CYLINDRICALLY SYMMETRIC FLUID DISTRIBUTION AND MAGNETIC ENERGY IN GENERAL RELATIVITY

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In the present paper, solution of Einstein-Maxwell field equations have been obtained for the cylindrically symmetric fluid distributions taking magnetic field along Z-direction.

1. INTRODUCTION

Banerji (1968) has investigated the problem of cylindrical symmetry of uncharged matter in absence of rotation. Som and Raychoudhury (1968) have also found a solution for charged dust with rotation where however Lorentz force vanishes. Here in the present paper the author has discussed the cylindrically symmetric fluid distribution taking magnetic field along Z-direction and has obtained the solution for ρ, p and H^2 by assuming

$$8\pi p = bH^2 - \tfrac{1}{2} r^{-2}\{n^2 - 2n + (1+b)(n^2+n)\}e^{-2(\alpha-\beta)}$$

and $\gamma = -n \log r$ where $b > 0$ and $0 < n < 0{\cdot}5$. However, the solution has singularity at $r = 0$ which case may be due to massive bodies which may continue to exist at infinite pressure and density at $r = 0$ and we are discussing one such case.

2. FIELD EQUATIONS AND THEIR SOLUTIONS

Choosing co-moving coordinates (r, ϕ, z) the line-element exhibiting cylindrical symmetry given by Marder (1968) can be put in the form

$$ds^2 = e^{2(\alpha-\beta)}(dt^2 - dr^2) - r^2 e^{-2\beta} d\phi^2 - e^{2(\beta+\gamma)} dz^2 \qquad ...(1)$$

where α, β and γ are functions of r alone.

The Einstein-Maxwell equations are

$$R^{\mu}_{\nu} - \tfrac{1}{2} R\delta^{\mu}_{\nu} = -8\pi T^{\mu}_{\nu} \qquad ...(2)$$

$$T^{\mu}_{\nu} = (\rho + p) V^{\mu} V_{\nu} - p\,\delta^{\mu}_{\nu} - (4\pi)^{-1}[F^{\mu\alpha} F_{\nu\alpha} - (4)^{-1}\delta^{\mu}_{\nu} F^{\alpha\beta} F_{\alpha\beta}] \qquad ...(3)$$

$$F^{\mu\nu}_{;\nu} = 4\pi J^{\mu} \qquad ...(4)$$

$$F_{[\mu\nu;\alpha]} = 0. \qquad ...(5)$$

The matter is at rest in the co-ordinate system of (1) so that $V^{\mu} = (g_{00})^{-1/2}\delta^{\mu}_{0}$. The surviving component of the magnetic field along Z direction is F^{12}.

The eqns. (2) are written out as follows

$$e^{2(\beta-\alpha)}\,(-\beta_1^2 - 2\beta_1\gamma_1 + \alpha_1\gamma_1 + r^{-1}\alpha_1 + r^{-1}\gamma_1) = 8\pi p + H^2 \qquad \ldots(6)$$

$$e^{2(\beta-\alpha)}\,(2\beta_1\gamma_1 + \alpha_{11} + \beta_1^2 + \gamma_{11} + \gamma_1^2) = 8\pi p + H^2 \qquad \ldots(7)$$

$$e^{2(\beta-\alpha)}\,(-2\beta_{11} - 2r^{-1}\beta_1 + \alpha_{11} + \beta_1^2) = 8\pi p - H^2 \qquad \ldots(8)$$

$$e^{2(\beta-\alpha)}\,(-\alpha_1\gamma_1 + 2\beta_1\gamma_1 - r^{-1}\alpha_1 + \beta_1^2 + \gamma_{11} + \gamma_1^2 + r^{-1}\gamma_1) = -(8\pi\rho + H^2) \qquad \ldots(9)$$

where the subscript 1 indicates differentiation w.r.t. r and $F^{12}F_{12} = H^2$.

Now since there are four equations and six variables, we can make two assumptions

(i) $8\pi p = bH^2 - \frac{1}{2} r^{-2}\{n^2 - 2n + (b+1)(n^2+n)\}\, e^{-2(\alpha-\beta)}$

(ii) $\gamma = -n\log r$

where $b > 0$ and $0 < n < 0.5$.

From eqns. (6) and (9) we have

$$8\pi\rho = 8\pi p + (r^{-2})\, n(1-n)\, e^{2(\beta-\alpha)}, \qquad \ldots(10)$$

where $0 < n < 0{\cdot}5$.

