CERTAIN SPECIFIED STRUCTURES IN A DIFFERENTIABLE MANIFOLD*

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In this paper the author has introduced certain structures in a differentiable manifold V_n and has obtained their properties.

1. INTRODUCTION

Legrand (1958, 1959) studied a generalization of an almost complex structure. We have studied a generalization of an almost contact structure.

Definition 1.1—Let there be defined in a differentiable manifold V_n a linear operator F, a vector field T and a 1-form A satisfying

$$\bar{\bar{X}} = \lambda^2 X - \lambda^2 A(X) T, \quad ...(1.1a)$$

$$\bar{X} \overset{\text{def}}{=} F X, \quad ...(1.1b)$$

$$\bar{T} = 0 \quad ...(1.1c)$$

and

$$\text{rank}(F) = n - 1 = \text{constant everywhere}, \quad ...(1.2)$$

where λ is a constant. Then $\{F, T, A\}$ will be known as an almost contact F-structure, briefly ACF structure and V_n will be called ACF manifold.

Agreement 1.1—In this and in what follows, the equations containing $X, Y, Z, U, \ldots$ hold for arbitrary vector fields $X, Y, Z, U, \ldots$

Let α be an eigenvalue of F, the corresponding eigenvector being P. Then from (1.1*a*)

$$(\lambda^2 - \alpha^2) P = \lambda^2 A(P) T.$$

Two cases arise:

Case 1—$\{P, T\}$ is a linearly independent set. In this case

$$\alpha = \pm \lambda, \quad ...(1.3a)$$

$$A(P) = 0. \quad ...(1.3b)$$

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Case 2—P is linearly dependent on T. Then

$$\alpha = 0 \qquad \ldots(1.4)$$

Consequently, there are p eigenvalues λ, q eigenvalues $-\lambda$ and one eigenvalue 0, such that

$$p + q + 1 = n.$$

If P is an eignevector corresponding to the eigenvalue λ or $-\lambda$, then

$$A(P) = 0, \qquad \ldots(1.5)$$

Theorem 1.1—We have

$$A(T) = 1 \qquad \ldots(1.6a)$$

$$A(\bar{X}) = 0. \qquad \ldots(1.6b)$$

PROOF :

Barring (1.1c) and using (1.1a), we get (1.6a).

Barring (1.1a) and using (1.1c), we get

$$\bar{\bar{\bar{X}}} = \lambda^2 \bar{X}.$$

Barring X in (1.1a), we get

$$\bar{\bar{\bar{X}}} = \lambda^2 \bar{X} - \lambda^2 A(\bar{X})\, T.$$

Comparing the last two equations, we get (1.6b).

We will now show that ACF structure $\{F, T, A, \lambda\}$ is not unique.

Theorem 1.2—Let μ be a non-singular tensor of the type (1, 1). Let

$$\mu F' X \overset{\text{def}}{=} \overline{\mu X}, \qquad \ldots(1.7a)$$

$$A'(X) \overset{\text{def}}{=} A(\mu X), \qquad \ldots(1.7b)$$

$$T' \overset{\text{def}}{=} \mu^{-1} T. \qquad \ldots(1.7c)$$

Then $\{F', T', A'\ \lambda\}$ is also ACF structure.

PROOF : We have from (1.7) and (1.1*a*),

$$\mu F'^2 X = \overline{\mu F' X} = \overline{\mu (X)} = \lambda^2 \mu X - \lambda^2 A (\mu X) T$$
$$= \lambda^2 \mu X - \lambda^2 A' (X) \mu T'.$$

Since μ is non-singular, this equation yields

$$F'^2 X = \lambda^2 X - \lambda^2 A' (X) T'.$$

Hence, we have the statement.

We can always introduce a non-singular symmetric metric tensor G in V_n. Let G satisfy

$$G (\bar{X}, \bar{X}) = \lambda^2 G (X, Y) - \lambda^2 A (X) A (Y). \quad ...(1.8)$$

Then we will call V_n a *ACF* metric manifold and the structure $\{F, T, A, \lambda, G\}$ a *ACF* metric structure.

