ON THE ORDER OF THE COEFFICIENTS AND CONVERGENCE OF A FOURIER-BESSEL SERIES OF A DIFFERENTIABLE FUNCTION

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Kito (1952*a*) has introduced the Fourier-Bessel series corresponding to a function f defined on (a, b), $a > 0$, as follows :

$$f(x) \sim \sum_{m=1}^{\infty} p_m Q_\nu (xk_m, ak_m), \; 0 < a \leqslant x \leqslant b \qquad \ldots(1)$$

where

$$Q_\nu(\alpha, \beta) = J_\nu(\alpha)\; Y'_\nu(\beta) - J'_\nu(\beta)\, Y_\nu(\alpha)$$

and $k_1 < k_2 < k_3 < \ldots$ are the positive zeros of

$$S(k) = J'_\nu(bk)\; Y'_\nu(ak) - J'_\nu(ak)\; Y'_\nu(bk).$$

In this paper we have proved the absolute and uniform convergence of (1) corresponding to a function f, which is differentiable several times, after estimating the order of the terms in the series (1).

The theorems that have been proved are similar to those proved in Tolstov (1962) for Fourier-Bessel series of the first type.

§1. Let $J_\nu(t)$ and $Y_\nu(t)$ denote the Bessel functions of the first and second kinds respectively of order $\nu \geqslant -1/2$. Denote by $j_1 < j_2 < j_3 < \ldots$ the positive zeros of $J_\nu(t)$.

Let

$$Q_\nu(\alpha, \beta) \equiv J_\nu(\alpha)\; Y'_\nu(\beta) - J'_\nu(\beta)\; Y_\nu(\alpha). \qquad \ldots(1.1)$$

Denote by $k_1 < k_2 < k_3 < \ldots$ the positive zeros of $S(k)$,

where

$$S(k) \equiv J'_\nu(bk)\; Y'_\nu(ak) - J'_\nu(ak)\; Y'_\nu(bk), \; 0 < a < b. \qquad \ldots(1.2)$$

Given a Lebesgue integrable function f, the Fourier-Bessel series

$$f(x) \sim \sum_{m=1}^{\infty} a_m J_\nu(xj_m), \qquad 0 \leqslant x \leqslant 1 \qquad \ldots(1.3)$$

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where

$$a_m = \frac{2}{J_{\nu+1}^2(j_m)} \int_0^1 tf(t) J_\nu(tj_m)\, dt$$

is well known.

Kito (1952*a*)* introduced the following series while studying the vibrations of a cylindrical shell immersed in water :

$$f(x) \sim \sum_{m=1}^{\infty} p_m Q_\nu(xk_m, ak_m), \quad a \leqslant x \leqslant b, \qquad \ldots(1.4)$$

where

$$p_m = \frac{\int_a^b tf(t)\, Q_\nu(tk_m, ak_m)\, dt}{\int_a^b tQ_\nu^2(tk_m, ak_m)\, dt}. \qquad \ldots(1.5)$$

Kito (1952*b*) has also studied the convergence of (1.4) to $\frac{1}{2}\{f(x+0) + f(x-0)\}$, $a < x < b$.

§2. Tolstov (1962, pp. 228–233) has studied the series (1.3) for a differentiable function to consider the order of coefficients of its terms and certain aspects of its uniform and absolute convergence.

In this paper we prove the following theorems concerning the series (1.4) that are similar to the theorems of Tolstov :

Theorem 1—Let $f(x)$ be a bounded and twice differentiable function defined for $0 < a \leqslant x \leqslant b$ such that $f(a) = f'(a) = 0$, $f(b) = f'(b) = 0$ and $f''(x)$ is bounded (the second derivative may not exist at certain points). Then the coefficients p_m of the series (1.4) of the function f satisfy the relation

$$|p_m| = O\left(\frac{1}{k_m}\right), \text{ as } m \to \infty.$$

Theorem 2—Let f be a function satisfying the conditions of Theorem 1. Then the series (1.4) converges uniformly and absolutely on $[a, b]$, for $\nu \geqslant -\frac{1}{2}$.

