ON RANDOM TRIGONOMETRIC POLYNOMIAL

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Consider the random trigonometric polynomial

$$T(\theta) = a_1 p_1 \cos\theta + a_2 p_2 \cos 2\theta + \ldots + a_n p_n \cos n\theta$$

where p_i are positive constants and a_i are normal dependent random variables with mean zero and joint density function

$$|M|^{1/2} (2\pi)^{-n/2} \exp[-(1/2)\bar{a}' M\bar{a}]$$

where M^{-1} is the moment matrix with $\sigma_i = 1, i = 1, 2, \ldots, n$, and $\bar{a}'$ is the transpose of the column vector $\bar{a}$.

Let $N(T; \alpha, \beta)$ denote the number of real zeros of $T(\theta) = 0$ in $\alpha \leq \theta \leq \beta$ where multiple roots are counted only once. Here we obtain the asymptotic value of the mathematical expectation of $N(T; 0, 2\pi)$ for large values of n, when $\rho_{ij} = \rho$, $0 < \rho < 1$, $i, j = 1, 2, \ldots, n$, $i \neq j$.

1. INTRODUCTION

Consider the random trigonometric polynomial

$$T(\theta) = a_1 p_1 \cos\theta + a_2 p_2 \cos 2\theta + \ldots + a_n p_n \cos n\theta \qquad \ldots(1.1)$$

where p_i are positive constants and a_i are normal dependent random variables with mean zero and joint density function

$$|M|^{1/2} (2\pi)^{-n/2} \exp[-(1/2)\bar{a}' M\bar{a}] \qquad \ldots(1.2)$$

where M^{-1} is the moment matrix with $\sigma_i = 1$, $i = 1, 2, \ldots, n$, and $\bar{a}'$ is the transpose of the column vector $\bar{a}$.

Let $N(T; \alpha, \beta)$ denote the number of real zeros of $T(\theta) = 0$ in $\alpha \leq \theta \leq \beta$ where multiple roots are counted only once. Here we obtain an asymptotic value of the mathematical expectation of $N(T; 0, 2\pi)$, for large values of n when $\rho_{ij} = \rho, 0 < \rho < 1$, $i, j = 1, 2, \ldots, n$, $i \neq j$.

The case when a_i are independent normally distributed and p_i are equal to one, has been discussed by Dunnage (1966). Das (1968) has discussed the case when a_i are independent normally distributed and p_i are positive constants.

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2. A Transformation

Here we introduce a suitable transformation which will be more useful in the calculation of average number of real zeros of (1.1).

Consider the linear transformation $\bar{a} = C\bar{b}$, where $C = (c_{ij})_{n \times n}$ and $\bar{b}' = (b_1, b_2, \ldots, b_n)$ such that $C'MC = I$. Under this transformation (1.2) reduces to

$$(1/|C|)(2\pi)^{-n/2} \exp\left[-(1/2)\sum_{i=1}^{n} b_i^2\right]. \qquad \ldots(2.1)$$

If $P' = (p_1 \cos\theta, p_2 \cos 2\theta, \ldots, p_n \cos n\theta)$, where P' is the transpose of the column vector P, under the above transformation we get

$$\begin{aligned} T(\theta) &= P'Cb \\ &= \sum_{k=1}^{n} (c_{1k} p_1 \cos\theta + c_{2k} p_2 \cos 2\theta + \ldots + c_{nk} p_n \cos n\theta)\, b_k \\ &= \sum_{k=1}^{n} X_k b_k, \end{aligned} \qquad \ldots(2.2)$$

where

$$X_k = c_{1k}p_1 \cos\theta + c_{2k}p_2 \cos 2\theta + \ldots + c_{nk} p_n \cos n\theta \qquad \ldots(2.3)$$

Since $C'MC = I$ we get $CC' = M^{-1}$. Equating the corresponding elements we get

$$c_{i1} c_{j1} + c_{i2} c_{j2} + \ldots + c_{in} c_{jn} = \begin{cases} 1 & \text{if } i = j \\ \rho & \text{if } i \neq j \\ i, j = 1, 2, \ldots, n. \end{cases} \qquad \ldots(2.4)$$