We are justified in assuming (1.8), because barring X and Y in (1.8), we again get the same equation.

We will now show that the structure $\{F, T, A, \lambda, G\}$ is not unique.

Theorem 1.3—Let us assume (1.7) and

$$G' (X, Y) \overset{\text{def}}{=} G (\mu X, \mu Y). \quad ...(1.9)$$

Then $\{F', T', G', A', \lambda\}$ is also *ACF* metric structure.

PROOF : We have from (1.9), (1.7*a*, *b*) and (1.8)

$$G' (F' X, F' Y) = G (\mu F' X), \mu (F' Y) = G (\overline{\mu X}, \overline{\mu Y})$$
$$= \lambda^2 G (\mu X, \mu Y) - \lambda^2 A (\mu X) A (\mu Y)$$
$$= \lambda^2 G' (X, Y) - \lambda^2 A' (X) A' (Y).$$

Remaining part of the proof is obvious.

Theorem 1.4—Let $\{F, T, A, \lambda\}$ be an *ACF* structure in an *ACF* manifold V_n. Let H be a symmetric metric in V_n, such that

$$H (X, T) = A (X). \quad ...(1.10)$$

Let us put

$$\lambda^2 G (X, Y) \overset{\text{def}}{=} \lambda^2 H (X, Y) + H (\bar{X}, \bar{Y}). \quad ...(1.11)$$

Then $\{F, T, A, \lambda, G\}$ is the *ACF* metric structure in *ACF* metric manifold V_n.

PROOF : From (1.11), (1.10) and (1.1) we have

$$\lambda^2 G(\bar{X}, \bar{Y}) = \lambda^2 H(\bar{X}, \bar{Y}) + H(\bar{\bar{X}}, \bar{\bar{Y}})$$

$$= \lambda^2 H(\bar{X}, \bar{Y}) + \lambda^4 H(X, Y) - \lambda^4 A(X) A(Y)$$

$$= \lambda^4 G(X, Y) - \lambda^4 A(X) A(Y).$$

This equation proves the statement.

2. SINGULAR METRIC

Theorem 2.1—We can always introduce a singular metric g in V_n, compatible with ACF structure.

PROOF : Let the metric g be such that

$$g(X, \bar{Y}) = g(\bar{X}, Y) = \lambda g(X, Y) - \lambda A(X) A(Y). \quad \ldots(2.1)$$

Then

$$g(X, T) = A(X). \quad \ldots(2.2)$$

Barring Y and using (1.1a) and (2.2), we again get (2.1). Thus we are justified in assuming (2.1). From (2.1) we again get

$$g(X, \bar{\bar{Y}}) = \lambda g(X, \bar{Y}).$$

This equation holds for arbitrary X and Y. Since $\bar{\bar{Y}} \neq \lambda \bar{Y}$. g is singular.

Let g be of rank $m < n$.

Definition 2.1—The structure $\{F, T, A, \lambda, g\}$ will be known as Riemannian almost contact F-structure ($RACF$ structure).

Theorem 2.2—Given an arbitrary symmetric quadratic form K such that

$$K(X, \bar{Y}) = K(\bar{X}, Y), \quad \ldots(2.3)$$

$$\lambda K(X, T) = A(X), \quad \ldots(2.4)$$

one is able to obtain $RACF$ structure.

PROOF : Let us put

$$g(X, Y) \overset{\text{def}}{=} K(X, \bar{Y}) + \lambda K(X, Y).$$

Then using (1.1) and (2.3) we get

$$g(X, \bar{Y}) = K(X, \bar{\bar{Y}}) + \lambda K(X, \bar{Y})$$

$$= \lambda^2 K(X, Y) - \lambda^2 A(Y) K(X, T) + \lambda K(X, \bar{Y})$$
$$= \lambda g(X, Y) - \lambda A(X) A(Y).$$

This equation proves the statement.

