SUMMABILITY MATRICES OVER NON-ARCHIMEDIAN FIELDS

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This paper studies (1) sequence to sequence, (2) series to sequence, (3) series to series and (4) sequence to series matrix transformations defined over a field K provided with non-trivial non-archimedian valuation. The results of Vermes (1946) and Ramanujan (1956) describing the algebraic properties of these matrices in the classical case are extended for the corresponding matrix transformations over a non-archimedian field K which is complete under the metric of valuation.

§1. Somasundram (1974) has studied recently some properties of limit preserving sequence to sequence matrix transformations known as T-matrices (Cooke 1955) defined over non-archimedian fields. The object of the present paper is to study the other summability matrices β, γ and δ (Cooke 1955) over non-archimedian fields and derive algebraic properties of these matrices which generalise theorems of Vermes (1946) and Ramanujan (1956) in the classical case.

§2. Let $A = (a_{np})$ $n, p = 1, 2, 3, \ldots$ be a matrix defined over a field K provided with non-trivial non-archimedian valuation. The field K is supposed to be complete under the metric of valuation. From a theorem of Monna (1963), we deduce as in the classical case the following theorem.

Theorem 1—A matrix $A = (a_{np})$ is a T-matrix over K called a $T(K)$ matrix if and only if

$$\sup_{n,p} |a_{np}| \leqslant M \text{ where } M \text{ is a constant.} \quad \ldots(2.1)$$

$$\lim_{n \to \infty} a_{np} = 0 \text{ for each fixed } p \quad \ldots(2.2)$$

$$\sum_{p=1}^{\infty} a_{np} = A_n \to 1 \text{ as } n \to \infty. \quad \ldots(2.3)$$

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§3. *Series to sequence transformations over K*—We shall consider the following transformation.

$$t_n = \sum_{p=1}^{\infty} g_{np} C_p, \; n = 1, 2, 3, \ldots \text{ of the series } \sum_{p=1}^{\infty} C_p \qquad \ldots(3.1)$$

The matrix $G = (g_{np})$ defined above transforms the series $\sum_{p=1}^{\infty} C_p$ into the sequence (t_n). The analogues of β and γ matrices for a series to sequence transformations over K are called $\beta(K)$ and $\gamma(K)$ matrices.

Theorem 2—The necessary and sufficient conditions that (t_n) defined by (3.1) should tend to a finite limit as $n \to \infty$ whenever $\sum_{p=1}^{\infty} C_p$ converges to s in metric of valuation are that

(i) $\sup_{n, p} | g_{np} - g_{np+1} | \leqslant N$ where N is a constant.

(ii) $\lim_{n \to \infty} g_{np} = \beta_p$ for each fixed p.

Moreover

(iii) $\lim t_n = \beta_1 s + \sum_{p=1}^{\infty} (\beta_p - \beta_{p+1}) (s_p - s)$

where $s_p = \sum_{r=1}^{p} C_r$ and the existence of either side of (iii) implies that of the other.

PROOF : The proof of the theorem depends upon the following Lemma.

Lemma— $\sum_{p=1}^{\infty} g_{np} C_p$ converges whenever $\sum_{p=1}^{\infty} C_p$ is convergent if and only if (g_{np}) is bounded in the metric of valuation as $p \to \infty$ for each fixed n.

The sufficiency follows easily because '$a_n \to 0$' is the necessary and sufficient condition for the convergence of a series $\sum_{n=1}^{\infty} a_n$ in a field with non-trivial non-archimedian valuation.

To prove the necessity of the condition, let us suppose that (g_{np}) is not bounded. Then there exists a sequence (p_r) such that

$$| g_{np_r} | > \frac{r}{\lambda^r} \quad r = 1, 2, 3, \ldots$$

where λ corresponds to an element Z of the field K such that $| Z | = \lambda < 1$. Such an element in K exists because the valuation is non-trivial. Let us define

$$C_p = 0 \text{ if } p \neq p_r$$
$$= Z^r \text{ if } p = p_r, \; r = 1, 2, \ldots$$

$\sum_{p=1}^{\infty} C_p = \sum_{r=1}^{\infty} C_{p_r} = \sum_{r=1}^{\infty} Z^r$ is convergent, since $Z^r \to 0$ as $r \to \infty$. But

$$|g_{np_r}C_{p_r}| = |g_{np_r}|\ |C_{p_r}| > \frac{r}{\lambda^r}\lambda^2 = r.$$

Hence $|g_{np_r}C_{p_r}| > r$. Therefore $g_{np}C_p$ does not tend to zero as $p \to \infty$ for every fixed n which shows that $\sum_{p=1}^{\infty} a_{np}C_p$ is not convergent whenever $\sum_{p=1}^{\infty} C_p$ is convergent. This contradiction proves the necessity condition of the lemma.

