LAMB'S LINE LOAD PROBLEM FOR A POROUS ELASTIC HALF-SPACE: NON-DISSIPATIVE CASE

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Lamb's line load problem for a porous elastic half-space is solved by the use of Laplace transform, neglecting the dissipation of the fluid on the basis of Biot's formulation of dynamic poroelasticity. All the dependent variables like the solid and the fluid displacements, stresses, etc., are expressed in terms of four displacement potentials which can be solved in the transformed space. The analysis is restricted only to the evaluation of the normal stress by Cagniard's technique. It is seen that the normal stress and similarly the other stresses, in general, consist of terms corresponding to three body waves and six head waves. In particular, when the fluid is absent or the velocities of the two dilatational waves are same the solutions are in agreement with those of Lamb's line load problem in classical elasticity. Lastly, some numerical results for the normal stress are given for different values of time and its comparison with non-porous classical case is shown graphically.

1. INTRODUCTION

The dynamical theory of poro-elasticity was established by Biot (1956) and was applied by Deresiewicz (1960) to analyse the effect of boundaries on the wave propagation in fluid-filled porous media. Some other problems considered by several authors are listed in a review article by Paria (1966). The present paper deals with the Lamb's line load problem for a porous elastic half-space in the non-dissipative case. The displacements and stresses have been expressed in terms of four displacement potentials as in Deresiewicz and Rice (1962). This has been followed by the application of Laplace-Fourier transform to solve the displacement potentials in the transformed domain. The evaluation of the normal stress is then effected by Cagniard's technique (Fung 1968). The expressions for the other stresses can similarly be obtained. The normal stress gives an idea of the stress-wave pattern in the medium. It turns out that the stress wave consists of three body waves (two dilatational wave and one shear wave) and six head waves while in the (non-porous) classical elastic case there are only two body waves and two head waves. The normal stress for various values of time for the porous and the non-porous classical cases has been evaluated numerically and exhibited graphically.

2. The Statement of the Problem and the Transformed Solution

The equations of motion of a liquid-filled porous elastic solid in the low frequency range and when the dissipation due to the flow of the liquid relative to solid is neglected, are given by Deresiewicz and Rice (1962) as

$$N\nabla^2 \vec{u} + \text{grad}\,[(D+N)\,\text{div}\,\vec{u} + Q\,\text{div}\,\vec{U}] = \frac{\partial^2}{\partial t^2}[\rho_{11}\vec{u} + \rho_{12}\vec{u}] \qquad ...(2.1)$$

$$\text{grad}\,[Q\,\text{div}\,\vec{u} + R\,\text{div}\,\vec{U}] = \frac{\partial^2}{\partial t^2}[\rho_{12}\vec{u} + \rho_{22}\vec{U}] \qquad ...(2.2)$$

where $\vec{u}$ and $\vec{U}$ are the solid and fluid displacement vector, D, N, Q and R non-negative elastic constants, ρ_{11}, ρ_{12} and ρ_{22}, dynamical coefficients.

The stress-strain relations are given by

$$\sigma_{ij} = (De + Q\varepsilon)\,\delta_{ij} + 2N\,\varepsilon_{ij}\begin{pmatrix} i = x, y, z \\ j = x, y, z \end{pmatrix}$$

and

$$S = Qe + R\varepsilon$$

with

$$\varepsilon_{ij} = \tfrac{1}{2}[u_{i,j} + u_{j,i}], \qquad e = \text{div}\,\vec{u}, \qquad \varepsilon = \text{div}\,\vec{U}$$

and δ_{ij}, the Kronecker symbol.

We consider a semi-infinite porous elastic medium occupying the region $z > 0$, which is bounded by the plane $z = 0$ and on the boundary surface of which a concentrated line load is applied along the y-axis. To this end we first consider the solutions of an elementary two-dimensional problem in the x-z plane in which the boundary conditions are given by

$$\sigma_{zz} = 0, \quad s = 0, \qquad \sigma_{zx} = -Q_0 e^{i\alpha x} H(t) \quad \text{on} \quad z = 0 \qquad ...(2.3)$$

where Q_0 is a constant, α a parameter and $H(t)$ the heaviside unit function. The Helmholtz resolution of each of two displacement vector $\vec{u}$, $\vec{U}$ can be taken as in Deresiewicz and Rice (1962) in the form

$$(a)\ \vec{u} = \text{grad}\,\phi + \text{curl}\,\vec{H}, \qquad (b)\ \vec{U} = \text{grad}\,\psi + \text{curl}\,\vec{G}. \qquad ...(2.4)$$

Subject to the condition

$$\text{div}\,\vec{H} = \text{div}\,\vec{G} = 0 \qquad ...(2.4a)$$

where

$$\phi = \phi\,(x, z, t), \qquad \psi = \psi\,(x, z, t)$$

$$\vec{H} = [H_1\,(x, z, t), \quad H_2\,(x, z, t), \quad H_3\,(x, z, t)]$$

$$\vec{G} = [G_1\,(x, z, t), \quad G_2\,(x, z, t), \quad G_3\,(x, z, t)].$$

Since $u_y = U_y = 0$, using (2.4) and the divergence conditions (2.4a) we have $H_1 = H_3 = G_1 = G_3 = 0$. We write henceforth H and G in place of H_2 and G_2 respectively. Substituting (2.4) in (2.1) and (2.2) we have

$$\left.\begin{aligned} P\nabla^2 \phi + Q\nabla^2 \psi &= \frac{\partial^2}{\partial t^2}[\rho_{11}\phi + \rho_{12}\psi] \\ Q\nabla^2 \phi + R\nabla^2 \psi &= \frac{\partial^2}{\partial t^2}[\rho_{12}\phi + \rho_{22}\psi] \end{aligned}\right\} \qquad \ldots(2.5)$$

and

$$\left.\begin{aligned} N\nabla^2 H &= \frac{\partial^2}{\partial t^2}[\rho_{11}H + \rho_{12}G] \\ 0 &= \frac{\partial^2}{\partial t^2}[\rho_{12}H + \rho_{22}G] \end{aligned}\right\} \qquad \ldots(2.6)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x^2}, \qquad P = D + 2N.$$