Theorem 3—Let $f(x)$ be a function defined for $0 < a \leqslant x \leqslant b$, such that f is differentiable $2s$ times ($s > 1$) and such that

$$f(a) = f'(a) = f''(a) = \ldots = f^{(2s-1)}(a) = 0;$$

$$f(b) = f'(b) = f''(b) = \ldots = f^{(2s-1)}(b) = 0;$$

* Dr. F. Kito used $G_n(z)$ instead of $Y_\nu(z)$, where n is a positive integer, and $G_n(z) = -\pi/2\, Y_n(z)$. Here, we use $Y_\nu(z)$, and $\nu \geqslant -1/2$, a real number, to study a more general case.

and $f^{(2s)}(x)$ is bounded (this derivative may not exist at certain points). Then the coefficients p_m of the series (1.4) of the function f satisfy the relation

$$|p_m| = O\left(\frac{1}{k_m^{2s-1}}\right), \text{ as } m \to \infty.$$

Theorem 4—Let f be a function satisfying the conditions of Theorem 3 for $s \geqslant 1$. Then for $\nu \geqslant -\frac{1}{2}$, the series (1.4) converges uniformly and absolutely on $[a, b]$.

§3. The following lemmas are needed to complete the proofs of the above theorems (K_1, K_2, K_3, ... etc., denote suitable positive constants):

Lemma 1—If $k > 0$ is sufficiently large, and ν is a real number, then

$$|Q_\nu(xk, ak)| < \frac{K_1}{k\sqrt{ax}},$$

where $0 < a \leqslant x \leqslant b$.

PROOF: By Watson (1966, p. 211), we have

$$|H_\nu^{(1)}(z)| < \frac{K_2}{|z|^{1/2}}, \quad |H_\nu^{(2)}(z)| < \frac{K_2}{|z|^{1/2}}, \qquad \text{...(3.1)}$$

for all sufficiently large real values of z, and $\nu \geqslant 0$.

Also, by Watson (1966, p. 74),

$$H_{-\nu}^{(1)}(z) = e^{\nu\pi i} H_\nu^{(1)}(z), \quad H_{-\nu}^{(2)}(z) = e^{-\nu\pi i} H_\nu^{(2)}(z). \qquad \text{...(3.2)}$$

By (3.1) and (3.2), it follows that

$$|H_\nu^{(1)}(z)| < \frac{K_2}{|z|^{1/2}}, \quad |H_\nu^{(2)}(z)| < \frac{K_2}{|z|^{1/2}} \qquad \text{...(3.3)}$$

where z is sufficiently large real number and ν is any real number.

Using recurrence relations [Watson 1966; p. 74], we obtain from (3.3),

$$|H_\nu^{(1)\prime}(z)| < \frac{K_2}{|z|^{1/2}}, \quad |H_\nu^{(2)\prime}(z)| < \frac{K_2}{|z|^{1/2}} \qquad \text{...(3.4)}$$

for sufficiently large real values of z, and ν any real number.

Again, by (1.1) and using relations (1) and (3), Watson (1966, §3.61, p. 74), we obtain

$$Q_\nu(xk, ak) = \frac{H_\nu^{(2)}(xk)\, H_\nu^{(1)\prime}(ak) - H_\nu^{(1)}(xk)\, H_\nu^{(2)\prime}(ak)}{2i}. \qquad \text{...(3.5)}$$

By (3.3), (3.4) and (3.5) the lemma follows straightaway.

Lemma 2—For $\nu \geqslant -\frac{1}{2}$ and sufficiently large real $k > 0$,

$$k\sqrt{x}\, Q_\nu(xk, ak) = K_3 \cos(xk - ak) + \frac{\sigma(k)}{xk}$$

where $0 < a \leqslant x$ and $\sigma(k)$ remains bounded as $k \to \infty$.

PROOF: Using the asymptotic expressions, Tolstov (1962, p. 213), and the recurrence relations, Watson (1966, p. 66), we obtain,

$$Q_\nu(xk, ak) = \tfrac{1}{2} J_\nu(xk)\{Y_{\nu-1}(ak) - Y_{\nu+1}(ak)\} -$$

$$- \tfrac{1}{2}\{J_{\nu-1}(ak) - J_{\nu+1}(ak)\}\, Y_\nu(xk)$$

$$= \frac{K_3 \cos(xk - ak)}{k\sqrt{x}} + \frac{\sigma(k)}{k^2 x^{3/2}},$$

where $\sigma(k)$ remains bounded as $k \to \infty$.

This proves the lemma.

Lemma 3—For $\nu \geqslant -\frac{1}{2}$ and sufficiently large real $k > 0$,

$$\frac{K_4}{k^2} \leqslant \int_a^b x Q_\nu^2(xk, ak)\, dx \leqslant \frac{K_5}{k^2},$$

where K_4 and K_5 are suitable positive constants depending at most on ν.