Proof of the Theorem—We first establish that the conditions are sufficient. It follows from the lemmas, condition (i) of the theorem and from the fact that $s_p \to s$ as $p \to \infty$ that $(g_{np} - g_{np} + 1)(s_p - s) \to 0$ as $p \to \infty$ in the metric of valuation for each fixed n so that the right hand side of (3.1) namely $\sum_{p=1}^{\infty} g_{np}C_p = sg_{n1} + \sum_{p=1}^{\infty} (g_{np} - g_{np+1})(s_p - s)$ is well defined. So the condition (iii) is well-defined for every n.

For any $\varepsilon > 0$, we can choose a p_0 such that

$$|s_p - s| < \frac{\varepsilon}{N} \text{ for all } p \geqslant p_0. \qquad \ldots(3.2)$$

Rewriting (3.1), we have

$$\sum_{p=1}^{\infty} g_{np}C_p = sg_{n1} + \{\sum_{p=1}^{p_0} + \sum_{p=p_0+1}^{\infty}\}(g_{np} - g_{np+1})(s_p - s)$$

$$= sg_{n1} + \Sigma_1 + \Sigma_2 \text{ (say)}$$

By condition (ii) Σ_1 tends to

$$\sum_{p=1}^{p_0} (\beta_p - \beta_{p+1})(s_p - s) \text{ as } n \to \infty$$

$$|\Sigma_2| \leqslant \sup_{p_0+1 \leqslant p < \infty} \{|g_{np} - g_{np+1}|\ |s_p - s|\}.$$

By condition (i) and (3.2) we have

$$|\Sigma_2| \leqslant \operatorname{Sup}\left(N\frac{\varepsilon}{N}\right).$$

Therefore $|\Sigma_2| < \varepsilon$ for every fixed $p \geqslant p_0$. By condition (ii) $\lim g_{n1} = \beta_1$. Using this we get $\lim_{n\to\infty} \sum_{p=1}^{\infty} g_{np}C_p = \beta_1 s + \sum_{p=1}^{\infty} (\beta_p - \beta_{p+1})(s_p - s)$. Hence the conditions are sufficient.

To prove the conditions are necessary, choose $C_p = 0$ if $p \neq q$ and $C_q = 1$. Then $t_n = g_{nq}$. When $n \to \infty$, $g_{nq} \to \beta_q$ which proves the necessity of (ii). We have

$$\sum_{p=1}^{r} g_{np} C_p = \sum_{p=1}^{r} g_{np} (s_p - s_{p-1})$$

$$= \sum_{p=1}^{r} g_{np} \{(s_p - s) - (s_{p-1} - s)\}$$

$$= \sum_{p=1}^{r-1} (g_{np} - g_{np+1})(s_p - s) + s g_{n_1} + (s_r - s) g_{nr}.$$

By the Lemma (g_{nr}) is bounded as $r \to \infty$ for every fixed n. By hypothesis $g_{np} C_p$ tends to a limit t_n as $r \to \infty$ and so we have

$$\sum_{p=1}^{r-1} (g_{np} - g_{np+1})(s_p - s) + s g_{n_1} \qquad \ldots(3.3)$$

tends to a limit t_n as $r \to \infty$ for all (s_p) such that $s_p \to s$ for every fixed n.

$$\lim_{n\to\infty} t_n \text{ exists by hypothesis.} \qquad \ldots(3.4)$$

Also we have

$$s g_{n_1} \to s\beta, \text{ as } n \to \infty. \qquad \ldots(3.5)$$

By using (3.4) and (3.5) in (3.3), we can consider (3.3) as the transformation of the null sequence into (t_n). So as in the proof a theorem of Monna (1963) dealing with the conservative matrices, the condition (i) is necessary. This completes the proof of the theorem.

Theorem 3—The necessary and sufficient condition that $t_n = \sum_{n=1}^{\infty} g_{np} C_p$ should tend to a finite limit s as $n \to \infty$ whenever $\sum_{p=1}^{\infty} C_p$ converges to the sum s are,

(i) $\sup_{n,p} | g_{np} - g_{np+1} | \leqslant N$

(ii) $\lim g_{np} = 1$ for each fixed p where 1 is the identity of the field K.