The non-vanishing stress components are

$$\left.\begin{aligned} \sigma_{yy} &= D\nabla^2 \phi + Q\nabla^2 \psi \\ \sigma_{xz} &= N\left[2\frac{\partial^2 \phi}{\partial x \partial z} + \frac{\partial^2 H}{\partial x^2} - \frac{\partial^2 H}{\partial z^2}\right] \\ \sigma_{zz} &= D\nabla^2 p + Q\nabla^2 \psi + 2N\left[\frac{\phi^2\, \partial}{\partial z^2} + \frac{\partial^2 H}{\partial z \partial x}\right] \\ \sigma_{xx} &= D\nabla^2 p + Q\nabla^2 \psi + 2N\left[\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 H}{\partial z \partial x}\right] \\ S &= Q\nabla^2 \phi + R\,\nabla^2 \psi \end{aligned}\right\} \qquad \ldots(2.7)$$

We define the Laplace transform of a function $f\,(x, z, t)$ by

$$\bar{f}\,(x, z, s) = \int_0^\infty f\,(x, z, t)\, e^{-st}\, dt$$

s being the Laplace transform parameter.

Applying the Laplace transform to the eqns. (2.5), we get the equations

$$\left.\begin{aligned} (P\nabla^2 - s^2\rho_{11})\bar{\phi} + (Q\nabla^2 - s^2\rho_{12})\bar{\psi} &= 0 \\ (Q\nabla^2 - s^2\rho_{12})\bar{\phi} + (R\nabla^2 - s^2\rho_{22})\bar{\psi} &= 0 \end{aligned}\right\} \qquad \ldots(2.8)$$

where we have assumed $\phi, \psi, \partial\phi/\partial t$ and $\partial\psi/\partial t$ their time tend to zero as $t \to 0$. Eliminating $\bar{\psi}$ from the two equations in (2.8), we have

$$A\nabla^4 \bar{\phi} - \delta_0^2 B \nabla^2 \bar{\phi} + \delta_0^4 C \bar{\phi} = 0 \qquad \ldots(2.9)$$

where

$$A = \sigma_{11}\sigma_{22} - \sigma_{12}^2, \qquad B = \gamma_{11}\sigma_{22} + \gamma_{22}\sigma_{11} - 2\gamma_{12}\sigma_{12},$$
$$C = \gamma_{11}\gamma_{22} - \gamma_{12}^2$$

and

$$H(\sigma_{11}, \sigma_{12}, \sigma_{22}) = (P, Q, R), \quad (\rho_{11}, \rho_{12}, \rho_{22}) = \rho(\gamma_{11}, \gamma_{12}, \gamma_{22})$$

$$H = P + 2Q + R, \qquad \rho = \rho_{11} + 2\rho_{12} + \rho_{22}, \qquad \delta_0^2 = \frac{\rho s^2}{H}.$$

It can be shown that each of the constants A, B, C is non-negative.

The solution of (2.9) which is bounded for all z is assumed in the form

$$\bar{\phi} = [A_1 e^{-\nu_1 z} + A_2 e^{-\nu_2 z}] e^{i\alpha x} \qquad \ldots(2.10)$$

which satisfy the equation (2.9) if

$$\nu_j^2 = \alpha^2 + \delta_j^2 \; (j = 1, 2) \qquad \ldots(2.11)$$

and

$$\delta_j^2 = \delta_0^2 \Lambda_j, \qquad \Lambda_{1,2} = \frac{B \mp \Delta}{2A}, \qquad \Delta^2 = B^2 - 4AC. \qquad \ldots(2.12)$$

It can also be shown that $B^2 - 4AC > 0$ and Λ_1 and Λ_2 are both positive.

The remaining scalar potential $\bar{\psi}$ is found to be

$$\bar{\psi} = [\mu_1 A_1 e^{-\nu_1 z} + \mu_2 A_2 e^{-\nu_2 z}] e^{i\alpha x} \qquad \ldots(2.13)$$

where

$$\mu_j = \frac{g - A\Lambda_j}{h} \qquad (j = 1, 2)$$

with

$$g = \sigma_{22}\gamma_{11} - \sigma_{12}\gamma_{12}, \qquad h = \sigma_{12}\gamma_{12} - \sigma_{22}\gamma_{12}.$$

Similarly from the pair (2.6) in the transformed space, we have

$$\bar{H} = B_1 e^{-\nu_3 z + i\alpha x} \qquad \text{and} \qquad \bar{G} = \mu_3 B\_ e^{-\nu_3 z + i\alpha x} \qquad \ldots(2\ 14)$$

with

$$\mu_3 = -\frac{\gamma_{12}}{\gamma_{22}}, \qquad \delta_3^2 = \delta_0^2 \Lambda_3, \qquad \nu_3^2 = \alpha^2 + \delta_3^2$$

$$\Lambda_3 = \frac{HC}{N\gamma_{22}}.$$

A_1, A_2 and β_1 are all function of s and α.

