m\alpha}.\,e^{-2h\alpha}\cdot\frac{\theta_1}{p+k}+\frac{\theta_2}{p}\cdot\frac{e^{-h\alpha}}{n+m\alpha}\cdot\frac{\delta(\xi)}{\xi}+O(e^{-3h\alpha}) \tag{3}$$

Next we consider the elastic field governed by Δ and Ω. As defined in eqn. (15) of Viswanathan and Sharma (1976) their Laplace transforms for time t are given by

$$\bar{\Delta}(p,r,z)=\int_0^\infty[E(\xi)\,e^{-z\alpha_1(\xi)}++F(\xi)\,e^{z\alpha_1(\xi)}]\,J_0(r\xi)\,d\xi+\bar{\Delta}_{(\theta)}$$

$$\bar{\Omega}(p,r,z)=\int_0^\infty[G(\xi)\,e^{-z\alpha_2}+H(\xi)\,e^{z\alpha_2(\xi)}]\,J_1(r\xi)\,d\xi \tag{4}$$

where

$$\bar{\Delta}_{(\theta)}=\int_0^\infty\widehat{\Delta}_{(\theta)}\,J_0(r\xi)\,d\xi$$

$$\widehat{\Delta}_{(\theta)}=\frac{\gamma}{\lambda+2\mu}\cdot\left[(Ae^{-\alpha z}+Be^{\alpha z})\cdot\frac{p}{\left(p-a\,\frac{p^2}{c_1^2}\right)}\right. \tag{5}$$

The functions $E(\xi)$, $F(\xi)$, $G(\xi)$ and $H(\xi)$ are given by eqn. 17 of Viswanathan and Sharma (1976) which involves certain functions $f_1^z(\xi)$ and $f_2^z(\xi)$ for $z=0$, h. These latter functions, defined in eqn (18) of the same paper, depend on A and B. After some simplifications we get, for the plate problem, retaining only upto first reflection effects,

$f_1^o(\xi) \approx \{f_1^o(\xi)\}_I$, as defined in eqn. (22) of Viswanathan and Sharma (1976), $f_2^o(\xi) \approx [\{f_2^o(\xi)\}_I$, as defined in (22) Viswanathan and Sharma (1976)]+

$$+4.\frac{\theta_2}{p} \cdot \delta(\xi) \cdot \frac{e^{-h\alpha}}{n+m\alpha} \cdot \frac{\gamma\,\xi\left(\xi^2+\frac{p}{a}\right)^{\frac{1}{2}}}{\rho\, c_1^2\left(\frac{p}{a}-\frac{p^2}{c_1^2}\right)} .$$

$$f_1^h(\xi) = \{f_1^o(\xi)\}_I \cdot e^{-h\alpha} \cdot \left(1-\frac{n-m\alpha}{n+m\alpha}\right) -$$

$$-\frac{\theta_2}{p}\ \frac{\delta(\xi)}{n+m\alpha} \cdot \frac{\mu\gamma\left(2\xi^2+\frac{p^2}{c_2^2}\right)}{\rho\, c_1^2\left(\frac{p}{a}-\frac{p^2}{c_1^2}\right)}$$

$$f_2^h(\xi) = \{f_2^o(\xi)\}_I \cdot e^{-h\alpha} \cdot \left(1+\frac{n-m\alpha}{n+m\alpha}\right) -$$

$$-\frac{\theta_2}{p} \cdot \frac{\delta(\xi)}{n+m\alpha} \cdot \frac{2\gamma\,\xi\left(\xi^2+\frac{p}{a}\right)^{\frac{1}{2}}}{\rho\, c_1^2\left(\frac{p}{a}-\frac{p^2}{c_1^2}\right)} . \tag{6}$$