3. Distributions in V_n

Let V_n be an ACF manifold. Let us put

$$2\lambda^2 L X = \bar{\bar{X}} + \lambda \bar{X}, \quad ...(3.1a)$$

$$2\lambda^2 M X = \bar{\bar{X}} - \lambda \bar{X}, \quad ...(3.1b)$$

$$\lambda^2 G X = \lambda^2 X - \bar{\bar{X}}. \quad ...(3.1c)$$

Then it can be easily verified that

$$X = L X + M X + G X \quad ...(3.2)$$

$$(a)\ L^2 X = L X, \qquad (b)\ M^2 X = M X, \qquad (c)\ G^2 X = G X, \quad ...(3.3)$$

$$L M X = L G X = M L X = M G X = G L X$$
$$= G M X = 0. \quad ...(3.4)$$

Let $\{\underset{x}{P}, x = 1, \ldots, p\}$ be a set of L.I. eigenvectors corresponding to the eigenvalue λ of F and $\{\underset{y}{Q}, y = 1, \ldots, q\}$ be a set of L.I. eigenvectors corresponding to the eigenvalue $-\lambda$ of F. Then we see that

$$(a)\ L\underset{x}{P} = \underset{x}{P}, \qquad (b)\ L\underset{y}{Q} = 0, \qquad (c)\ L\,T = 0, \quad ...(3.5)$$

$$(a)\ M\underset{x}{P} = 0, \qquad (b)\ M\underset{y}{Q} = \underset{y}{Q}, \qquad (c)\ M\,T = 0, \quad ...(3.6)$$

$$(a)\ G\underset{x}{P} = 0, \qquad (b)\ G\underset{y}{Q} = 0, \qquad (c)\ G\,T = 1. \quad ...(3.7)$$

Theorem 3.1—The necessary and sufficient condition that V_n be an ACF manifold is that there exists a distribution π_p of dimension p, a distribution π_q of dimension $q, p + q + 1 = n$, and a distribution π_1 of real dimension 1 such that π_p, π_q and π_1 have no direction in common and span together a manifold of dimension n.

Proof : Since $\{\underset{x}{P}\}$ and $\{\underset{y}{Q}\}$ are L. I.,

$$\overset{x}{a}\underset{x}{P} = 0 \Rightarrow \overset{x}{a} = 0 \;\forall\; x,$$

$$\overset{y}{b}\underset{y}{Q} = 0 \Rightarrow \overset{y}{b} = 0 \;\forall\; y.$$

Now

$$\overset{x}{c}\,\underset{x}{P} + \overset{y}{d}\,\underset{y}{Q} + eT = 0 \Rightarrow \overset{x}{c}\underset{x}{\bar{P}} + \overset{y}{d}\underset{y}{\bar{Q}} = 0$$

$$\overset{x}{c}\,\underset{x}{p} - \overset{z}{d}\,\underset{y}{Q} = 0 \Rightarrow \overset{x}{c}\,\underset{x}{\bar{P}} - \overset{y}{d}\,\underset{y}{\bar{Q}} = 0$$

$$\overset{x}{c}\,\underset{x}{P} + \overset{y}{d}\,\underset{y}{Q} = 0.$$

All these equations imply

$$\overset{x}{c} = \overset{y}{d} = e = 0.$$

Thus $\{\underset{x}{P}, \underset{y}{Q}, T\}$ is a L.I. set. Consequently, we can define the inverse set $\{\overset{x}{p}, \overset{y}{q}, A\}$ such that

$$(a)\ \overset{x_1}{p}\,(\underset{x_2}{P}) = \overset{x_1}{\underset{x_2}{\delta}}, \qquad (b)\ \overset{x}{p}\,(\underset{y}{Q}) = 0, \qquad (c)\ \overset{x}{p}\,(T) = 0 \qquad \ldots(3.8)$$

$$(a)\ \overset{y}{q}\,(\underset{x}{P}) = 0, \qquad (b)\ \overset{y_1}{q}\,(\underset{y_2}{Q}) = \overset{y_1}{\underset{y_2}{\delta}}, \qquad (c)\ \overset{y}{q}\,(T) = 0 \qquad \ldots(3.9)$$

$$(a)\ A\,(\underset{x}{P}) = 0, \qquad (b)\ A\,(\underset{y}{Q}) = 0, \qquad (c)\ A\,(T) = 1. \qquad \ldots(3.10)$$

$$\overset{x}{p}\,(X)\underset{x}{P} + \overset{y}{q}\,(X)\,\underset{y}{Q} + A\,(X)\,T = X. \qquad \ldots(3.11)$$