The Laplace transformed boundary conditions (2.3) yield

$$\left.\begin{aligned}
\bar{\phi} &= \frac{Z}{N}\frac{(2\alpha^2+\delta_3^2)}{F(\alpha,s)}\left[2K_2 e^{-\nu_1 z} - 2K_1 e^{-\nu_2 z}\right] e^{i\alpha x} \\
\bar{\psi} &= \frac{Z}{N}\frac{(2\alpha^2+\delta_3^2)}{F(\alpha,s)}\left[2\mu_1 K_2 e^{-\nu_1 z} - 2\mu_2 k_1 e^{-\nu_2 z}\right] e^{i\alpha x} \\
\bar{H} &= \frac{Z}{N}\cdot\frac{4i\alpha(\nu_2 K_1 - \nu_1 k_2)}{F(\alpha,s)}\, e^{-\nu_3 z + i\alpha x} \\
\bar{G} &= \frac{Z}{N}\cdot\frac{4\mu_3 i\alpha(\nu_2 k_1 - \nu_1 k_2)}{F(\alpha,s)}\, e^{-\nu_3 z + i\alpha x}
\end{aligned}\right\} \qquad \ldots(2.15)$$

where

$$F(\alpha,s) = \left(2\alpha^2 + \frac{s^2}{V_3^2}\right)^2 - 4\alpha^2\left(\alpha^2+\frac{s^2}{V_3^2}\right)^{1/2}\left[K_1\left(\alpha^2+\frac{s^2}{V_2^2}\right)^{1/2} - K_2\left(\alpha^2+\frac{s^2}{V_1^2}\right)^{1/2}\right]$$

$$K_i = \frac{\alpha_i\Lambda_i}{\alpha_1\Lambda_1 - \alpha_2\Lambda_2}, \qquad Z = \frac{Q_0}{2s}, \qquad \alpha_i = Q + R\mu_i \quad (i = 1, 2)$$

and where we have introduced the phase velocities of the three body waves defined by (Deresiewicz and Rice 1962)

$$\delta_j^2 = \frac{s^2}{V_j^2} \qquad (j = 1, 2, 3).$$

The expressions for the stresses for the elementary problem given by (2.3) can now be obtained. The solutions for the original line load problem are obtained by integrating those for the elementary problem with respect to α from $-\infty$ to $+\infty$ after multiplication with a factor $\left(\frac{1}{2\pi}\right)$. The normal stress, for example for the line load problem is therefore given by

$$\bar{\sigma}_{zz} = \bar{I}_1 + \bar{I}_2 + \bar{I}_3 \qquad \ldots(2.16)$$

where

$$\bar{I}_1 = \frac{Q_0}{4\pi}\int_{-\infty}^{+\infty} \frac{\left(2\alpha^2 + \frac{X_1}{N}\frac{s^2}{V_1^2}\right)\cdot 2K_2\left(2\alpha^2+\frac{s^2}{V_3^2}\right) e^{-\left(\alpha^2+\frac{s^2}{V_1^2}\right)^{1/2} z + i\alpha x}}{s\,.\,F(\alpha,s)}\, d\alpha$$

$$\bar{I}_2 = -\frac{Q_0}{4\pi}\int_{-\infty}^{+\infty} \frac{\left(2\alpha^2 + \frac{X_2}{N}\cdot\frac{s^2}{V_2^2}\right)\cdot 2k_1\left(2\alpha^2+\frac{s^2}{V_3^2}\right) e^{-\left(\alpha^2+\frac{s^2}{V_2^2}\right)^{1/2} z + i\alpha x}}{s\,.\,F(\alpha,s)}\, d\alpha$$

$$\bar{I}_3 = \frac{Q_0}{4\pi}\int_{-\infty}^{+\infty} \frac{8\alpha^2\left(\alpha^2+\frac{s^2}{V_3^2}\right)^{1/2}\left[K_1\left(\alpha^2+\frac{s^2}{V_2^2}\right)^{1/2} - K_2\left(\alpha^2+\frac{s^2}{V_1^2}\right)^{1/2}\right] e^{-\left(\alpha^2+\frac{s^2}{V_3^2}\right)^{1/2} z + i\alpha x}}{(s\,.\,F\alpha, s)}$$

with

$$\chi_i = P + Q\mu_i \quad (i = 1, 2).$$

3. Evaluation of the Normal Stress

The normal stress can be evaluated by taking inverse Laplace transforms. However, we shall make use of Cagniard's Techique (Fung 1968) to put the right side of (2.16) in terms of Laplace transform of known functions so that σ_{zz} can be obtained by inspection. Since each of μ_i and K_i $(i = 1, 2)$ is real, by putting in the above integral I_1 we have

$$I_1(x, z, s) = \frac{Q_0}{2\pi} \int_{-\infty}^{+\infty} M_1(k)\, e^{-s[\sqrt{k^2+V_1^{-2}}\, z - ikx]}\, dk \qquad \ldots(3.1)$$

where

$$M_1(k) = \left[\left(2k^2 + \frac{\chi_1}{N} V_1^{-2}\right) k_2 (2k^2 + V_3^{-2}) R_0^{-1}\right]$$

and

$$R_0 = (2k^2 + V_3^{-2})^2 - 4k^2\sqrt{k^2 + V_3^{-2}}\,(k_1\sqrt{k^2 + V_2^{-2}} - k_2\sqrt{k^2 + V_1^{-2}}) \qquad \ldots(3.2)$$

denotes Rayleigh function.