PROOF : The conditions are sufficient as in Theorem 2. The conditions are necessary because if $\beta_p - \beta_{p+1} \neq 0$ for any particular p, let us define

$C_i = 0$ for $i < p$

$C_p = 1$, $C_{p+1} = -1$

$C_{p+1} = 0$ for $i > 1$,

Then

$$\lim_{n\to\infty} t_n = (\beta_p - \beta_{p+1}) \neq 0.$$

Hence it is necessary that $\beta_p - \beta_{p+1} = 0$ for every p. Therefore we have from (iii) of Theorem 2, $\lim_{n\to\infty} t_n = \beta_1 s$ so that $\beta_1 = 1$ is a necessary condition. Hence $\beta_p = 1$ for every p so that (ii) is necessary. The necessity of (i) can be proved exactly as in Theorem 2. This completes the proof of the theorem.

Examples of $\beta(K)$ and $\gamma(K)$ Matrices

(1) Consider the matrix $G = (g_{np})$ defined over the π-adic field for any prime π.

Let

$$g_{np} = \begin{cases} p\pi^n & \text{for} \quad n \geqslant p \\ 0 & \text{for} \quad n < p. \end{cases}$$

We shall show that G is a $\beta(K)$ matrix

$$\begin{aligned} g_{np} - g_{np+1} &= n\pi^n \quad \text{when} \quad n = p \\ &= \pi^n \quad \text{when} \quad n > p \\ &= 0 \quad \text{when} \quad n < p. \end{aligned}$$

Since $|n| < 1$ and $\pi^n \to 0$ as $n \to \infty$ in the metric of valuation, $\operatorname{Sup}_{n,p} |g_{np} - g_{np+1}| \leqslant 1$ which satisfies the condition (i) of Theorem 2 for a $\beta(K)$ matrix.

Now

$$\lim_{n\to\infty} g_{np} = \lim_{n\to\infty} n\pi^n = 0$$

This proves condition (ii) of Theorem 2.

We have $\beta_1 = \beta_2 = \ldots = \beta_K = \ldots = 0$ which shows that the right hand member of conditions (iii) of Theorem 2 is zero.

If t_n is the g-transform of the series $1 + \pi + \pi^2 + \ldots$, then we get

$$t_n = \pi^n (1 + 2\pi + 3\pi^2 + \ldots + n\pi^{n-1})$$

$$|t_n| \leqslant |\pi^n| \operatorname{Max}(|1|, |2||\pi|, \ldots |n||\pi|^{n-1}).$$

Therefore we have $|t_n| < |\pi|^n \to 0$ as $n \to \infty$. Hence condition (iii) of Theorem 2 is satisfied which shows that G is a β matrix defined over K.

(2) Consider the matrix $F = (f_{np})$ defined as follows

$$f_{np} = \begin{cases} 1 - p\pi^n & \text{for} \quad n \geqslant p \\ 0 & \text{for} \quad n < p. \end{cases}$$

As in the previous case, we can verify the condition (i) and (ii) of a $\gamma(K)$-matrix. The series $1 + \pi + \pi^2 + \ldots$ converges to $1/1 - \pi$ in the metric of valuation. We shall show that its transform by the above matrix converges to the same limit. Now we have

$$t_n = (1 + \pi + \pi^2 + \pi^3 + \ldots + \pi^{n-1}) - \pi^n (1 + 2\pi + 3\pi^2 + \ldots + n\pi^{n-1})$$

Since

$$\lim_{n\to\infty} \pi^n (1 + 2\pi + 3\pi^2 + \ldots + n\pi^{n-1}) = 0$$

and

$$\lim_{n\to\infty} 1 + \pi + \pi^2 + \ldots + \pi^{n-1} = \frac{1}{1-\pi}.$$

we have

$$\lim_{n\to\infty} t_n = \frac{1}{1-\pi}.$$

§4. *Series to Series matrix transformations over K*—We shall consider the matrix transformation

$$V_n = \sum_{p=1}^{\infty} h_{np} u_p, \quad h_{np} \in k \qquad \ldots(4.1)$$

of a convergent series $\sum_{p=1}^{\infty} u_p$ into a sequence V_n such that $\sum_{n=1}^{\infty} V_n$ is convergent. The matrix which preserves the convergence of a series to series transformation is called a $\delta(K)$ matrix where as the limit preserving series to series transformation is called a $\alpha(K)$ matrix. The following theorem dealing with series to series matrix transformation defined over K can be proved exactly as in the classical case by Vermes (1946).