The solutions for $E(\xi)$, $F(\xi)$, $G(\xi)$ and $H(\xi)$ are then found to be

$$E(\xi) = E_1(\xi) - e^{-h\alpha_1} F_1(\xi) . \frac{\Delta_2(\xi)}{\Delta(\xi)} + $$

$$+\frac{4\, c_2^2\, \alpha_2\, \xi\,(\xi^2+\alpha_2^2)\; e^{-h\alpha_2}\; H_1(\xi)}{c_1^2\, \Delta(\xi)}$$

$$G(\xi) = G_1(\xi) + \frac{4\, c_1^2\, \alpha_1\, \xi\left(2\xi^2+\frac{p^2}{c_2^2}\right) e^{-h\alpha_1} F_1(\xi)}{c_2^2\, \Delta(\xi)} -$$

$$-e^{-h\alpha_2} . \frac{H_1(\xi)\,\Delta_2(\xi)}{\Delta(\xi)}$$

$$F(\xi) = e^{-h\alpha_1}\left[F_1(\xi) - E_1(\xi) . e^{-h\alpha_1} \frac{\Delta_2(\xi)}{\Delta(\xi)} - \right.$$

$$\left. - \frac{4 c_2^2\, \alpha_2\, \xi\,(\xi^2+\alpha_2^2)\, e^{-h\alpha_2}\, G_1(\xi)}{c_1^2\, \Delta(\xi)}\right]$$

$$H(\xi) = e^{-h\alpha_2}\left[H_1(\xi) + \frac{4 c_1^2\, \alpha_1\, \xi\left(2\xi^2+\frac{p^2}{c_2^2}\right) .\; e^{-h\alpha_1} E_1(\xi)}{c_2^2\, \Delta(\xi)} + \right.$$

$$\left. + e^{-h\alpha_2} \frac{G_1(\xi)\, \Delta_2(\xi)}{\Delta(\xi)}\right]$$

$$\left[\Delta_2(\xi) = \left(2\xi^2+\frac{p^2}{c_2^2}\right)^2 + 4\xi^2 \alpha_1 \alpha_2\right] \tag{7}$$

where $E_1(\xi)$ and $G_1(\xi)$ are effectively the same as in eqn. (20) or (25) of Viswanathan and Sharma (1974) owing to their additional terms of the type $\delta(\xi).\ \xi^N$, $(N>0)$, making no contributions to the integrals involved later. The other functions in (7) are given by

$$F_1(\xi)=e^{-h\alpha}F_{11}(\xi)+F_{12}(\xi)\,.\,\frac{\delta(\xi)}{\xi}$$

$$H_1(\xi)=e^{-h\alpha}H_{11}(\xi)+H_{12}(\xi)\,.\,\frac{\delta(\xi)}{\xi} \tag{8}$$

with

$$F_{11}(\xi)=\Big[\Big(2\xi^2+\frac{p^2}{c_2^2}\Big)\Big(1-\frac{n-m\alpha}{n+m\alpha}\Big)\Big\{f_1^0(\xi)\Big\}_I+$$
$$+2\mu\xi\alpha_2\Big(1+\frac{n-m\alpha}{n+m\alpha}\Big)\Big\{f_2^0(\xi)\Big\}_I\Big]\Big/\Big\{\mu.\frac{c_1^2}{p^2}\triangle(\xi)\Big\}$$

$$H_{11}(\xi)=\Big[\mu\Big(2\xi^2+\frac{p^2}{c_2^2}\Big)\Big(1+\frac{n-m\alpha}{n+m\alpha}\Big)\Big\{f_2^0(\xi)\Big\}_I+$$
$$+2\xi\alpha_1\Big(1-\frac{n-m\alpha}{n+m\alpha}\Big)\Big\{f_1^0(\xi)\Big\}_I\Big]\Big/\Big\{\mu.\frac{c_2^2}{p^2}.\triangle(\xi)\Big\}$$

$$F_{12}(\xi)=\frac{\Big(2\xi^2+\frac{p^2}{c_2^2}\Big)\Big\{f_1^0(\xi)\Big\}_I-2\mu\xi\alpha_2\{f_2^0(\xi)\}_I}{\mu.\frac{c_1^2}{p^2}.\triangle(\xi)}\,.$$
$$.\frac{\theta_2}{p(n+m\alpha)}\,.\,\frac{1}{\Big(\frac{\theta_1}{p+k}\Big)}$$

$$H_{12}(\xi)=\frac{-\mu\Big(2\xi^2+\frac{p^2}{c_2^2}\Big)\{f_2^0(\xi)\}_I+2\xi\alpha_1\{f_1^0(\xi)\}_I}{\mu\,.\,\frac{c_2^2}{p^2}\,.\,\triangle(\xi)}\,.$$
$$.\frac{\theta_2}{p(n+m\alpha)}\,.\,\frac{1}{\Big(\frac{\theta_1}{p+k}\Big)} \tag{9}$$

This completes the form of the solution for the plate which includes terms of order upto first reflection.