In the expressions (3.1), the Laplace transform parameter only appears in the argument of the exponent. This form is necessary to use Cagniard's technique (Fung 1968).

Now we introduce the transformation,

$$t = \sqrt{k^2 + V_1^{-2}}\, z - ikx\,. \qquad \ldots(3.3)$$

The singularities of the integrand in the complex half-plane (Im $k > 0$) are the branch points at $+\, iV_1^{-1}$, $+\, iV_2^{-1}$ and $+\, iV_3^{-1}$. If in the equation (3.2) we put $k = i/c$, this lead to a seventh degree equation in C whose real root c_R corresponds to Rayleigh wave velocity (Deresiewicz and Rice 1962). Hence in the k-plane the point $+\, iC_R^{-1}$; a simple pole of the integrand. It is clear that $V_1 > V_2$; also we assume that $V_2 > V_3 > C_R$.

The positions of the singularities of $M_1(k)$ in the complex k-plane are shown in Fig. 1.

solving for k from (3.3), we get

$$k = k_1^{\pm} = \pm\sqrt{\frac{t^2}{r^2} - V_1^{-2}}\, \sin\theta + i\frac{t}{r}\cos\theta$$

$$(r^2 = x^2 + z^2, \qquad \theta = \tan^{-1}(z/x), \qquad 0 \le \theta \le \pi)$$

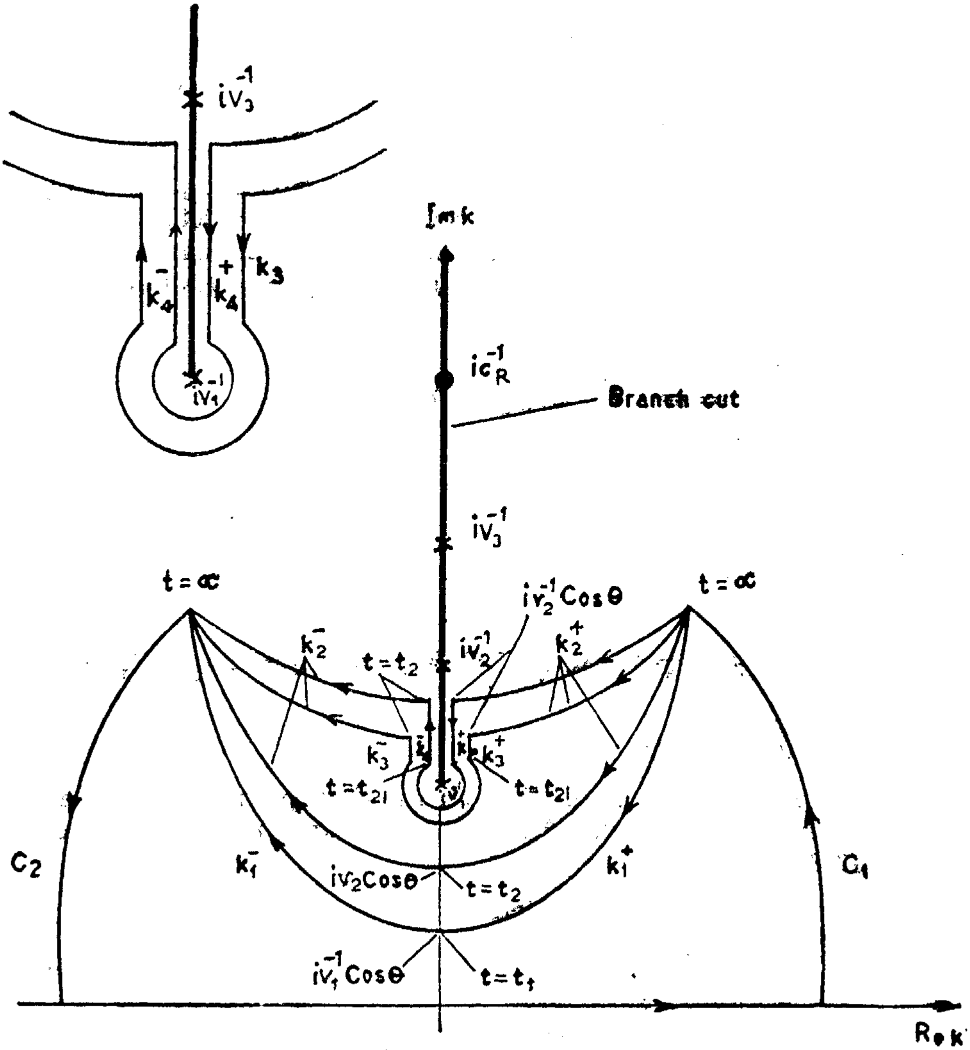

FIG. 1. Contour in the k plane.

which defines one branch of a hyperbola with the vertex $+\, iV_1^{-1} \cos\theta$ and whose asymptotes make an angle θ with the imaginary k-axis. Let t be real and positive. The hyperbola $k = k_1^{\pm}$ is parametrically described by t varies from $t_1 \,(= V_1^{-1} r)$ to ∞. Then proceeding as in Fung (1968) it can be shown that the integral

$$\frac{Q_0}{2\pi} \int_{-\infty}^{+\infty} M_1(k)\, e^{-st}\, dk$$

can be replaced by the integrals along the hyperbolic path $\mathrm{Re}\, k + C_1 + k_1^{+} + k_1^{-} + C_2$ in a direction opposite to that shown in Fig. 1.