Now, from (3.1) and (1.1)

$$L\,\bar{X} = \overline{L\,X} = \lambda L\,X, \qquad \ldots(3.12a)$$

$$M\,\bar{X} = \overline{M\,X} = -\,\lambda M\,X, \qquad \ldots(3.12b)$$

$$G\,\bar{X} = \overline{G\,X} = 0. \qquad \ldots(3.12c)$$

Consequently, from (3.2) and (3.11)

$$X = \overset{x}{p}\,(X)\underset{x}{P} + \overset{y}{q}\,(X)\,\underset{y}{Q} + A\,(X)\,T = L\,X + m\,X + G\,X, \qquad \ldots(3.13a)$$

$$\bar{X} = \lambda\overset{x}{p}\,(X)\underset{x}{P} - \lambda\overset{y}{q}\,(X)\,\underset{y}{Q} = \lambda\,L\,X - \lambda\,M\,X, \qquad \ldots(3.13b)$$

$$\bar{\bar{X}} = \lambda^2\overset{x}{p}\,(X)\underset{x}{P} + \lambda^2\overset{y}{q}\,(X)\,\underset{y}{Q} = \lambda L\,X + \lambda M\,X. \qquad \ldots(3.13c)$$

From these equations, we get

$$(a)\ L X = \overset{x}{p}(X)\underset{x}{P}, \qquad (b)\ M X = \overset{y}{q}(X)\underset{y}{Q}, \qquad (c)\ G X = A(X)T. \qquad \ldots(3.14)$$

Thus we see that in V_n, there exists a distribution π_p of dimension p, a distribution π_q of dimension q and a distribution π_1 of real dimension 1 such that π_p, π_q, π_1 have no direction in common and span together a manifold of dimension n, projections on π_p, π_q and π_1 being L, M, G respectively.

Conversely, suppose that in V_n, there exists a distribution π_p of dimension p, a distribution π_q of dimension q and a distribution π_1 of real dimension 1 such that π_p, π_q and π_1 have no direction in common and span together a manifold of dimension $p + q + 1 = n$.

We take p L.I. vectors $\underset{x}{P}$ in π_p, q L.I. vectors $\underset{y}{Q}$ in π_q and T a vector in π_1. Since π_p, π_q, π_1 have no direction in common and span a manifold of dimension n, $\{\underset{x}{P}, \underset{y}{Q}, T\}$ are L.I. Let $\{\overset{x}{p}, \overset{y}{q}, A\}$ be the set inverse to $\{\underset{x}{P}, \underset{y}{Q}, T\}$. Let us put

$$\bar{X} \overset{\text{def}}{=} \lambda\overset{x}{p}(X)\underset{x}{P} - \lambda\overset{y}{q}(X)\underset{y}{Q}. \qquad \ldots(3.15)$$

Then

$$\bar{\bar{X}} = \lambda\overset{x}{p}(\bar{X})\underset{x}{P} - \lambda\overset{y}{q}(\bar{X})\underset{y}{Q}$$

$$= \lambda\overset{x}{p}(X)\{\lambda\overset{x_1}{p}(\underset{x}{P})\underset{x_1}{P} - \lambda\overset{y}{q}(\underset{x}{P})\underset{y}{Q}\}$$

$$- \lambda\overset{y_1}{q}(X)\{\lambda\overset{x}{p}(\underset{y_1}{Q})\underset{x}{P} - \lambda\overset{y}{q}(\underset{y_1}{Q})\underset{y}{Q}\}$$

$$= \lambda\overset{x}{p}(X)\underset{x}{P} + \lambda^2\overset{y}{q}(X)\underset{y}{Q}$$

$$= \lambda^2 X - \lambda^2 A(X)T.$$

Thus the condition is sufficient also.