3. Axial Displacement ω of the Plate

The Laplace transform of the axial displacement is given by

$$\bar{w}=\frac{c_1^2}{p^2}\,.\,\frac{\partial\triangle}{\partial z}-\frac{c_2^2}{p^2}\,.\,\frac{1}{r}\,\frac{\partial}{\partial r}(r\bar{\Omega})-\frac{\gamma}{\rho\,p^2}\,\frac{\partial\bar{\theta}}{\partial z}\,. \tag{10}$$

Using the results given in the preceding section, we can write

$$\bar{w} = \bar{w}_E + \bar{w}_F + \bar{w}_G + \bar{w}_H + \bar{w}_\theta \tag{11}$$

where

$$\bar{w}_E = \frac{1}{p^2} \int_0^\infty E_1(\xi)\, e_1(\xi)\, J_0(\xi r)\, d\xi$$

$$\bar{w} = \frac{1}{p^2} \int_0^\infty F_1(\xi) f_1(\xi)\, J_0(\xi r)\, d\xi$$

$$\bar{w}_G = \frac{1}{p^2} \int_0^\infty G_1(\xi)\, g_1(\xi)\, J_0(\xi r)\, d\xi$$

$$\bar{w}_H = \frac{1}{p^2} \int_0^\infty H_1(\xi)\, h_1(\xi)\, J_0(\xi r)\, d\xi$$

$$\bar{w}_\theta = -\frac{\gamma\theta_1}{\rho(p+k)} \int_0^\infty \frac{1}{c_1^2\left(\frac{p}{a} - \frac{p^2}{c_1^2}\right)} \cdot \left\{e^{-\alpha z} + \frac{n-m\alpha}{n+m\alpha} \cdot e^{-(2h-z)\alpha}\right\} \cdot \alpha\xi\, J_0(r\xi)\, d\xi -$$

$$-\frac{\gamma\theta_2}{\rho p^3} \cdot e^{-h\sqrt{\frac{p}{a}}} \cdot \frac{\sqrt{\frac{p}{a}}}{n+m\sqrt{\frac{p}{a}}} \cdot \left(e^{-\sqrt{\frac{p}{a}}\cdot z} + e^{\sqrt{\frac{p}{a}}\cdot z}\right) +$$

$$+\frac{\gamma\theta_2}{\rho p^3} \cdot \frac{1}{1-\frac{ap}{c_1^2}} \cdot \frac{\sqrt{\frac{p}{a}}}{n+m\sqrt{\frac{p}{a}}} \cdot \left(e^{-\sqrt{\frac{p}{a}}\cdot(h-z)} + e^{-\sqrt{\frac{p}{a}}\cdot(h+z)}\right) \tag{12}$$

where

$$e_1(\xi) = -c_1^2\alpha_1 e^{-z\alpha_1} - c_1^2\alpha_1 \cdot \frac{\Delta_2(\xi)}{\Delta(\xi)} e^{-(2h-z)\alpha_1} - \frac{4c_1^2\alpha_1\xi^2\left(2\xi^2 + \frac{p^2}{c_2^2}\right)e^{-h\alpha_1-(h-z)\alpha_2}}{\Delta(\xi)}$$

$$f_1(\xi) = c_1^2\alpha_1 e^{-(h-z)\alpha_1} + c_1^2\alpha_1 \frac{\Delta_2(\xi)}{\Delta(\xi)} e^{-(h+z)\alpha_1} - \frac{4c_1^2\alpha_1\xi^2\left(2\xi^2 + \frac{p^2}{c_2^2}\right)e^{-h\alpha_1 - z\alpha_2}}{\Delta(\xi)}$$