So we may write

$$\bar{I}_1 = \frac{Q_0}{2\pi} \int_{t_1}^{+\infty} \left[M_1(k_1^+) \frac{\partial k_1^+}{\partial t} - M_1(k_1^-) \frac{\partial (k_1^-)}{\partial t} \right] e^{-st}\, dt$$

where

$$k_1^\pm = \pm \sqrt{\frac{t^2}{r^2} - V_1^{-2}} \sin\theta + i\frac{t}{r}\cos\theta, \quad 0 \leq \theta \leq \pi.$$

And the inverse transform of $\bar{I}_1$, is

$$I_1(x, z, t) = \frac{Q_0}{2\pi} H(t - t_1) \left[M_1(k_1^+) \frac{\partial k_1^+}{\partial t} - M_1(k_1^-) \frac{\partial k_1^-}{\partial t} \right]. \tag{3.4}$$

Considering the second term in the right-hand expression of $\bar{\sigma}_{zz}$ and integrating similarly as in Fung (1968) along the contour $\mathrm{Re}\, k + C_1 + k_2^+ + k_2^- + C_2$ $\left(\text{as } \cos^{-1}\frac{V_2}{V_3} \leq \theta \leq \pi - \cos^{-1}\frac{V_2}{V_1}\right)$, $\mathrm{Re}\, k + C_1 + k_2^+ + k_3^+ + k_3^- + k_2^- + C_2$ $\left(\text{as } \delta_1 \to 0) \text{ and } 0 \leq \theta \leq \cos^{-1}\frac{V_2}{V_1}\right)$ and $\mathrm{Re}\, k + C_1 + k_2^+ + k_4^+ + k_4^- + k_2^- + C_2$ $\left(\text{as } \delta_2 \to 0 \text{ and } \pi - \cos^{-1}\frac{V_2}{V_1} \leq \theta \leq \pi\right)$ of Fig. 1 respectively, we have in the range $0 \leq \theta \leq \pi$

$$\begin{aligned} I_2(x, z, t) = -\frac{Q_0}{2\pi}\Bigg[& H(t - t_2) \left\{ M_2(k_2^+) \frac{\partial k_2^+}{\partial t} - M_2(k_2^-) \frac{\partial k_2^-}{\partial t} \right\} \\ & + f_\theta^{(1)} f_t^{(1)} \{M_2(k_3^+) - M_2(k_3^-)\} \frac{\partial k_3^+}{\partial t} \\ & + f_\theta^{(2)} f_t^{(2)} \{M_2(k_4^+) - M_2(k_4^-)\} \frac{\partial k_4^+}{\partial t} \Bigg] \end{aligned} \tag{3.5}$$

$$M_2(k) = \left[k_1 \left(2k^2 + \frac{x_2}{N} V_2^{-2} \right) (2k^2 + V_3^{-2}) R_0^{-1} \right]$$

$$k_2^\pm = \pm \sqrt{\frac{t^2}{r^2} - V_2^{-2}} \sin\theta + i\frac{t}{r}\cos\theta$$

$$k_3^\pm = i\left[-\sqrt{V_2^{-2} - \frac{t^2}{r^2}} \sin\theta + \frac{t}{r}\cos\theta \right] + \delta_1$$

$$k_4^\pm = i\left[\sqrt{V_2^{-2} - \frac{t^2}{r^2}} \sin\theta + \frac{t}{r}\cos\theta \right] \pm \delta_2$$

$$t_2 = V_2^{-1} r$$

$f_\theta^{(1)} = 1$ in the range $0 \leq \theta < \cos^{-1}\frac{V_2}{V_1}$

$= 0$ elsewhere.

$$f_t^{(1)} = 1 \text{ for } t_2 \geq t \geq t_{21}$$
$$= 0 \text{ elsewhere}$$
$$f_\theta^{(2)} = 1 \text{ for } \pi - \cos^{-1}\frac{V_2}{V_1} < \theta \leq \pi$$
$$= 0 \text{ elsewhere}$$
$$f_t^{(2)} = 1 \text{ for } t_2 \geq t \geq t'_{21}$$
$$= 0 \text{ elsewhere.}$$

where

$$t_{21} = V_1^{-1} r \cos\theta + r\sqrt{V_2^{-2} - V_1^{-2}} \sin\theta$$
$$t'_{21} = V_1^{-1} r \,|\cos\theta\,| + r\sqrt{V_2^{-2} - V_1^{-2}} \sin\theta$$

Lastly, when we invert $\bar{I}_3$, we see that the vertex of the hyperbola $k = k_5^{\pm}$ can lie on the branch out between the branch points at $k = iV_3^{-1}$ and $k = iV_2^{-1}$, and at $k = iV_3^{-1}$ and $k = iV_1^{-1}$ as well as at $k = iV_2^{-1}$ and $k = iV_1^{-1}$ depending on different values of θ. Then the contours in the k-plane which are necessary for inverting $\bar{I}_3$. though complicated and have multiple configurations, proceeding in a manner similar to that used in inverting $\bar{I}_2$, we get

$$I_3(x, z, t) = \frac{2Q_0}{\pi}\Bigg[H(t - t_3)\left\{M_3(k_5^+)\frac{\partial k_5^+}{\partial t} - M_3(k_5^-)\frac{\partial k_5^-}{\partial t}\right\}$$
$$+ f_\theta^{(3)} f_t^{(3)} \{M_3(k_6^+) - M_3(k_6^-)\}\frac{\partial k_6^+}{\partial t}$$
$$+ f_\theta^{(4)} f_t^{(4)} \{M_3(k_7^+) - M_3(k_7^-)\}\frac{\partial k_7^+}{\partial t}$$
$$+ f_\theta^{(5)} f_t^{(5)} \{M_3(k_8^+) - M_3(k_8^-)\frac{\partial k_8^+}{\partial t}$$
$$+ f_\theta^{(6)} f_t^{(6)} \{M_3(k_9^+) - M_3(k_9^-)\}\frac{\partial k_9^+}{\partial t}\Bigg] \quad \ldots(3.6)$$