4. Integrability Conditions of V_n

Theorem 4.1—The necessary and sufficient conditions that π_p be integrable are

$$\overline{\overline{N(\bar{X},\bar{Y})}} - \lambda\,\overline{N(\bar{X},\bar{Y})} = 0 \qquad \ldots(4.1a)$$

$$A(N(\bar{X},\bar{Y})) + \lambda\{A(N(\bar{X}, Y)) + A(N(X,\bar{Y}))\} + \lambda^2 A(N(X, Y)) = 0. \qquad \ldots(4.1b)$$

PROOF : From (3.2) we see that π_p is given by

$$X = L\,X, \qquad M\,X = 0, \qquad G\,X = 0.$$

In order that $M = 0$ is integrable it is necessary and sufficient that

$$(dM)(X, Y) = 0,$$

holds for $X = L(X)$. Hence, we have

$$(dM)\big(L\,X, L\,Y\big) = 0,$$

which in consequence of (3.4) assumes the form

$$M\,[L\,X, L\,Y] = 0.$$

Substituting from (3.1) in this equation, we get

$$\overline{[L\,X, L\,Y]} = \lambda\,\overline{[L\,X, L\,Y]}.$$

Use of (3.1) in this equation yields.

$$\overline{\overline{[\bar{\bar{X}}, \bar{\bar{Y}}]}} + \lambda\,\overline{\overline{[\bar{\bar{X}}, \bar{Y}]}} + \lambda\,\overline{\overline{[\bar{X}, \bar{\bar{Y}}]}} + \lambda^2\,\overline{\overline{[\bar{X}, \bar{Y}]}}$$

$$= \lambda\,\overline{[\bar{\bar{X}}, \bar{\bar{Y}}]} + \lambda^2\,\overline{[\bar{\bar{X}}, \bar{Y}]} + \lambda^2\,\overline{[\bar{X}, \bar{\bar{Y}}]} + \lambda^3\,\overline{[\bar{X}, \bar{Y}]}.$$

Since Nijenhuis tensor N is given by

$$N(X, Y) = [\bar{X}, \bar{Y}] + \overline{\overline{[X, Y]}} - \overline{[\bar{X}, Y]} - \overline{[X, \bar{Y}]}, \qquad \ldots(4.2)$$

the above equation yields, in consequence of (1.1*a*)

$$\overline{\overline{N(\bar{X}, \bar{Y})}} = \lambda\,\overline{N(\bar{X}, \bar{Y})}$$

In order that $G = 0$ is integrable, it is necessary and sufficient that

$$(dG)(X, Y) = 0,$$

holds for $X = L\,X$. Hence, we have

$$(dG)(L\,X, L\,Y) = 0,$$

which, in consequence of (3.4), assumes the form

$$G\,[L\,X, L\,Y] = 0.$$

In consequence of (3.1*c*) and (1.1*a*), this equation takes the form

$$A\,(L\,X, L\,Y) = 0.$$

Substituting from (3.1) in this equation, we get

$$A\,([\bar{\bar{X}}, \bar{\bar{Y}}]) + \lambda A\,([\bar{\bar{X}}, \bar{Y}]) + \lambda A\,([\bar{X}, \bar{\bar{Y}}]) + \lambda^2 A\,([\bar{X}, \bar{Y}]) = 0.$$

Use of (4.2) in the last equation yields (4.1*b*).

Theorem 4.2—The necessary and sufficient conditions that π_q be integrable are

$$\overline{\overline{N(\bar{X},\bar{Y})}} + \lambda \overline{N(\bar{X},\bar{Y})} = 0, \qquad \ldots(4.3a)$$

$$A\,(N(\bar{X},\bar{Y})) - \lambda\,\{A\,(N(\bar{X},Y)) + A\,(N(X,\bar{Y}))\} + \lambda^2 A\,(N(X,Y)) = 0. \qquad \ldots(4.3b)$$

PROOF : The proof follows the pattern of the proof of Theorem 4.1.