Theorem 4—The necessary and sufficient condition that the matrix H defined by (4.1) is a $\delta(K)$ matrix or $\alpha(K)$ matrix is that the matrix $G = (g_{np})$ defined by

$$g_{np} = h_{1p} + h_{2p} + \ldots + h_{np}$$

or

$$h_{np} = g_{np} - g_{n-1p} \text{ is a } \beta(K) \text{ matrix or } \gamma(K) \text{ matrix.}$$

§5. *Sequence to series matrix transformation over K*—Consider the matrix transformation.

$$V_n = \sum_{p=1}^{\infty} f_{np} s_p, \quad f_{np} \in K, \ n = 1, 2, 3, \ldots \qquad \ldots(5.1)$$

of the sequence (s_p) into V_n such that $\sum_{n=1}^{\infty} V_n$ is convergent. The matrix preserving convergence in this case is called a $\lambda(k)$ matrix and the limit preserving matrix is called a $\gamma(k)$ matrix. In this connection, we have the following theorem.

Theorem 5—The necessary and sufficient condition for the matrix F in (5.1) to be a $\lambda(k)$ matrix are

(i) $\operatorname{Sup}_{n,p} \left| \sum_{m=1}^{n} f_{mp} \right| \leq M$

(ii) $\lim_{n\to\infty} \sum_{m=1}^{n} f_{mp} = f$ exists for each $p = 1, 2, 3, \ldots$

(iii) $\lim_{n\to\infty} \Sigma \sum_{m=1}^{n} f_{mp} = f$ exists

Moreover if $s_p \to s$, then we get (iv) $\lim_{n\to\infty} V_n = fs + \sum_{p=1}^{\infty} f_p (s_p - s)$.

PROOF: Since we are considering the sequence to series transformation, let us consider the sequence of partial sums of the series $\sum_{n=1}^{\infty} V_n$. Let $V_n = \sum_{m=1}^{n} V_m$. Then $V_n = \sum_{p=1}^{\infty} (\sum_{m=1}^{n} f_{mp}) s_p$. Hence the matrix $(\sum_{m=1}^{n} f_{mp})$ $n, p = 1, 2, 3 \ldots$ should transform the convergent sequence (s_p) into the convergent sequence (V_n). Hence applying a theorem of Monna (1946) we get the required result.

Remark.—If $F = (f_{np})$ is a $\mu(k)$ matrix then (i), (ii) and (iii) with $f = 1$ and $f_p = 0$ for each fixed p are the necessary and sufficient conditions.

Note: Theorem 5 may be stated in a more convenient form as follows.

Theorem 6—The necessary and sufficient condition for the matrix F to be a $\lambda(k)$ matrix or a $\mu(k)$ matrix is that the matrix defined by

$$a_{np} = f_{1p} + f_{2p} + \ldots + f_{np}$$

or

$$f_{np} = a_{np} - a_{n-1} p$$

is a conservative matrix or a $T(K)$ matrix.

This theorem is a generalization of a theorem of Hill.

§6. *Algebraic Properties of the above matrix over K*—Vermes (1946) establishes in the classical case that the product $H = AG$ of matrix A and a γ-matrix G is a γ-matrix if and only if A is a T-matrix. We shall generalise this for matrices defined over K in the following.

Theorem 7—Let $A = (a_{np})$ be a matrix defined over K. And let $G = (g_{np})$ be $\gamma(k)$ matrix. Then the product $H = AG$ is a $\gamma(K)$ matrix if and only if A is a $T(K)$ matrix.

PROOF : From the condition (ii) of Theorem 3 (g_{np}) is a bounded sequence for every fixed p. Hence $|g_{n_1}| \leqslant M_1$ (say)

$$g_{np} = g_{n_1} - (g_{n_1} - g_{n_2}) \ldots - (g_{np-1} - g_{np}).$$

Therefore we have from the above,

$$|g_{np}| \leqslant \text{Max}\ (|g_{n1}|,\ |g_{n1} - g_{n2}|, \ldots |g_{np-1} - g_{np}|)$$

So we get

$$|g_{np}| \leqslant \text{Max}\ \{M_1, N\} \qquad \ldots(6.1)$$

For condition (iii) of Theorem 1 to be defined,

$$a_{np} \to 0 \text{ as } p \to \infty \text{ for each fixed } n. \qquad \ldots(6.2)$$

So the general term of the product $h_{np} = \sum_{j=1}^{\infty}{}' a_{nj} g_{jp}$ is well-defined by using the Lemma given in the proof of Theorem 2, the condition of the Lemma being satisfied by virtue of (6.1) and (6.2). So $H = AG$ exists.