$$g_1(\xi) = -c_2^2\,\xi\, e^{-z\alpha_2} - \frac{4c_2^2\alpha_1\alpha_2\xi\left(2\xi^2+\frac{p^2}{c_2^2}\right)}{\Delta(\xi)} \cdot e^{-h\alpha_1-(h-z)\alpha_1} - $$

$$-c_2^2\,\xi \cdot \frac{\Delta_2(\xi)}{\Delta(\xi)} \cdot e^{-(2h-z)\alpha_2}$$

$$h_1(\xi) = -c_2^2\,\xi\, e^{-(h-z)\alpha_2} - \frac{4c_2^2\alpha_1\alpha_2\xi\left(2\xi^2+\frac{p^2}{c_2^2}\right)}{\Delta(\xi)} e^{-h\alpha_2-z\alpha_1} +$$

$$+ c_2^2\,\xi \cdot \frac{\Delta_2(\xi)}{\Delta(\xi)} \cdot e^{-z\alpha_2-h\alpha_2}.$$

The integrals appearing in this results are evaluated by the saddle-point approximation

$$\int_0^\infty J_0(r\xi)\,\phi(\xi)\,e^{-\lambda\alpha_1-\mu\alpha_2-\nu\alpha} \cdot d\xi \approx$$

$$\approx \frac{i\,\phi(\xi_s)\exp.\left(-\frac{pr\sin\chi_1}{c_1} - \frac{p\lambda\cos\chi_1}{c_1} - \frac{p\mu\cos\chi_2}{c_2} - \nu\sqrt{\frac{p}{a}}\cos\chi_3\right)}{\left[\frac{r\sin\chi_1}{c_1}\left(\lambda c_1\sec^3\chi_1+\mu c_2\sec^3\chi_2+\nu\sqrt{ap}\,\sec^3\chi_3\right)\right]^{\frac{1}{2}}} \tag{13}$$

where

$$\lambda\tan\chi_1+\mu\tan\chi_2+\nu\tan\chi_3 = r$$

$$\text{and}\quad \xi_s = \frac{ip\sin\chi_1}{c_1} = \frac{ip\sin\chi_2}{c_2} = \sqrt{\frac{p}{a}}\sin\chi_3\,.$$

Subsequently all the expressions are inverted for Laplace transform using the approximations such as :

(a) If $\bar{f}(p) = \frac{1}{(p+k)} \cdot e^{-p.\frac{R}{c_1}} \cdot \left[g^{(1)}(p)+\alpha.g^{(2)}(p)\right]$

then

$$f(t) \approx g^{(1)}(-k) \cdot e^{-k\left(t-\frac{R}{c_1}\right)} +$$

+ a branch integral term of the form

$$\left(\frac{2c_0}{\Gamma(\frac{1}{2})}\left(t-\frac{R}{c_1}\right)^{\frac{1}{2}}\right) \tag{14}$$

where the second term will arise due to any factor of order that may be present $g^{(2)}(p) \cdot \frac{\alpha}{(p+k)}$ term when approximating* for large $|p|$ and therefore

[* Take $\alpha \approx \sqrt{\xi^2+\frac{p}{a}} \approx \sqrt{\frac{p}{a}}$ since $\xi \to 0$ as $r \to 0$].

for small values of $\left| t - \frac{R}{c_1} \right|$. There is no pole effect in the $g^{(2)}$ term.

(b) If $\bar{f}(p) = \dfrac{g(p)\, e^{-A-pB\sqrt{p}}}{(p+k)}$

then the inversion is carried out by applying the saddle point method yielding

$$f(t) \approx \frac{Bg(p_s)\, e^{-\frac{B^2}{4(t-A)}}}{2\sqrt{\pi}\,(t-A)^{\frac{3}{2}}(p_s+k)} + e^{-(t-A)k} \,.\, \mathrm{Re.}\, \{g\,(k\, e^{i\pi})\, e^{-ik^{\frac{1}{2}}\cdot B}\} \tag{15}$$

where

$$p_s = \frac{B^2}{4\,(t-A)^2}\,.$$

Using the above methods, the axial displacement (11) can be inverted finally to yield the solution in terms of r, z and t. We do not write them in detail here as these are now straightforward.