$$k_5^{\pm} = \pm\sqrt{\frac{t^2}{r^2} - V_9^{-2}}\sin\theta + i\frac{t}{r}\cos\theta,$$

$$k_6^{\pm} = k_8^{\pm} = i\left[-\sqrt{V_3^{-2} - \frac{t^2}{r^2}}\sin\theta + \frac{t}{r}\cos\theta\right] + \delta_1^1$$

$$k_7^{\pm} \pm k_9^{\pm} = i\left[\sqrt{V_3^{-2} - \frac{t^2}{r^2}}\sin\theta + \frac{t}{r}\cos\theta\right] + \delta_2^1, \qquad t_3 = V_3^{-1} r$$

$$M_3(k) = [k^2\sqrt{k^2 + V_3^{-2}}\,(k_1\sqrt{k^2 = V_2^{-2}} - k_2\sqrt{k^2 + V_1^{-2}})\,R_0^{-1}]$$

$$f_\theta^{(3)} = 1 \text{ in the range } 0 \leq \theta < \cos^{-1}\frac{V_3}{V_1}$$
$$= 0 \text{ elsewhere}$$
$$f_t^{(3)} = 1 \text{ for } t_2 \geq t \geq t_{31}$$
$$= a = \text{elsewhere}$$

$$f_\theta^{(4)} = 1 \text{ in the range } \pi - \cos^{-1}\frac{V_3}{V_1} < \theta \leq \pi$$
$$= 0 \text{ elsewhere}$$
$$f_\theta^{(4)} = 1 \text{ for } t_3 \geq t \geq t_{31}^1$$
$$= 0 \text{ elsewhere}$$

$$f_\theta^{(5)} = 1 \text{ in the range } 0 \leq \theta < \cos^{-1}\frac{V_3}{V_2}$$
$$= 0 \text{ elsewhere}$$
$$f_t^{(5)} = 1 \text{ for } t_3 \geq t \geq t_{32}$$
$$= 0 \text{ elsewhere}$$

$$f_\theta^{(6)} = 1 \text{ in the range } \pi - \cos^{-1}\frac{V_3}{V_2} > \theta > \pi$$
$$= 0 \text{ elsewhere}$$
$$f_t^{(6)} = 1 \text{ for } t_3 \geq t \geq t_{32}^1$$
$$= 0 \text{ elsewhere.}$$

Where

$$t_{31} = V_1^{-1}\, t \cos\theta + r\sqrt{V_3^{-2} - V_1^{-2}}\, \sin\theta$$
$$t_{31}^1 = V_1^{-1}\, r\,|\cos\theta| + r\sqrt{V_3^{-2} - V_1^{-2}}\, \sin\theta$$
$$t_{32} = V_2^{-1}\, r\cos\theta + r\sqrt{V_3^{-2} - V_2^{-2}}\, \sin\theta$$
$$t_{32}^1 = V_2^{-1}\, r\,|\cos\theta| + r\sqrt{V_3^{-2} - V_2^{-1}}\, \sin\theta$$

Hence combining (3.3), (3.4) and (3.5), we have

$$\sigma_{zz} = \frac{Q_0}{2\pi} H(t - t_1)\left[M_1(k_1^+)\frac{\partial k_1^+}{\partial t} - M_1(k_1^-)\frac{\partial k_1^-}{\partial t}\right]$$
$$- \frac{Q_0}{2\pi} H(t - t_2)\left[M_2(k_2^+)\frac{\partial k_2^+}{\partial t} - M_2(k_2^{-1})\frac{\partial k_2^-}{\partial t}\right]$$
$$+ \frac{2Q_0}{\pi} H(t - t_3)\left[M_3(k_5^+)\frac{\partial k_5^+}{\partial t} - M_3(k_5^-)\frac{\partial k_5^-}{\partial t}\right]$$

$$
\begin{aligned}
&-\frac{Q_0}{2\pi} f_\theta^{(1)}\; f_t^{(1)} \Big[M_2(k_3^+) - M_2(k_3^-) \Big] \frac{\partial k_3^+}{\partial t} \\
&-\frac{Q_0}{2\pi} f_\theta^{(2)}\; f_t^{(2)} [M_2(k_4^+) - M_2(k_4^-)] \frac{\partial k_4^+}{\partial t} \\
&+\frac{2Q_0}{\pi} f_\theta^{(3)}\; f_t^{(3)} [M_3(k_6^+) - M_3(k_6^-)] \frac{\partial k_6^+}{\partial t} \\
&+\frac{2Q_0}{\pi} f_\theta^{(4)}\; f_t^{(4)} [M_3(k_7^+) - M_3(k_7^-)] \frac{\partial k_7^+}{\partial t} \\
&+\frac{2Q_0}{\pi} f_\theta^{(5)}\; f_t^{(5)} [M_3(k_8^+) - M_3(k_8^{-1})] \frac{\partial k_8^+}{\partial t} \\
&+\frac{2Q_0}{\pi} f_\theta^{(6)}\; f_t^{(6)} [M_3(k_9^+) - M_3(k_9^-)] \frac{\partial k_9^+}{\partial t}
\end{aligned}
\qquad \ldots(3.7)
$$

Similarly the expressions for the other stresses can be found out. It may be seen that the expressions for s and σ_{zz} are not effected by the body shear wave.