3. To Find the Number of Zeros in $w - \varepsilon \leq \theta \leq w + \varepsilon$.

Let the w-set be $(0, \pm\pi, \pm 2\pi, \ldots)$. Here we show that the probability of $T(\theta)$ having an appreciable number of zeros in a small interval $w - \varepsilon \leq \theta \leq w + \varepsilon$ is small. Let us take $L(n)$ to be any positive valued function of n such that $L(n)$ and $n/L(n)$ approach infinity with n. We take

$$\varepsilon = \varepsilon(n) = (L(n))/n.$$

Let

$$T(w + z) = T_W(z) = \sum_{k=1}^{n} a_k p_k (\cos kw \cos kz - \sin kw \sin kz)$$

The distribution function of the random variable

$$g_n = T_W(0) = \sum_{k=1}^{n} a_k p_k \cos kw$$

is

$$(2\pi \Lambda^2)^{-1/2} \int_{-\infty}^{x} \exp(-u^2/2\Lambda^2)\, du,$$

where

$$\Lambda^2 = \sum_{k=1}^{n} p_k^2 \cos^2 kw + \rho \sum_{\substack{k=1 \\ k\neq j}}^{n} \sum_{j=1}^{n} p_k p_j \cos kw \cos jw.$$

Since $\Lambda^2 > 0$, g_n has a continuous probability function and

$$Pr\,(g_n = 0) = 0.$$

Now we take $a_1, a_2, \ldots, a_n$ as any fixed set of vectors such that $T_w(0) \neq 0$. Let $n(r)$ denote the number of zeros of $T_W(z)$ in $|z| \leq n$. Applying Jensen's theorem to entire function $T_W(z)$ we obtain

$$\begin{aligned} n(\varepsilon) \log 2 &\leq \int_{\epsilon}^{2\epsilon} [n(t)/t]\, dt \\ &\leq \int_{0}^{2\epsilon} [n(t)/t]\, dt \\ &= (2\pi)^{-1} \int_{0}^{2\pi} \log \left| \frac{T_W(2\varepsilon e^{i\theta})}{T_W(0)} \right| d\theta . \end{aligned}$$

Thus

$$n(\varepsilon) \leq (2\pi \log 2)^{-1} \int_{0}^{2\pi} \log \left| \frac{T_W(2\varepsilon e^{i\theta})}{T_W(0)} \right| d\theta \qquad \ldots(3.1)$$

holds for every set of $(a_1, a_2, \ldots, a_n)$ such that $T_W(0) \neq 0$.

Now

$$|\cos (2n\varepsilon\, e^{i\theta})| \leq 2e^{2n\epsilon}, \text{ and so}$$

$$|T_w(2\varepsilon e^{i\theta})| \leq 2e^{2n\epsilon}(p_1 + \cdots + p_n) \max_{1 \leq k \leq n} |a_k|$$

Now if $\max_{1 \leq k \leq n} |a_k| > n$ then $|a_k| > n$ for atleast one value of $k \leqslant n$, so that

$$\begin{aligned} P(\max |a_k| > n) &\leq \sum_{k=1}^{n} P(|a_k| > n) \\ &= n(2/\pi)^{1/2} \int_{n}^{\infty} e^{-t^2/2}\, dt \leqslant e^{-n^2/3} \end{aligned} \qquad \ldots(3.2)$$

Hence denoting $(p_1 + p_2 + \ldots + p_n)$ by D_n, and using (3.2) we get

$$Pr\,[|T_W(2\varepsilon e^{i\theta})| \leq 2D_n n\, e^{2n\epsilon}] \geq 1 - e^{-n^2/3}. \qquad \ldots(3.3)$$

Using the distribution function of $T_W(0)$ we find that

$$\begin{aligned} Pr\,[|T_w(0)| \leq e^{-2n\epsilon}] &= (2/\pi\Lambda^2)^{1/2} \int_{0}^{e^{-2n\epsilon}} \exp\,[-(1/2)\,(u^2/\Lambda^2)]\, du \\ &\leq (2/\pi\Lambda^2)^{1/2}\, e^{-2n\epsilon} < e^{-n\epsilon}. \end{aligned} \qquad \ldots(3.4)$$

From (3.3) and (3.4) we obtain

$$Pr\left[\left|\frac{T_w(2\varepsilon e^{i\theta})}{T_w(0)}\right| \leq 2n\, D_n\, e^{4n\epsilon}\right] \geq 1 - 2e^{-n\epsilon}.$$

From the relation (3.1) we get

$$Pr\,[n(\varepsilon) \leq 1 + (\log 2)^{-1} (\log n + \log D_n + 4n\varepsilon] > 1 - 2e^{-n\varepsilon}.$$

It can be easily verified that the number of zeros of $T(\theta)$ in $w - \varepsilon \leq \theta \leq w + \varepsilon$ does not exceed $n(\varepsilon)$. Hence

Lemma 1—The probability that $T(\theta)$ has more than

$$1 + (\log 2)^{-1} [2 \log n + \log D_n + 4n\varepsilon]$$

zeros in $w - \varepsilon \leq \theta \leq w + \varepsilon$ does not exceed $2e^{-n\epsilon}$.