4. Numerical Results

The various terms of (11) denote actually the contributions to axial displacements by various means, viz. due to parabolic thermal conduction and the hyperbolic elastic compressional and shear waves which arise directly from either of the boundaries $z=0$, h or the once reflected waves as the case may be. In part I we have calculated the axial displacement due to the direct effects from $z=0$. Here we shall calculate the once reflected effects from $z=h$. We restrict, further, to the case when $\theta_2=0$ in the boundary conditions (1) of Viswnathan and Sharma (1974).

It is of interest to give a physical explanation of the various terms which we propose to compute here. The $\bar{w}_E$ term in (11) which is defined in (12) can be split into three terms in the order in which these appear in the expression for $e_1(\xi)$ defined in the same equation. These three terms denote the direct P, the once reflected PP and the once reflected PS waves which arise due to the source of heat on $z=0$, the reflections being subsequently from $z=h$. Here P and S denote the nature of the elastic wave, viz. compression and shear respectively. Similarly $\bar{w}_G$ consists of three terms due to the three terms in $g_1(\xi)$ which can be interpreted as the direct S, the once reflected SP and the once reflected SS waves all originally starting from $z=0$ due to the heat source applied.

In a similar way, $\bar{w}_F$ represents a set of three terms, due to those of $f_1(\xi)$, which are the direct P, the once reflcted PP and the once reflected

PS waves all starting from $z=h$ after this surface acquires a distribution of temperature from the source on $z=0$. The reflections mentioned here are against $z=0$. $\bar{w}_H$ has the corresponding interpretation in terms of direct *S*, the once reflected *SP* and once reflected *SS* waves starting from $z=h$ as just outlined.

Recalling that we have assumed $\theta_2=0$, the first term of $\bar{w}_\theta$ gives the purely thermal conduction effects due to the source $z=0$ and due to the presence of $z=h$ respectively where we include only upto first order interaction between boundaries.

As for numerical results we compute only the following :

$$\begin{aligned} &\text{(i)} \quad w=(w_\theta)_I + (w_\theta)_{II} \\ &\text{(ii)} \quad w= (w_F)_{II} \\ &\text{(iii)} \quad w= (w_E)_I + (w_E)_{II} \end{aligned} \tag{16}$$

where $(w_\theta)_I$ is same sa (I, 34) being the direct thermal effect from $z=0$, $(w_E)_I$ is the same as w_c of eqn. (35) of Viswanathan and Sharma (1974) being the direct *P*-wave effect from $z=0$ and the remaining functions are such that $(w_\theta)_{II}$ is the thermal field contribution arising from $z=h$ on the latter being thermally induced by the source on $z=0$, $(w_F)_{II}$ is the *P*-wave arising from $z=h$ after the latter is thermally induced as above, and finally $(w_E)_{II}$ is the once reflected *PP*-wave which originally starts at $z=0$ and gets reflected from $z=h$.

The expressions $(w_\theta)_{II}$, $(w_E)_{II}$ and $(w_F)_{II}$ are given by the saddle point approximations :

$$\begin{aligned} (w_\theta)_{II} = & \frac{\gamma\theta_1}{2\rho\, c_1^2 \sqrt{(\pi a)}} \cdot \frac{e^{-\{r^2+(2h-z)^2\}/(4at)}}{t^{\frac{3}{2}}\left\{k+\dfrac{r^2+(2h-z)^2}{4at^2}\right\}} \cdot \\ & \cdot \frac{(2h-z)^2}{\left\{1-\dfrac{r^2+(2h-z)^2}{4\, c_1^2\, t^2}\right\}\{r^2+(2h-z)\}^2} \cdot \\ & \cdot \frac{n-m\,\dfrac{2h-z}{2at}}{n+m\,\dfrac{2h-z}{2at}} + \\ & + \frac{\gamma\theta_1}{\rho\, c_1^2} \cdot \frac{\dfrac{k}{a}}{\left(\dfrac{k}{a}+\dfrac{k^2}{c_1^2}\right)} \cdot \frac{(2h-z)^2}{\left\{r^2+(2h-z)^2\right\}^{\frac{3}{2}}} \cdot e^{-kt} \\ & . \operatorname{Re} . \left\{ e^{-i\cdot\sqrt{\frac{k}{a}}\cdot\sqrt{r^2+(2h-z)^2}} \cdot \frac{n-\dfrac{im\sqrt{(k/a)}\,(2h-z)}{\sqrt{r^2+(2h-z)^2}}}{n+\dfrac{im\sqrt{(k/a)}\,(2h-z)}{\sqrt{r^2+(2h-z)^2}}} \right\}, \end{aligned}$$