4. Discussion and Numerical Results

We shall now analyse the stress-wave pattern governed by the equation (3.7). It is clear from equation (2.7) that at a time after the application of the force Q_0 at the origin, the first three terms are non-vanishing only when $r \leq V_1 t$, $r \leq V_2 t$ and $r \leq V_3 t$ respectively. These are effects of the two dilatational waves and one shear wave. The remaining six terms in the expressions for σ_{zz} are non-vanishing at a point which lies in the regions IV, V, VI, VII, VIII and IX in Fig. 2. These terms may be considered as the head wave contributions. The corresponding results in the Lamb's line load problem for the (non-porous) classical case is that there are two body waves and two head waves. So the effect of the fluid (in the non-dissipative case) may be considered so as to introduce an additional body wave and four additional head waves.

It should, however, be noted that the expression for σ_{zz} in equation (3.7) has been obtained on the assumption that $V_1 > V_2 > V_3$. The corresponding results for the case $V_1 > V_3 > V_2$ can similarly be written down. In fact the expression for σ_{zz} is obtained from (3.7) by interchanging V_2 by V_3 wherever they occur. As a check, if we make (i) $Q \to 0$, $R \to 0$, $\rho_{12} \to 0$ $\rho_{22} \to 0$ such that

$$
\frac{H}{\rho} \cdot \frac{\rho_{11}\rho_{22} - \rho_{12}^2}{\rho_{11}R + \rho_{22}P - 2\rho_{12}Q} \to 1
$$

which imply

$$
V_2 \to 0,\; V_1 \to \sqrt{\frac{D+2N}{\rho_{11}}} = V_1^*,\; V_3 \to \sqrt{\frac{N}{\rho_{11}}} = V_3^*
$$

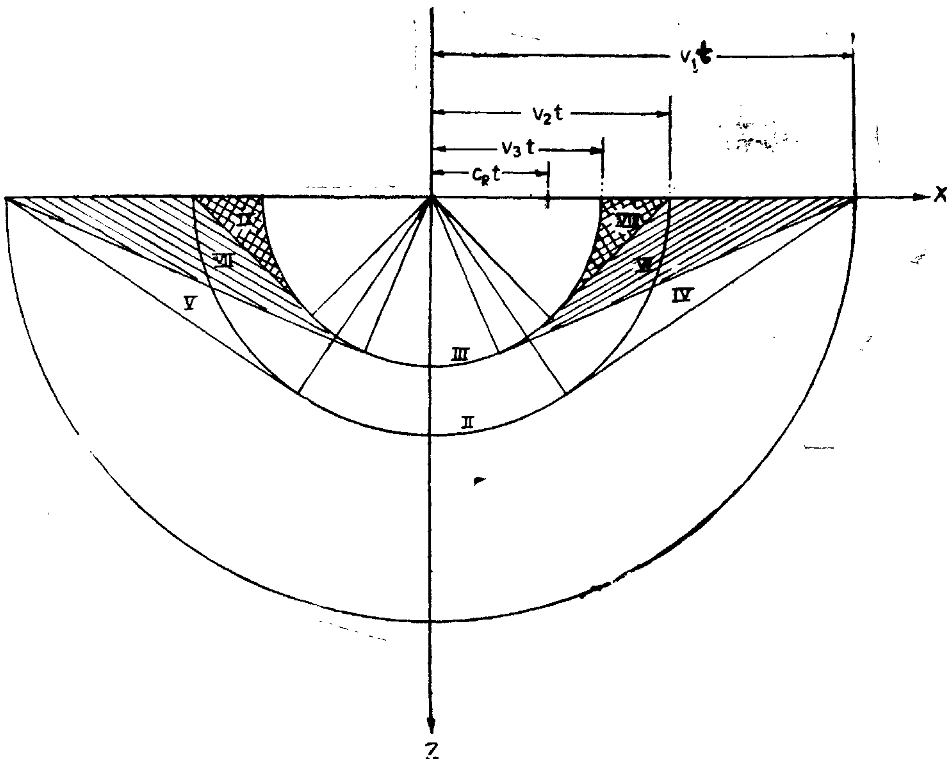

FIG. 2. Stress wave pattern.

(ii) $\sigma_{11} = \sigma_{22}$, $\gamma_{11} = \gamma_{22}$, $\gamma_{12} = 0$, $\sigma_{12} = 0$
which imply $V_1 = V_2$

may be shown to correspond with that for the classical case (Fung 1968). In particular, the expression for σ_{zz} on the z-axis ($\theta > \pi/2$) is given by

$$\sigma_{zz} = \frac{Q_0}{2\pi} H(t - t_1)\left[M_1(k_1^+)\frac{\partial k_1^+}{\partial t} - M_1(k_1^-)\frac{\partial k_1^-}{\partial t}\right]$$
$$+ \frac{2Q_0}{\pi} H(t - t_3)\left[M_3(k_5^+)\frac{\partial k_5^+}{\partial t} - M_3(k_5^-)\frac{\partial k_5^-}{\partial t}\right]$$
$$- \frac{Q_0}{2\pi} H(t - t_2)\left[M_2(k_2^+)\frac{\partial k_2^+}{\partial t} - M_2(k_2^-)\frac{\partial k_2^-}{\partial t}\right] \qquad \ldots(4.1)$$