Now

$$h_{np} - h_{np+1} = \sum_{j=1}^{\infty} a_{nj} (g_{jp} - g_{jp-1}).$$

Therefore

$$|h_{np} - h_{np+1}| \leqslant \underset{1 \leqslant j < \infty}{\text{Max}}\ (|a_{nj}|,\ |g_{jp} - g_{jp+1}|)$$

By using condition (i) of Theorem 1 and condition (i) of Theorem 3 in the above, we have $|h_{np} - h_{np+1}| \leqslant \text{Max}\,(M, N)$.

Therefore

$$\sup_{n,p} |h_{np} - h_{np+1}| \leqslant L \text{ where } L = \text{Max}\,(M, N)$$

So the condition (i) of Theorem 3 is satisfied for $H = (h_{np})$. We shall now prove that $h_{np} \to 1$ as $n \to \infty$ for each fixed p so that the condition (ii) of Theorem 3 is satisfied for $H = (h_{np})$.

Since $g_{jp} \to 1$ as $j \to \infty$ for each fixed p, by condition (ii) of Theorem 3 we have

$$|g_{jp} - 1| < \varepsilon/M \text{ for all } j \geqslant j_0. \qquad \ldots(6.3)$$

Let us consider the partial sum of the series $\sum_{j=1}^{\infty} a_{nj}(g_{jp}-1)$. Let

$$S_m(n) = \sum_{j=1}^{n} a_{nj}(g_{jp}-1).$$

Now

$$|S_m(n)| \leqslant \sup_{1\leqslant j<\infty} |a_{nj}|\ |g_{jp}-1|.$$

By using condition (i) of Theorem 1 and (6.3), we get

$|S_m(n)| \leqslant M \cdot \varepsilon/M$ for sufficiently large m. Hence $S_m(n) \to 0$ as $m \to \infty$. From this we conclude that $\sum_{j=1}^{\infty} a_{nj}g_{jp} = A_n$ which tends to 1 as $n \to \infty$. These prove that $H = (h_{np})$ is a $\gamma(K)$ matrix.

To prove the necessity of the condition, consider the matrix, $V = (V_{np})$ where $V_{np} = 1$ for $p \leqslant n$ and $V_{np} = 0$ for $p > n$. This is easily seen to be a $\gamma(K)$ matrix. Since (h_{np}) is a $\gamma(K)$ matrix, we have

$$\sup_{n,p} |h_{np} - h_{np+1}| \leqslant N. \qquad \ldots(6.4)$$

The general term in the product AV is

$$h_{np} = \sum_{j=1}^{\infty} a_{nj}V_{jp} = \sum_{j=p}^{\infty} a_{nj}.$$

Therefore

$$h_{np} - h_{np+1} = a_{np}$$

and we have

$$\sup_{n,p} |a_{np}| = \sup_{n,p} |h_{np} - h_{np+1}| \leqslant N \text{ by (6.4).}$$

Hence we have from the above

$$\sup_{n,p} |a_{np}| \leqslant N \qquad \ldots(6.5)$$

$$h_{np} \to 1 \text{ for every fixed } p. \qquad \ldots(6.6)$$

Taking $p = 1$, we have

$$h_{n1} = \sum_{j=1}^{\infty} a_{nj} \to 1 \text{ as } p \to \infty. \qquad \ldots(6.7)$$

From (6.6), we have

$$\lim_{n\to\infty} (h_{np} - h_{np+1}) = \lim_{n\to\infty} a_{np} = 0 \text{ for each fixed } p. \qquad \ldots(6.8)$$

(6.5), (6.7) and (6.8) show that A is a $T(K)$ matrix. This establishes the necessity of the condition.