4. To Find $E[N(T;\ \alpha, \beta)]$

The mathematical expectation of $N(T;\ \alpha, \beta)$ is

$$M_n(\alpha, \beta) = \int_{R_n} N(T;\ \alpha, \beta)\, dP(\bar{a}).$$

It is simpler to work with

$$N^*(T;\ \alpha, \beta) = N(T;\ \alpha, \beta) - (1/2)[v(\alpha) + v(\beta)]$$

where

$$v(\theta) = \begin{cases} 1 & \text{if } T(\theta) = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Since the probability that $T(\alpha)\, T(\beta)$ vanishes is null, we have

$$M_n(\alpha, \beta) = \int_{R_n} N^*(T;\ \alpha, \beta)\, dP(\bar{a}).$$

We state the result of Kac (1959), viz.:

If $f(\theta)$, continuous for $\alpha \leq \theta \leq \beta$, is continuously differentiable in $\alpha < \theta < \beta$ and if $f'(\theta)$ vanishes only at a finite number of points in the interval (α, β) then

$$N^*(f;\ \alpha, \beta) = (2\pi)^{-1} \int_{-\infty}^{\infty} d\xi \int_{\alpha}^{\beta} \cos(\xi f(\theta))\, |f'(\theta)|\, d\theta.$$

Thus

$$M_n(\alpha, \beta) = |M|^{1/2} (2\pi)^{-n/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} N^*(T;\ \alpha, \beta)$$

$$\exp[-(1/2)(\bar{a}'M\bar{a})]\, da_1\, da_2 \ldots da_n$$

$$= |M|^{1/2} (2\pi)^{-n/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp[-(1/2)\, \bar{a}'M\bar{a}]$$

$$[(2\pi)^{-1} \int_{-\infty}^{\infty} \cos(\xi\, T(\theta))\, |T'(\theta)|\, d\theta]\, da_1 \ldots da_n$$

$$= (2\pi)^{-1} \int_{\alpha}^{\beta} d\theta \int_{-\infty}^{\infty} d\xi R_n(\xi, \theta) \qquad \ldots(4.1)$$

where

$$R_n(\xi,\theta) = (2\pi)^{-n/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\left[-(1/2) \sum_{i=1}^{n} b_i^2\right] \cos(\xi T(\theta)) \, |T'(\theta)| \, db_1 \ldots db_n.$$

Using the identity $\pi^{-1} \int_{-\infty}^{\infty} \eta^{-2}(1-\cos\eta y)\, d\eta = |y|$, for $y = T'(\theta)$ we get

$$R_n(\xi,\theta) = (\pi)^{-1} \int_{-\infty}^{\infty} \eta^{-2}\, d\eta\, (2\pi)^{-n/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\left[-(1/2)\sum_{i=1}^{n} b_i^2\right] [\cos \xi T(\theta) - \cos \xi T(\theta) \cos \eta T'(\theta)]\, db_1 \ldots db_n.$$

Since

$$\cos(\xi T(\theta)) \cos(\eta T'(\theta)) = (1/2) \operatorname{Re} [\exp i(\xi T(\theta) + \eta T'(\theta)) + \exp i(\xi T(\theta) - \eta T'(\theta))]$$

we get

$$(2\pi)^{-n/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\left[-(1/2)\sum_{i=1}^{n} b_i^2\right] \cos(\xi T(\theta)) \cos(\eta T'(\theta))\, db_1 \ldots db_n$$

$$= (1/2) \operatorname{Re} \Big[(2\pi)^{-n/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\left(-(1/2)\sum_{i=1}^{n} b_i^2\right) \Big\{\exp i \sum_{k=1}^{n} (\xi X_k + \eta X_k') b_k + \exp i \sum_{k=1}^{n} (\xi X_k - \eta X_k')\, b_k\Big\} \times db_1\, db_2 \ldots db_n\Big]$$

$$= (1/2) \left[\exp\left(-(1/2)\sum_{i=1}^{n} U_i^2\right) + \exp\left(-(1/2)\sum_{i=1}^{n} V_i^2\right)\right] \qquad \ldots(4.2)$$

where X_k is as in (2.2), X_k' is the derivative of X_k with respect to x and $U_k = \xi X_k + \eta X_k'$; $V_k = \xi X_k - \eta X_k'$, $k = 1, 2, \ldots, n$.