here

$$t \neq \frac{\sqrt{r^2+(2h-z)^2}}{2c_1} \tag{17}$$

$$(w_F)_{II} = \frac{\gamma\theta_1}{\rho c_1^2 \sqrt{\pi}\, a^{\frac{5}{4}}} \cdot \frac{\frac{p_s^2}{c_1}}{(k+p_s)\left(\frac{p_s}{a} - \frac{p_s^2}{c_1^2}\right)} \cdot \frac{h p_s^{\frac{1}{4}}}{(h^2+r^2)^{\frac{1}{4}}} \cdot$$

$$\cdot \frac{\exp\left\{-\frac{r^2+h^2}{4a\left(t-\frac{h-z}{c_1}\right)}\right\}}{\left\{(h-z)\,c_1+\sqrt{a\,p_s}\;\frac{(r^2+h^2)^{3/2}}{h^2}\right\}^{\frac{1}{2}}\left\{n+m\sqrt{\frac{p_s}{a}}\cdot\frac{h}{\sqrt{h^2+r^2}}\right\}} \cdot$$

$$\cdot \frac{m\cdot\left(\frac{p_s}{c_2^2}-\frac{2r^2}{a(r^2+h^2)}\right)^2-4n\cdot\frac{r^2}{a\,c_2(r^2+h^2)}}{\left(\frac{p_s}{c_2^2}-\frac{2r^2}{a(r^2+h^2)}\right)^2+4p_s\cdot\frac{r^2}{ac_1c_2(r^2+h^2)}} \cdot \frac{1}{\left(t-\frac{h-z}{c_1}\right)^{3/2}}$$

$$-\frac{2\gamma\,\theta_1}{\rho c_1^2\, a^{\frac{3}{4}}} \cdot \frac{\frac{k^2}{c_1}}{\frac{k}{a}+\frac{k^2}{c_1^2}} \cdot \frac{h}{(r^2+h^2)^{\frac{3}{4}}} \cdot e^{-k(t-(h-z)/c_1)} \,.$$

$$\cdot \frac{m\cdot\left(\frac{k}{c_2^2}+2\cdot\frac{r^2}{a(r^2+h^2)}\right)^2-4n\cdot\frac{r^2}{ac_2\,(r^2+h^2)}}{\left(\frac{k}{c_2^2}+2\cdot\frac{r^2}{a(r^2+h^2)}\right)^2-4k\cdot\frac{r^2}{ac_1\,c_2(r^2+h^2)}} \cdot$$

$$.\,\mathrm{Re}\,.\left\{\frac{e^{-i\sqrt{\frac{k}{a}}\cdot\sqrt{r^2+h^2}}\cdot e^{i\frac{\pi}{4}}\cdot k^{\frac{1}{4}}}{\left\{(h-z)c_1+i\sqrt{a\,k}\cdot\frac{(r^2+h^2)^{3/2}}{h^2}\right\}^{\frac{1}{2}}} \cdot \frac{1}{n+im\sqrt{\frac{k}{a}}\cdot\frac{h}{r^2+h^2}} \cdot\right\}$$