PROOF: We have by Lemma 1,

$$\int_a^b x Q_\nu^2(xk, ak)\, dx \leqslant \frac{K_1^2}{ak^2} \int_a^b dx = \frac{K_5}{K^2}.$$

This proves the right-hand part of the inequality.

Also

$$\int_a^b x Q_\nu^2(xk, ak)\, dx = \frac{1}{k^2} \int_{ak}^{bk} t Q_\nu^2(t, ak)\, dt \qquad \text{...(3.6)}$$

and from Lemma 2,

$$tk Q_\nu^2(t, ak) \geqslant K_3^2 \cos^2(t - ak) - \frac{K_6}{t}. \qquad \text{...(3.7)}$$

By using (3.6) and (3.7), we get

$$\int_a^b x Q_\nu^2(xk, ak)\, dx \geqslant \frac{K_4}{k^2},$$

where K_4 is positive when k is sufficiently large. This completes the proof of the lemma.

The following lemma is due to Naylor (1966, p. 70)

Lemma 4—For sufficiently large m and any real ν,

$$k_m = \frac{m\pi}{(b-a)} + \frac{(4\nu^2+3)(b-a)}{8m\,\pi ab} + O(m^{-3}).$$

§4. *Proof of Theorem* 1—Let $F(x) = x^{1/2} f(x)$. Then the conditions of the theorem are also true for F, *i.e.*,

$$\begin{cases} F(a) = F'(a) = 0, \; F(b) = F'(b) = 0, \\ \text{and } F''(x) \text{ is bounded (save at certain points).} \end{cases} \quad \ldots(4.1)$$

It is known that $C_1 J_\nu(x) + C_2 Y_\nu(x)$ is a general solution of the Bessel's equation

$$x^2 y'' + xy' + (x^2 - \nu^2) y = 0$$

for arbitrary C_1 and C_2. It follows that $Q_\nu(x, ak)$ is a solution of the equation. Putting $z = x^{1/2} y$, it is obvious that $z(x) = x^{1/2} Q_\nu(x, ak)$ is a solution of

$$x^2 z'' + (x^2 - \mu) z = 0,$$

where $\mu = \nu^2 - \frac{1}{4}$. Further, by substituting $x = tk$, we observe that $z(t) = (tk)^{1/2}$. $Q_\nu(tk, ak)$ is a solution of

$$z'' + \left(k^2 - \frac{\mu}{t^2}\right) z = 0. \quad \ldots(4.2)$$

Putting $u(t) = t^{1/2} Q_\nu(tk, ak)$, since $u(t)$ differs from $z(t)$ only by a constant, $u(t)$ is also a solution of (4.2). Hence, we have

$$u'' + (k^2 - \mu/t^2) u = 0,$$

i.e.,

$$u = \frac{1}{k^2}\left(\frac{\mu}{t^2} u - u''\right). \quad \ldots(4.3)$$

Now, consider

$$I = \int_a^b x f(x)\, Q_\nu(xk, ak)\, dx$$

$$= \int_a^b F(x)\, u(x)\, dx$$

$$= \frac{1}{k^2} \int_a^b F(x) \left\{ \frac{\mu}{x^2} u(x) - u''(x) \right\} dx, \quad \text{[using (4.3)]},$$

$$= \frac{1}{k^2} \int_a^b \left\{ \frac{\mu}{x^2} F(x) - F''(x) \right\} u(x)\, dx +$$

$$+ \frac{1}{k^2} \int_a^b \{F''(x) u(x) - F(x) u''(x)\}\, dx$$

$$= \frac{1}{k^2} \int_a^b \left\{ \frac{\mu}{x^2} F(x) - F''(x) \right\} u(x)\, dx +$$

$$+ \frac{1}{k^2} [F'(x) u(x) - F(x) u'(x)]_a^b. \qquad \text{...(4.4)}$$

Since

$$u'(x) = \tfrac{1}{2} x^{-1/2} Q_\nu(xk, ak) + kx^{1/2} \{J'_\nu(xk)\, Y'_\nu(ak) - Y'_\nu(xk)\, J'_\nu(ak)\},$$

we have

$$u'(a) = \tfrac{1}{2} a^{-1/2} Q_\nu(ak, ak) \text{ and } u'(b) = \tfrac{1}{2} b^{-1/2} Q_\nu(bk, ak),$$

if k is a zero of $S(k)$. Hence, by Lemma 1, we conclude that $u(a)$, $u(b)$, $u'(a)$ and $u'(b)$ are all bounded. In view of (4.1), it, therefore, follows that

$$[F'(x) u(x) - F(x) u'(x)]_a^b = 0. \qquad \text{...(4.5)}$$

From (4.4) and (4.5) we obtain

$$I = \frac{1}{k_m^2} \int_a^b \left\{ \frac{\mu}{x^2} F(x) - F''(x) \right\} u(x)\, dx.$$