Putting $\eta = 0$ in (4.2) we get

$$(2\pi)^{-n/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\left[-(1/2)\sum_{i=1}^{n} b_i^2\right] \cos(\xi T(\theta))\, db_1 \ldots db_n$$

$$= \exp\left[-(1/2)\sum_{k=1}^{n} W_k^2\right]$$

where

$$W_k = \xi X_k,\ k = 1, 2, \ldots, n.$$

Thus we get

$$R_n(\xi,\theta) = \pi^{-1} \int_{-\infty}^{\infty} \eta^{-2}\, d\eta \left[\exp\left(-(1/2)\sum_{k=1}^{n} W_k^2\right) - \exp\left(-(1/2)\sum_{k=1}^{n} V_k^2\right)\right]$$

$$= \pi^{-1} \int_{-\infty}^{\infty} \eta^{-2} \left[\exp\left(-(1/2)\, \xi^2 \sum_{k=1}^{n} X_k^2\right) - \exp\left(-(1/2) \sum_{k=1}^{n} (\xi X_k - \eta X_k')^2\right)\right] d\eta$$

$$= \pi^{-1} \int_{-\infty}^{\infty} \eta^{-2} [\exp \ (-(1/2)\, \xi^2 A)$$
$$- \exp \ (-(1/2) \ (\xi^2 A - 2\xi\eta B + \eta^2 C))]\, d\eta,$$

where

$$A = \sum_{k=1}^{n} X_k^2,$$

$$B = \sum_{k=1}^{n} X_k X_k'$$

and

$$C = \sum_{k=1}^{n} X_k'^2.$$

Using (2.2), (2.3) and (2.4) we get

$$\left.\begin{aligned} A &= (1-\rho) \sum_{k=1}^{n} p_k^2 \cos^2 k\theta + \rho \left(\sum_{k=1}^{n} p_k \cos k\theta \right)^2 \\ B &= (1-\rho) \sum_{k=1}^{n} p_k^2 k \cos k\theta \sin k\theta \\ &\quad + \rho \left(\sum_{k=1}^{n} p_k \cos k\theta \right) \left(\sum_{k=1}^{n} p_k k \sin k\theta \right) \\ C &= (1-\rho) \sum_{k=1}^{n} p_k^2 k^2 \sin^2 k\theta + \rho \left(\sum_{k=1}^{n} p_k k \sin k\theta \right)^2 \end{aligned}\right\} \quad \ldots(4.3)$$

Here always $AC - B^2 > 0$. Because

$AC = B^2$ only if

$$\frac{X_1'}{X_1} = \frac{X_2'}{X_2} = \ldots = \frac{X_n'}{X_n}$$

That is $\frac{X_1}{X_2}, \frac{X_1}{X_3}, \ldots, \frac{X_1}{X_n}$ are each constants.

That is $X_1, X_2, \ldots, X_n$ are each constant multiples of X_1. This is not the case always. Therefore

$$AC - B^2 \geqslant 0.$$

We assume that (α, β) is an interval in which $AC - B^2 > 0$. It can be shown, following Kac (1959) that

$$\int_{-\infty}^{\infty} d\xi \ Rn(\xi, \theta) = \frac{2}{\pi} (AC - B^2)^{1/2}$$

and hence from (4.1) we get

$$Mn(\alpha, \beta) = \pi^{-1} \int_{\alpha}^{\beta} [(AC - B^2)^{1/2} / A] \, d\theta. \quad \ldots(4.4)$$

5. To Find the Asymptotic Value of $E[N(T;\ 0, 2\pi)]$

The set of values w for which either

$$A(w)\,C(w) = B^2(w) \text{ or } C(w) = 0$$

will be called an w-set. In the present case, this set is $(0, \pm\pi, + 2\pi, \ldots)$. We take (α, β) as $(\varepsilon, \pi - \varepsilon)$ and $(\pi + \varepsilon, 2\pi - \varepsilon)$ successively. This yields

$$M_n(\varepsilon, \pi - \varepsilon) + M_n(\pi + \varepsilon, 2\pi - \varepsilon) = (2/\pi) \int_{\epsilon}^{\pi-\epsilon} [(AC - B^2)^{1/2}/A]\,d\theta.$$

From section 3 we get

$$M_n(0,\ 2\pi) = (2/\pi) \int_{\epsilon}^{\pi-\epsilon} [(AC - B^2)^{1/2}/A]\,d\theta$$

$$+ 0(\log n + \log D_n + L(n) + n\,e^{-L(n)}). \qquad \ldots(5.1)$$

Now we consider $M_n(0, 2\pi)$ for $p_k = k^m$, $m \geq 0$, $\varepsilon \leq \theta \leq \pi - \varepsilon$.