Theorem 4.3—π_1 is completely integrable.

PROOF : From (3.2), π_1 is given by

$$X = G\,X, \quad L\,X = 0, M\,X = 0.$$

In order that $L = 0$ is integrable, it is necessary and sufficient that

$$(dL)\,(G\,X, \quad G\,Y) = 0,$$

which, in consequence of (3.4), becomes

$$L\,[G\,X, G\,Y] = 0.$$

This equation is automatically satisfied by virtue of (3.1*a*, *c*) and (1.1*c*).

Theorem 4.4—The necessary and sufficient conditions that V_n be completely integrable are

$$\overline{N(\bar{X},\bar{Y})} = 0, \qquad \ldots(4.4a)$$

$$A\,(N\,(X,\bar{Y})) + A\,(N\,(\bar{X},Y)) = 0. \qquad \ldots(4.4b)$$

PROOF : From (4.1) and (4.3) we have

$$(a)\ \overline{N(\bar{X},\bar{Y})} = 0, \qquad (b)\ \overline{\overline{N(\bar{X},\bar{Y})}} = 0 \qquad \ldots(4.5)$$

$$(a)\ A\,(N(X,\bar{Y})) + A\,(N(\bar{X},Y)) = 0, \quad (b)\ A\,(N(\bar{X},\bar{Y})) + \lambda^2 A\,(N\,(X,Y)) = 0. \qquad \ldots(4.6)$$

Each one of (4.5) implies the other. Similarly each one of (4.6) implies the other.

5. *ACF* 2-STRUCTURES

Definition 5.1—Let $\{F_1, T_1, A_1\}$ and $\{F_3, T_2, A_2\}$ be two *ACF* structures in V_n and satisfy

$$F_1\,F_2\,X + \lambda^2 A_2\,(X)\,T_1 = F_2\,F_1\,X + \lambda^2 A_1\,(X)\,T_2. \qquad \ldots(5.1)$$

Then $\{F_1, T_1, A_1\}$ and $\{F_2, A_2, T_2\}$ are said to define *ACF* 2-structures in V_n.

Theorem 5.1—Equation (5.1) implies

$$F_1 T_2 - F_2 T_1 = 0, \qquad \ldots(5.2a)$$

$$A_1 (F_2 X) - A_2 (F_1 X) = 0. \qquad \ldots(5.2b)$$

PROOF: Putting $F_2 X$ for X in (5.1), we get

$$F_1 F_2^2 X = F_2 F_1 F_2 X + \lambda^2 A_1 (F_2 X) T_2.$$

Using (5.1) in this equation, we get

$$F_1 F_2^2 X = F_2^2 F_1 X - \lambda^2 A_2 (X) F_2 T_1 + \lambda^2 A_1 (F_2 X) T_2.$$

In consequence of (1.1*a*), this equation takes the form

$$A_2 (X) \{F_2 T_1 - F_1 T_2\} = \{A_1 (F_2 X) - A_2 (F_1 X)\} T_2.$$

Since A_2 and T_2 do not vanish, we have (5.2).

Theorem 5.2—Equations (1.1) and (5.1) imply

$$A_1 (T_2) - A_2 (T_1) = 0. \qquad \ldots(5.3)$$

PROOF: Putting T_1 for X in (5.1), we get

$$F_1 (F_2 T_1) + \lambda^2 A_2 (T_1) T_1 = \lambda^2 T_2.$$

From this equation, we at once get (5.3).

Theorem 5.3—Let $\{F_1, T_1, A_1, \lambda\}$ and $\{F_2, T_2, A_2, \lambda\}$ define ACF 2-structures. Let us put

$$\lambda F_3 \; X \overset{\text{def}}{} F_1 F_2 X + \lambda^2 A_2 (X) T_1 = F_2 F_1 X + \lambda^2 A_1 (X) T_2, \qquad \ldots(5.4a)$$

$$\lambda T_3 \overset{\text{def}}{=} F_1 T_2 = F_2 T_1, \qquad \ldots(5.4b)$$

$$\lambda A_3 (X) \overset{\text{def}}{=} A_2 (F_1 X) = A_1 (F_2 X). \qquad \ldots(5.4c)$$

Then any two of the three structure $\{F_x, T_x, A_x\}$, $x = 1, 2, 3$, define ACF 2-structure.

