# TRIPLE SERIES EQUATIONS INVOLVING JACOBI AND LAGUERRE POLYNOMIALS

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The solution of certain triple series equations involving Jacobi polynomials has been obtained by reducing them to a Fredholm integral equation of second kind. Results for similar triple series equations involving Laguerre polynomials have also been obtained by applying a limit process.

## 1. INTRODUCTION

Recently, there has been considerable interest in the triple series equations involving Jacobi and Laguerre polynomials. By going through the literature, however, one feels that once a result has been obtained for Jacobi polynomials, it becomes a routine matter to work out similar results for Laguerre polynomials (see for example Lowndes 1968 *a*, 1969). The source of this symmetry seems to be the fact that Laguerre polynomials are certain limiting cases of Jacobi polynomials. Indeed, if adequate care is taken in presenting the results on Jacobi polynomials it is possible to avoid duplication of work and similar results for Laguerre polynomials can be deduced through a limit process.

In this paper we consider certain triple series equations involving Jacobi polynomials which are generalizations of those considered by Lowndes (1968*a*). We deduce results for similar triple series equations involving Laguerre polynomials by applying a limit process. In order to emphasize that in most of the cases it is unnecessary to consider dual series equtions for Jacobi and Laguerre polynomials separately, the proofs have been carried out in such a way that the limit process can be applied not only to the final results but to any intermediate step and to any formula being used thereof. In the next section we give, for ready reference, some results which will be needed in the course of analysis. Sections 3 and 4 have been devoted to triple series equations and section 5 to their particular cases.

## 2. Some Results

In Szegö notation the Jacobi polynomials may be defined ( Rainville 1960, p. 254 ) in terms of hypergeometric function, as

$$P_n^{(\alpha,\beta)}\left(1-\frac{2x}{c}\right)=\frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\,\Gamma(\alpha+1)}\,{}_2F_1\left(-n,\ n+\alpha+\beta+1\ ;\ \frac{x}{c}\right). \tag{2.1}$$

We shall at time need the following relation between the Szegö notation and the one used by Noble (1963) :

$$P_n^{(\alpha,\beta)}\left(1-\frac{2x}{c}\right)=\frac{\Gamma\,(n+\alpha+1)}{\Gamma(n+1)\ \Gamma(\alpha+1)}\,J_n\left(\alpha+\beta+1,\ \alpha+1\ ;\frac{x}{c}\right). \tag{2.2}$$

A limit formula involving gamma functions which follows from Erdélyi [1953*a*. p. 47, (4)] is given by

$$\lim_{\beta\to\infty}[\beta^{(q_1-q_2)}\ \Gamma(\beta+q_2)/\Gamma(\beta+q_1)]=1. \tag{2.3}$$

From Erdélyi (1954, p. 191, (43) and (44) ) we have the following formulae which are similar to the Sonine integrals of the first and second kind :

$$\int_0^y \frac{x^{\alpha}\,P_n^{(\alpha,\delta)}\left(1-\dfrac{2x}{c}\right)}{(y-x)^{1-\mu}}dx=\frac{\Gamma(\mu)\,\Gamma(\alpha+n+1)}{\Gamma(\alpha-\mu+n+1)}\,y^{\alpha+\mu}\ \times$$

$$\times\,P_n^{(\alpha+\mu,\,\delta-\mu)}\left(1-\frac{2y}{c}\right),\quad \alpha>-1,\ \mu>0, \tag{2.4}$$

$$\int_y^c \frac{\left(1-\dfrac{x}{c}\right)^{\delta}P_n^{(\alpha,\delta)}\left(1-\dfrac{2x}{c}\right)}{(x-y)^{1-\mu}}dx=\frac{c^{\mu}\,\Gamma(\mu)\,\Gamma(\delta+n+1)}{\Gamma(\delta+\mu+n+1)}\times$$

$$\times\left(1-\frac{y}{c}\right)^{\delta+\mu}P_n^{(\alpha-\mu,\,\delta+\mu)}\left(1-\frac{2y}{c}\right),\ \alpha>-1,\ \mu>0. \tag{2.5}$$

The orthogonality relation for Jacobi polynomials (Rainville 1960, p. 135) may be written as

$$\int_0^c x^{\alpha}\left(1-\frac{x}{c}\right)^{\delta}P_n^{(\alpha,\delta)}\left(1-\frac{2x}{c}\right)P_m^{(\alpha,\delta)}\left(1-\frac{2x}{c}\right)dx$$

$$=\frac{c^{\alpha+1}\Gamma(\alpha+n+1)\Gamma\,(\delta+n+1)}{\Gamma\,(n+1)(2n+\alpha+\delta+1)\Gamma(n+\alpha+\delta+1)}\,\delta_{mn},\quad \alpha>-1,\ \delta>-1, \tag{2.6}$$

where $\delta_{mn}$ is the Kronecker delta. It follows from the orthogonality relation (2.6) and the formula (2.4) that

$$\sum_{n=0}^{\infty} \frac{n!\,(2n+\delta+\alpha+1)\,\Gamma(n+\alpha+\delta+1)}{c^{\epsilon+1}\,\Gamma(\delta+n+1)\,\Gamma(\alpha+\mu+n+1)} P_n^{(\alpha,\delta)}\left(1-\frac{2x}{c}\right)\times$$

$$\times\ P_n^{(\alpha+\mu,\delta-\mu)}\left(1-\frac{2y}{c}\right)=$$

$$=\frac{H(y-x)\,(y-x)^{-1}}{\left(1-\frac{x}{c}\right)^{\delta} y^{1+\mu}\,\Gamma(\mu)},\quad \alpha>-1,\ \delta>-1,\ \mu>0, \tag{2.7}$$

where $H(x)$ is the the Heaviside's Unit function such that

$$H(y-x)\ =\ \begin{cases}1, & \text{if } y \geq x\\ 0, & \text{if } y < x.\end{cases}$$