with

$$t > \frac{h-z}{c_1}$$

and

$$p_s = \frac{r^2+h^2}{4a\left(t-\frac{h-z}{c_1}\right)^2} \tag{18}$$

Finally

$$(w_E)_{II} = \frac{\gamma\theta_1}{\rho c_1^4} \cdot \frac{k\, e^{-k\left(t-\frac{\sqrt{r^2+(2h-z)^2}}{c_1}\right)}}{\left(\frac{1}{a}+\frac{k}{c_1^2}\right)} \cdot$$

$$\cdot\, g(r,z) \,.\, \left\{1-2r^2 \cdot \frac{\frac{c_2^2}{c_1^2}}{r^2+(2h-z)^2}\right\}^2 +$$

$$+\frac{2\gamma\,\theta_1}{\rho\, c_1^2\sqrt{\pi a}} \cdot c_2 \cdot \left\{ t-\frac{\sqrt{r^2+(2h-z)^2}}{c_1}\right\}^{\frac{3}{2}}$$

$$\cdot\; g(r,z) \,.\, 4\,\frac{c_1^3}{c_1^3} \cdot \frac{r^2(2h-z)}{\{r^2+(2h-z)^2\}^{\frac{3}{2}}} \tag{19}$$

where

$$g(r,z) = \frac{(2h-z)^2}{\{r^2+(2h-z)^2\}^{\frac{3}{2}}} \,.$$

$$\cdot\frac{\left(1-2r^2 \cdot \frac{\frac{c_2^2}{c_1^2}}{r^2+(2h-z)^2}\right)^2 - 4r^2(2h-z) \cdot \frac{c_2^3}{c_1^3} \cdot \frac{\sqrt{1-\frac{c_2^2\, r^2}{c_1^2\,\{r^2+(2h-z)^2\}}}}{\{r^2+(2h-z)^2\}^{3/2}}}{\left[\left(1-2r^2 \cdot \frac{\frac{c_2^2}{c_2^1}}{r^2+(2h-z)^2}\right)^2 + 4r^2(2h-z) \cdot \frac{c_3^2}{c_1^3} \cdot \frac{\sqrt{1-\frac{c_2^2\, r^2}{c_1^2\,\{r^2+(2h-z)^2\}}}}{r^2+(2h-z)^2\}^{3/2}}\right]^2}$$

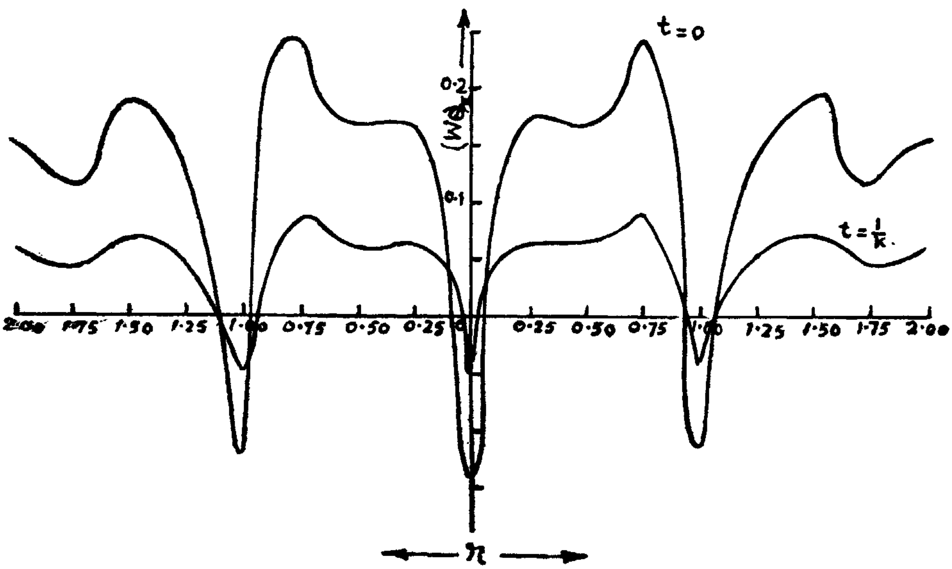

FIG. 1.

and

$$t \geqslant \frac{\sqrt{r^2+(2h-z)^2}}{c_1}$$

The axial displacements $(w_\theta)_{II}$, $(w_E)_{II}$ and $(w_F)_{II}$ are numerically computed and shown in Figs. 1, 2 and 3 for various points of location on the plane $z=1$ cm. The plate thickness h is taken to be 2 cms. and the ratio $m:n=1:2$.