PROOF: From (5.4*a*) and (1.1*a*),

$$\lambda^2 F_3^2 X = F_1 F_2^2 F_1 X + \lambda^4 A_1 (X) T_1$$

$$= \lambda^2 F_1^2 X - \lambda^2 A_2 (F_1 X) F_1 T_2 + \lambda^4 A_1 (X) T_1$$

$$= \lambda^4 X - \lambda^4 A_3 (X) T_3.$$

Hence

$$F_3^2 X = \lambda^2 X - \lambda^2 A_3 (X) T_3. \qquad \ldots(5.5)$$

Thus $\{F_3, A_3, T_3, \lambda\}$ is an ACF structure.

Now from (5.4) and (1.1a)

$$\lambda F_1 F_2 X + \lambda^3 A_3(X) T_1$$
$$= F_1^2 F_2 X + \lambda^3 A_3(X) T_1$$
$$= \lambda^2 F_2 X - \lambda^2 A_1 (F_2 X) T_1 + \lambda^3 A_3(X) T_1$$
$$= \lambda^2 F_2(X).$$

Hence

$$\lambda F_2(X) = F_1 F_3 X + \lambda^2 A_3(X) T_1. \quad \ldots(5.6a)$$

Also, in consequence of (5.4) and (1.1)

$$\lambda F_3 F_1 X + \lambda^3 A_1(X) T_3$$
$$= F_2 F_1^2 X + \lambda^3 A_1(X) T_3$$
$$= \lambda^2 F_2 X - \lambda^2 A_1(X) F_2 T_1 + \lambda^3 A_1(X) T_3$$
$$= \lambda^2 F_2(X).$$

Therefore

$$\lambda F_2(X) = F_3 F_1 X + \lambda^2 A_1(X) T_3. \quad \ldots(5.6b)$$

We can similarly prove that

$$\lambda F_1 X = F_2 F_3 X + \lambda^2 A_3(X) T_2 = F_3 F_2 X + \lambda^2 A_2(X) T_3. \quad \ldots(5.7)$$

Equations (5.5), (5·6) and (5.7) prove the statement.

Corollary 5.1—We have

$$A_1(T_3) = A_2(T_3) = A_3(T_1) = A_3(T_2) = A_1(T_2) = A_2(T_1) = 0, \quad \ldots(5.8)$$

$$\lambda T_2 = F_3 T_1 = F_1 T_3, \quad \ldots(5.9)$$

$$\lambda T_1 = F_2 T_3 = F_3 T_2, \quad \ldots(5.10)$$

$$\lambda A_2(X) = A_3(F_1 X) = A_1(F_3 X), \quad \ldots(5.11)$$

$$\lambda A_1(X) = A_3(F_2 X) = + A_2(F_3 X). \quad \ldots(5.12)$$

PROOF : Putting T_1 for X in (5.4 a), and using (5.4b), we get

$$F_1 T_3 + \lambda A_2(T_1) T_1 = \lambda T_2 = + F_3 T_1 \quad \ldots(5.13)$$

or

$$A_2(F_1 T_3) + \lambda A_2(T_1) A_2(T_1) = \lambda.$$

Using (5.4*c*) and (1.6*a*) in this equation we get $A_2(T_1) = 0$. We similarly get $A_1(T_2) = 0$. Remaining equations (5.8) are obtained from (5.4*b*, *c*). Putting $A_2(T_1) = 0$ in (5.13) we get (5.9). We similarly ootain (5.10).

From (5.4*a*), we have

$$\lambda A_2(X) = A_1(F_3 X),$$

and from (5.4*c*) we have

$$\lambda A_2(X) = A_3(F_1 X).$$

Thus we have (5.11). We can similarly prove (5.12).

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