Taking $\varepsilon = n^{-3/13}$ so that $L(n) = n^{10/13}$ we get

$$\sum_{k=1}^{n} k^{2m} \cos^2 k\theta = \frac{1}{2}\left(\frac{n^{2m+1}}{2m+1}\right)(1 + O(n^{-10/13})),$$

$$\sum_{k=1}^{n} k^{2m+2} \sin^2 k\theta = \frac{1}{2}\left(\frac{n^{2m+3}}{2m+3}\right)(1 + O\,n^{-16/13}))$$

$$\sum_{k=1}^{n} k^{2m+1} \sin k\theta \cos k\theta = O(n^{2m+16/13})$$

$$\sum_{k=1}^{n} k^{m} \cos k\theta = O(n^{m+3/13})$$

$$\sum_{k=1}^{n} k^{m+1} \sin k\theta = O(n^{m+16/13}) \qquad \ldots(5.2)$$

Hence for large n from (5.2) we get

$$\frac{(AC - B^2)^{1/2}}{A} = \left(\frac{2m+1}{2m+3}\right)^{1/2} [n(1 + O(n^{-7/13}))].$$

Therefore, from (5.1) we get

$$m_n(0, 2\pi) = \left(\frac{2m+1}{2m+3}\right)^{1/2} [2n + O(n^{10/13})].$$

6. The preceding argument can be applied to trigonometric polynomials of the form $\sum_{k=1}^{n} a_k p_k \cos m_k\theta$, $(m_1 > m_2 > \ldots)$.

The case when the m_i form an arithmetic progression all the preceding results hold. For general m_i and p_i not very large, we obtain results for the average number of real zeros in an arbitrary closed intervals.

The situation for the trigonometric polynomials of the form

$$S(\theta) + T(\theta) \qquad \ldots(6.1)$$

where $S(\theta) = \sum_{k=1}^{m} c_k q_k \sin k\theta$, where c_i are distributed as a_i in section (1) and q_k form a set of positive constants. Here

$$M_n(\theta, 2\pi) = (2/\pi) \int_0^{\pi} [(A_1 C_1 - B_1^2)^{1/2}/A_1]\, d\theta$$

where

$$\left.\begin{aligned} A_1 &= (1-\rho)\Big(\sum_{k=1}^{n} p_k^2 \cos^2 k\theta + \sum_{k=1}^{m} q_k^2 \sin^2 k\theta\Big) \\ &\quad + \rho\Big[\Big(\sum_{k=1}^{n} p_k \cos k\theta\Big)^2 + \Big(\sum_{k=1}^{m} q_k \sin k\theta\Big)^2\Big] \\ B_1 &= (1-\rho)\Big(\sum_{k=1}^{n} k p_k^2 \cos k\theta \sin k\theta - \sum_{k=1}^{m} k q_k^2 \cos k\theta \sin k\theta\Big) \\ &\quad + \rho\Big[\Big(\sum_{k=1}^{n} k p_k \sin k\theta\Big)\Big(\sum_{k=1}^{n} p_k \cos k\theta\Big) \\ &\quad - \Big(\sum_{k=1}^{m} k q_k \cos k\theta\Big)\Big(\sum_{k=1}^{m} q_k \sin k\theta\Big)\Big] \\ &\text{and} \\ C_1 &= (1-\rho)\Big(\sum_{k=1}^{n} k^2 p_k^2 \sin^2 k\theta + \sum_{k=1}^{m} k^2 q_k^2 \cos^2 k\theta\Big) \\ &\quad + \rho\Big[\Big(\sum_{k=1}^{n} k p_k \sin k\theta\Big)^2 + \Big(\sum_{k=1}^{m} k q_k \cos k\theta\Big)^2\Big]. \end{aligned}\right\} \qquad \ldots(6.2)$$

If p_k increases as rapidly as $k^{\log k}$ and $1 \leq k \leq n$ then

$$M_n(0, 2\pi) \sim 2n, \quad n \to \infty. \qquad \ldots(6.3)$$

If $T(\theta)$ has multiple roots it can be shown that M_n is unchanged. Similar result holds for $S(\theta) + T(\theta)$ also.

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