Remark : The product of $\gamma(K)$ matrix and a $T(K)$-matrix need not be a $\gamma(K)$ matrix as seen from the following example.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\ \pi^2 & 1-\pi^2 & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\ \pi^3 & \pi^3 & 1-2\pi^3 & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\ \pi^4 & \pi^4 & \pi^4 & 1-3\pi^4 & 0 & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \pi^n & \pi^n & \pi^n & \cdots & \cdots & & 1-(n-1)\pi^n & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

We can easily verify that A is a $T(K)$ matrix. Take $\gamma(K)$-matrix given in the proof of the Theorem 7. $V = (V_{np})$ where $V_{np} = 1$ for $p \leqslant n$ and $V_{np} = 0$ for $p > n$. Then we have

$$VA = \begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1+\pi^2 & 1-\pi^2 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1+\pi^2+\pi^3 & 1-\pi^2+\pi^3 & 1-2\pi^3 & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1+\pi^2+\ldots+\pi^n & \cdots & \cdots & \cdots & \cdots & 1-(n-1)\pi^n & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

In this matrix if $VA = (g_{np})$, $g_{np} = 0$ for $p > n$ and g_{np} is given above for $n \geqslant p$.

Now

$$|g_{np} - 1| = |1 + \pi^2 + \pi^3 + \ldots + \pi^n - 1|$$
$$\leqslant \text{Max}\,(|\pi|^2, |\pi|^3, \ldots |\pi|^n).$$

Since

$$|\pi| < 1, \ |g_{np} - 1| < 1,$$

This shows that (g_{np}) is bounded for each p as $n \to \infty$ but it does not converge to 1. Hence the product of a $\gamma(K)$ matrix and a $T(K)$ matrix is not necessarily a $\gamma(k)$ matrix.

The following two theorems of Ramanujan (1956) can be proved exactly as in the classical analysis.

Theorem 8—The product $C = FA$ of a $\lambda(K)$-matrix F with a matrix A exists and it is a $\lambda(K)$-matrix if and only if A is a conservative matrix defined over the field K denoted by $\bar{K}$.

Theorem 9—The product of two $\lambda(K)$ matrices is always a $\lambda(K)$-matrix.

Theorem 10—The set of all $\lambda(K)$ matrices F forms a non-archimedian Banach algebra under the norm $\| F \| = 2 \operatorname{Sup}_{n,p} | \sum_{m=1}^{n} f_{mp} |$.

PROOF : The norm is well defined because of condition (i) of Theorem 5. If F and G are two $\lambda(K)$-matrices, by using the properties of non-archimedian valuation on K, we can easily verify that $\| F + G \| \leqslant \operatorname{Max} \{\| F \|, \| G \|\}$.

Let (F^r) be a Cauchy sequence of $\lambda(K)$ matrices where $F^r = (f^r_{ip})$. The product $F^r \cdot F^s$ and sum $F^r + F^s$ of two $\lambda(K)$ matrices are $\lambda(K)$ matrices. We have therefore to prove only that the space of the above matrices is complete under the above norm and $\| F^r \cdot F^s \| \leqslant \| F^r \| \; \| F^s \|$. Since (F^r) is a Cauchy sequence, we have $\| F^r - F^s \| < \varepsilon$, $r, s > r_0$. Let A^r denote the $\bar{K}$ matrix corresponding to the $\lambda(K)$ matrix F^r. Now given the matrix $D = (d_{ik})$ over K, let

$$\| D \|_\lambda = 2 \operatorname{Sup}_{n,p} \left| \sum_{i=1}^{n} d_{ip} \right|$$

and

$$\| D \|_{\bar{K}} = \operatorname{Sup}_{n,p} | d_{np} |$$

Then in the above notation if A^r is the K-matrix associated with matrix F^r, we have $\| F^r \|_\lambda = 2 \| A^r \|_{\bar{K}}$

$$\| F^r - F^s \| = 2 \| A^r - A^s \|_{\bar{K}}.$$

Hence

$$\| A^r - A^s \| < \varepsilon/2 \text{ for } r, s > r_0.$$

Since the set of all $\bar{K}$ matrices over K forms a non-archimedian Banach algebra (Rangachari and Srinivasan 1964), there exists a $\bar{K}$ matrix A defined over K to which the sequence of matrices A^r converges. Corresponding to this $\bar{K}$-matrix A, there exists a $\lambda(K)$ matrix F which is the limit of the sequence F^r. Hence the space is complete under the above norm. The proof of the inequality $\| F^r \cdot F^s \| \leqslant \| F^r \| \; \| F^s \|$ can be established as in the classical case of Ramanujan (1956) by making use of Theorem 8.

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