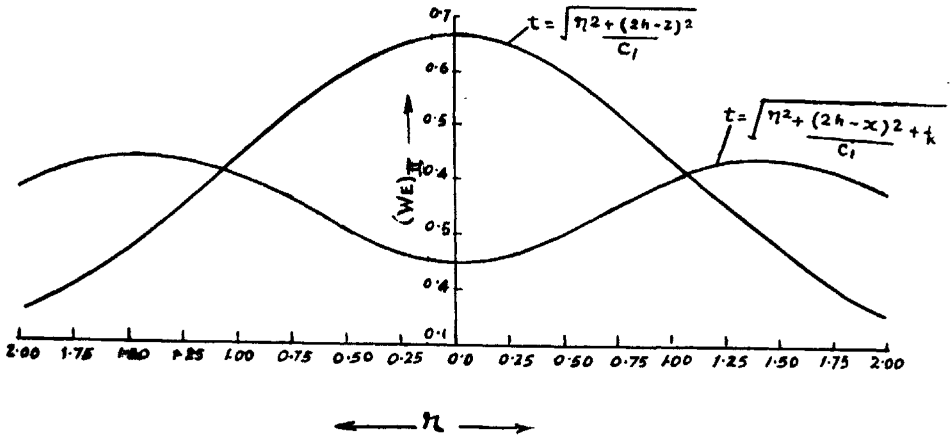

FIG. 2.

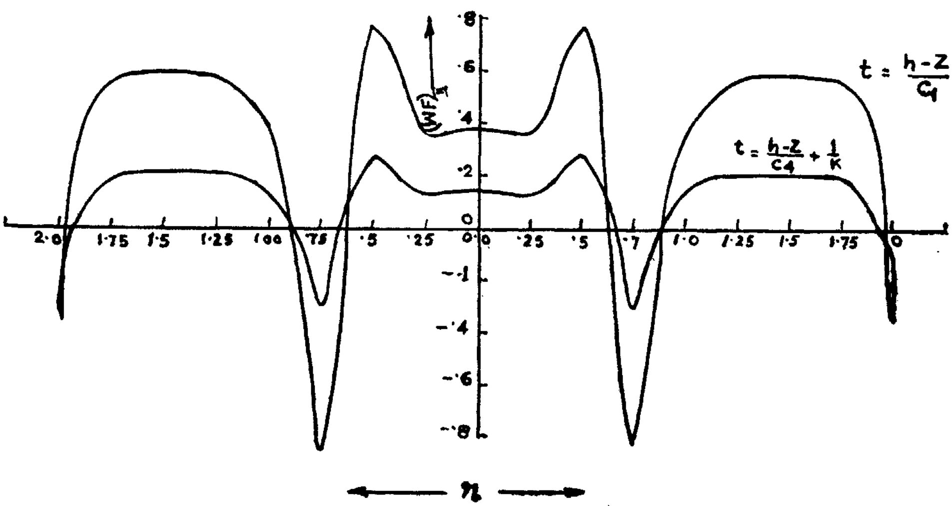

FIG. 3.

The constants of the material are already given in Part I, viz appropriate to those of steel. As seen from the graphs the strongest displacements are due to $(w_F)_{II}$ and next in order are the $(w_\theta)_{II}$ and $(w_E)_{II}$ contributions. That is, the elastic compressional wave generated from the thermal secondary effects from $z = h$ is dominant among the group of terms generated by the presence of $z = h$. Another feature is that while $(w_E)_{II}$ and $(w_F)_{II}$ give positive (downward) deflections near $r = 0$, $(w_\theta)_{II}$ gives a negative (upward) deflection. As noted already (Viswanathan and Sharma 1974) the parameter k is so sensitive as can possibly change some of these altogether.

It may be worth pointing out that our method and results can be usefully exploited with some connected works in the field (Jepps 1965).

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