Choosing K_7, such that,

$$\left| \frac{\mu}{x^2} F(x) - F''(x) \right| \leqslant K_7,$$

we have by Schwartz inequality,

$$|I| \leqslant \frac{K_7}{k_m^2} \sqrt{b - a} \sqrt{\int_a^b |u(x)|^2\, dx} =$$

$$= \frac{K_7 \sqrt{b-a}}{k_m^2} \sqrt{\int_a^b xQ_\nu^2 (xk_m, ak_m)\, dx}$$

$$\leqslant \frac{K_8}{k_m^3} \qquad \ldots(4.6)$$

by using Lemma 3. Again, by Lemma 3,

$$\int_a^b xQ_\nu^2 (xk_m, ak_m)\, dx \geqslant \frac{K_4}{k_m^2}, \qquad \ldots(4.7)$$

Hence by (1.5), (4.6) and (4.7),

$$| p_m | = \left| \frac{\int_a^b xf(x)\, Q_\nu (xk_m, ak_m)\, dx}{\int_a^b xQ_\nu^2 (xk_m, ak_m)\, dx} \right|$$

$$\leqslant \frac{K_9}{k_m}.$$

The proof of Theorem 1 is now complete.

§5. *Proof of Theorem* 2—Since $f(x)$ satisfies the hypothesis of Theorem 1, using its conclusion together with Lemma 1, we get

$$| p_m Q_\nu (xk_m, ak_m) | \leqslant \frac{K_{10}}{k_m^2}. \qquad \ldots(5.1)$$

Also, by Lemma 4,

$$\frac{1}{k_m} \leqslant \frac{2}{m}, \qquad \ldots(5.2)$$

if m is sufficiently large.

Consequently, for large m,

$$| p_m Q_\nu (xk_m, ak_m) | \leqslant \frac{4K_{10}}{m^2}.$$

Therefore, the series (1.4) converges uniformly and absolutely for $0 < a \leqslant x \leqslant b$.

§6. *Proof of Theorem* 3—Let $F(x) = x^{1/2} f(x)$ and $u(x) = x^{1/2}\, Q_\nu (xk_m, ak_m)$. Then as in the proof of Theorem 1, since F satisfies the conditions of Theorem 1, we have

$$I = \frac{1}{k_m^2} \int_a^b \left\{ \frac{\mu}{x^2} F(x) - F''(x) \right\} u(x)\, dx$$

$$= \frac{1}{k_m^2} \int_a^b F_1(x)\, u(x)\, dx,$$

where

$$F_1(x) = \frac{\mu}{x^2} F(x) - F''(x).$$

F_1 also satisfies the conditions of Theorem 1. So

$$I = \frac{1}{k_m^4} \int_a^b F_2(x)\, u(x)\, dx,$$

where

$$F_2(x) = \frac{\mu}{x^2} F_1(x) - F_1''(x).$$

If $s > 2$, F_2 satisfies the conditions of Theorem 1, and the argument can be repeated till we obtain

$$I = \frac{1}{k_m^{2s}} \int_a^b F_s(x)\, u(x)\, dx,$$

where

$$F_s(x) = \frac{\mu}{x} F_{s-1}(x) - F''_{s-1}(x).$$

Hence, as in the proof of Theorem 1,

$$|I| < \frac{K_{11}}{k_m^{2s}} \int_a^b |u(x)|\, dx$$

$$< \frac{K_{12}}{k_m^{2s+1}}. \qquad \text{...(6.1)}$$

Using (4.7) and (6.1), we obtain,

$$|p_m| \leqslant \frac{K_{13}}{k_m^{2s-1}}.$$

Theorem 3 is, now, proved.

§7. *Proof of Theorem* 4—Since $f(x)$ satisfies the hypothesis of theorem 3, using its conclusion together with Lemma 1, we get

$$|p_m\, Q_\nu(xk_m, ak_m)| \leqslant \frac{K_{14}}{k_m^{2s}}. \qquad \text{...(7.1)}$$

By (5.2) and (7.1), it follows that

$$|p_m Q_\nu (xk_m, ak_m)| \leqslant \frac{2^{2s} K_{14}}{m^{2s}}.$$

Hence the series (1.4) converges uniformly and absolutely for $0 < a \leqslant x \leqslant b$. This proves Theorem 4.

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