where

$$k_1^{\pm} = \pm\sqrt{\frac{t^2}{r^2} - V_1^{-2}}$$

$$k_5^{\pm} = \pm\sqrt{\frac{t^2}{r^2} - V_2^{-2}}$$

$$k_2^{\pm} = \pm\sqrt{\frac{t^2}{r^2} - V_3^{-2}}$$

This may be put in the non-dimensional form

$$\frac{\sigma_{zz}\,.\,\pi}{Q_0} = \tau_{zz} = H(\tau - 1)\,M_1\,(k_1^+)\,\frac{\tau}{\sqrt{\tau^2 - 1}}$$
$$- 4H\,(\tau - l_2)\,M_3\,(k_5^+)\,\frac{\tau}{\sqrt{\tau^2 - l_2^2}}$$
$$- H\,(\tau - l_1)\,M_2\,(k_2^+)\,\frac{\tau}{\sqrt{\tau^2 - l_1^2}} \qquad \ldots(4.2)$$

$$\tau = \frac{V_1\,t}{r}, \qquad l_1 = \frac{V_1}{V_3}, \qquad l_2 = \frac{V_1}{V_2}.$$

For the numerical evaluation of τ_{zz} as a function of τ, we use the values of the parameters from Fatt's experimental result we get $l_2 = 1\cdot9208$, $l_1 = 2\cdot1552$

$$\frac{V_1^*}{V_3^*} = l_2^* = 1\cdot9012.$$

The values of τ_{zz} in porous case (case 1) and the non-porous classical case (case 2) for different τ are given in Table I which is represented graphically in Fig. 3.

TABLE I

	1·1	1·2	1·3	1·4	1·5	1·6
(Case 1)	−1·5352	−1·3570	−1·3560	−1·4574	−1·5849	−1·7535
(Case 2)	−2·6996	−2·2914	−2·2304	−2·2872	−2·4062	−2·5657

	1·7	1·8	1·9	2·0	2·2	2·4
(Case 1)	−1·9290	−2·2190	−2·3729	−1·9677	−3·8462	−2·6273
(Case 2)	−2·7562	−2·9733	−3·2144	−2·7340	−2·4935	−2·4067

	2·6	2·8	3·0	3·2	3·4
(Case 1)	−2·4929	−2·4024	−2·3419	−2·3081	−2·2465
(Case 2)	−2 3395	−2·2867	−2·2523	−2·2161	−2·1954

	3·6	3·8	4·0
(Case 1)	−2·2252	−2·1889	−2·1587
(Case 2)	−2·1753	−2·1652	−2·1494

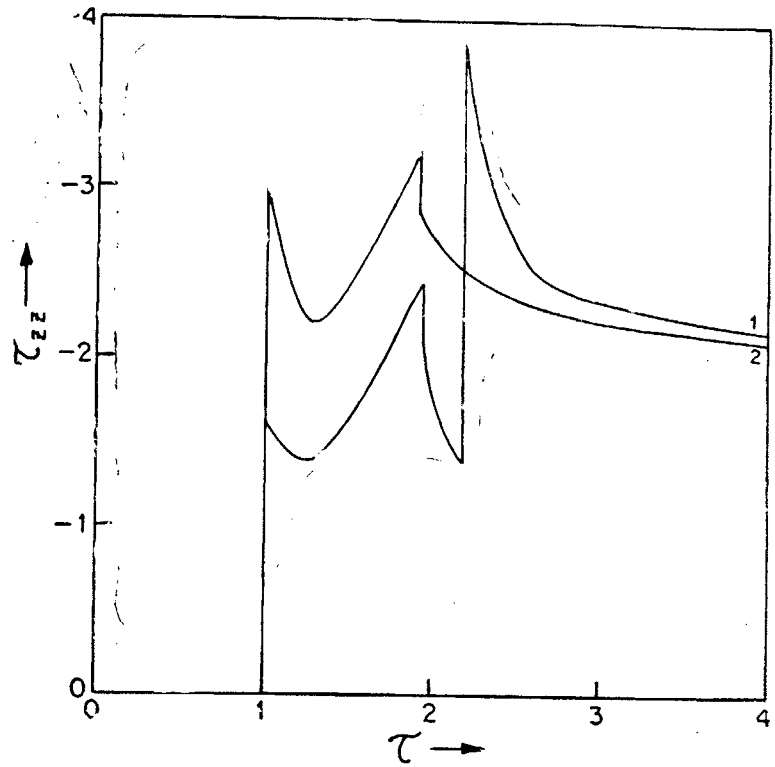

FIG. 3

It is seen that there are three stress wavefronts at $\tau = 1$, $\tau = l_2$, $\tau = l_1$ in the porous case and two wavefronts $\tau = 1$, $\tau = l_2^*$ in the non-porous classical case.

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REFERENCES

Biot, M. A. (1956). Theory of propagation of elastic waves in a fluid-saturated porous solid. *J. acoust. Soc. Am.*, **28**, 168.

Deresiewicz, H. (1960). The effect of boundaries on wave propagation in a liquid-filled porous solid—I: Reflection of plane waves at a free plane boundary (non-dissipative case). *Bull. Seism. Soc. Am.*, **50**, 599.

Deresiewicz, H., and Rice, J. T. (1962). The effect of boundaries on wave propagation in a liquid-filled porous solid—III: Reflection of plane waves at a free plan boundary (general case). *Bull. Seism. Soc. Am.*, **52**, 595.

Fung, Y. C. (1968). Foundations of Solid Mechanics. Prentice Hall of India.

Paria, G. (1966). Flow of Fluids through Porous Deformable Solids. *Applied Mechanic Surveys* (edited by H. N. Abramson and others). Sparton Books, Washington.