From eqns. (6) and (7)

$$\beta_1^2 = (r)^{-1} 2n\beta_1 - \tfrac{1}{2}\alpha_{11} + (2r)^{-1}\alpha_1(1-n) - (2r^2)^{-1}(n^2+2n). \qquad \ldots(11)$$

From eqns. (6) and (8) after substraction

$$H^2 = [\beta_{11} + r^{-1}(1-n)\beta_1 + (2r^2)^{-1}(n^2+n)]\, e^{2(\beta-\alpha)}\,. \qquad \ldots(12)$$

From eqn. (9) using eqns. (11) and (12)

$$[(1+b)\beta - \tfrac{1}{2}\alpha]_{11} + (r)^{-1}(1-n)[(1+b)\beta - \tfrac{1}{2}\alpha]_1 = 0. \qquad \ldots(13)$$

Solution of eqn. (13)

$$\alpha = 2(1+b)\beta - 2l. \qquad \ldots(14)$$

Here l can be reduced to zero by a suitable choice of co-ordinate. Now from eqn, (11) using eqn. (14)

$$\beta = (1+b)\log(1 + Br^{2A/(1+b)} - k\log r + \log C \qquad \ldots(15)$$

where

$$k = A - [n + \tfrac{1}{2}(1+b)(2-n)]$$

and

$$A = \{[n + \tfrac{1}{2}(1+b)(2-n)]^2 - \tfrac{1}{2}(n^2+2n)\}^{1/2}$$

and C is a constant of integration.

Now eqn. (12) gives H^2 and then

$$8\pi p = \left[\frac{E + FBr^{2A/(1+b)} + G\,B^2 r^{4A/(1+b)}}{C^{2(1+b)}\, r^{2-2k(1+b)}\,(1 + Br^{2A/(1+b)})^{2+2(1+b)\,(1+2b)}}\right] \qquad \ldots(16)$$

where

$$E = b\left[\tfrac{1}{2}(n^2+n) + nk\right] - \tfrac{1}{2}\left[n^2 - 2n + (1+b)(n^2+n)\right]$$

$$F = b\left[n^2 + n - 2An + 2kn + (1+b)^{-1} 4A^2\right] - \tfrac{1}{2}\left[n^2 - 2n + (1+b)(n^2+n)\right]$$

$$G = b\left[\tfrac{1}{2}(n^2+n) - 2An + nk\right] - \tfrac{1}{2}\left[n^2 - 2n + (1+b)(n^2+n)\right]. \quad \ldots(17)$$

Equation (10) will now give $8\pi\rho$.

At the boundary since $p = 0$ we have

$$H_0^2 = (2br_0^2)^{-1}\left[n^2 - 2n + (1+b)(n^2+n)\right] e^{2(\beta-\alpha)} \quad \ldots(18)$$

For H_0^2 to be positive, $b > (1-2n)(1+n)^{-1}$. Thus $n < 0{\cdot}5$. Hence n lies between 0 and $0{\cdot}5$. With these values of n and b, k is negative. A will be real. E and F will be positive and G will be negative.

The boundary at which $p = 0$ is given by

$$2GBr_0^{2A/(1+b)} = \left[-F - (F^2 - 4EG)^{1/2}\right]. \quad \ldots(19)$$

For $b = n = 0$, we have the empty-space solution

$$\alpha = 2\log C(1 + Br^2) \quad \ldots(20)$$

$$\beta = \log C(1 + Br^2) \quad \ldots(21)$$

and

$$H^2 = 4B(1 + Br^2)^{-1} e^{2(\beta-\alpha)}. \quad \ldots(22)$$

The conduction current-density along ϕ direction

$$J^2 = (4\pi r)^{-1} e^{2\beta-\alpha}\left[\alpha_1 \sqrt{H^2} + (\sqrt{H^2})_1\right]. \quad \ldots(23)$$

3. Continuity with the Exterior Solution

The exterior solutions obtained from equations (6)–(9) with $\gamma = 0$ are

$$\alpha = \log r^{(2+f-2\sqrt{1+f})}(1 + hr^{2\sqrt{1+f}})^2 \quad \ldots(24)$$

$$\beta = \log(r^{1-\sqrt{1+f}} + hr^{1+\sqrt{1+f}}) \quad \ldots(25)$$

where f and h are constants of integration. For the solution (14) and (15) to be continuous with the exterior solution (24) and (25) at $r = r_0$ we have

$$C(1 + Br_0^{2A/1+b})^{2(1+b)^2} r_0^{-k} = r_0^{2+f-2\sqrt{1+f}}(1 + hr_0^{2\sqrt{1+f}})^2 \quad \ldots(26)$$

$$\left[\frac{d}{dr} C(1 + Br^{2A/1+b})^{2(1+b)^2} r^{-k}\right]_{r=r_0} = \left[\frac{d}{dr} r^{2+f-2\sqrt{1+f}}(1 + hr^{2\sqrt{1+f}})^2\right]_{r=r_0} \quad \ldots(27)$$

$$C(1 + Br_0^{2A/1+b})^{(1+b)} r_0^{-k} = (r_0^{1-\sqrt{1+f}} + hr_0^{1+\sqrt{1+f}}) \quad \ldots(28)$$

$$\left[\frac{d}{dr}\, C\left(1+Br^{2A/1+\delta}\right)^{(1+\delta)} r^{-k}\right]_{r=r_0} = \left[\frac{d}{dr}\, r^{1-\sqrt{1+f}} + hr^{1+\sqrt{1+f}}\right]_{r=r_0} \qquad \ldots(29)$$

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