Further, if

$$S(x,y) = \sum_{n=0}^{\infty} \frac{n!\,(2n+\alpha+\delta+p+1)\,\Gamma(n+\alpha+\delta+p+1)\,\Gamma(\alpha-\sigma+n+1)}{c^{\alpha-\sigma+1}\,\Gamma(\delta+p+\sigma+n+1)\,\Gamma(\alpha+p+n+1)\,\Gamma(\alpha+n+1)}\times$$

$$\times\ P_n^{(\alpha+q,\delta)}\left(1-\frac{2y}{c}\right) P_n^{(\alpha,\delta+p)}\left(1-\frac{2x}{c}\right) \tag{2.8}$$

$$S_\omega(x,y) = \int_0^w n(t)\,(x-t)^{\sigma-1}\,(y-t)^{\sigma+p-1}\,dt \tag{2.9}$$

where $\omega = \min(x, y)$ and

$$n(t) = t^{\alpha-\sigma}\left(1-\frac{t}{c}\right)^{-(\delta+p+\sigma)} \tag{2.9}$$

then, it can be readily shown using (2.9) and (2.6) that

$$S(x,y) = S_\omega(x,y) \mid \{\Gamma(\sigma)\,\Gamma(\sigma+p)\,x^{\alpha}\,y^{\alpha+p}\},$$

$$\alpha-\sigma>-1,\ p+\sigma>0,\ \sigma>0,\ \delta>-1. \tag{2.11}$$

We shall frequently need the following integrals which may be evaluated by integration by parts or may be found in Lowndes (1968*a*, p. 104) :

$$\frac{d}{dt}\int_a^t \frac{(x-r)^{\sigma-1}}{(t-x)^{\sigma}}\,dx = \frac{(a-r)^{\sigma}}{(t-r)\,(t-a)^{\sigma}},\quad 0<\sigma<1 \tag{2.12}$$

$$\int_a^t \frac{(y-r)^{\sigma}}{(t-y)^{\sigma}}\,dy = \frac{(t-a)^{1-\sigma}}{(t-r)\,(a-r)^{1-\sigma}},\quad 0<\sigma<1. \tag{2.13}$$

For $p = 0$, $c = 1$, the results (2.6) to (2.11) may be found in Noble (1963). On the other hand if we put $p = 0$, $c = \delta$ in (2.6) to (2.11) and let $\delta$ approach infinity then, using (2.3), it can be easily seen by making appropriate changes in the parameters that these results have appeared in Lowndes (1968*b*, 1969).

## 3. Triple Series Equations of First Kind

$$\sum_{n=0}^{\infty} \frac{A_n \Gamma(\delta+\sigma+p+n+1)}{c^{\sigma+p}\, \Gamma(\delta+n+1)} P^{(\alpha+p,\delta)}\left(1-\frac{2x}{c}\right) = 0,\ 0<x<a \tag{3.1}$$

$$\sum_{n=0}^{\infty} \frac{A_n \Gamma(\alpha-\sigma+n+1)}{\Gamma(\alpha+n+1)} P_n^{(\alpha,\delta+p)}\left(1-\frac{2x}{c}\right) = f(x),\ a<x<b \tag{3.2}$$

$$\sum_{n=0}^{\infty} \frac{A_n \Gamma(\delta+\sigma+p+n+1)}{c^{\sigma+p}\, \Gamma(\delta+n+1)} P_n^{\alpha(+p,\delta)}\left(1-\frac{2x}{c}\right) = 0,\ b<x<c \tag{3.3}$$

where the parameters $\alpha$, $p$, $\sigma$ and $\delta$ satisfy

$$(i)\ \alpha+1>\sigma,\quad (ii)\ 0<\sigma<1,\quad (iii)\ \delta+1>\max,\ (0,-p),\quad (iv)\ 0<\sigma+p<1. \tag{3.4}$$

The equations (3.1) to (3.3) have been discussed by Lowndes (1968) for the case $p=0$, $c=1$. We reduce the problem of solving the triple series equations of the first kind to that of solving a Fredholm integral equation of the second kind.

To solve (3.1) to (3.3), we set

$$\sum_{n=0}^{\infty} \frac{A_n \Gamma(\delta+\sigma+p+n+1)}{c^{\sigma+p}\, \Gamma(\delta+n+1)} P_n^{(\alpha+p,\delta)}\left(1-\frac{2x}{c}\right)$$

$$=\left(1-\frac{x}{c}\right)^{-\delta} \phi(x),\ a<x<b, \tag{3.5}$$

wher $\phi(x)$ is as yet an unspecified function. In view of the orthogonality relation (2.6) it follows from equations (3.1), (3.3) and (3.5) that

$$A_n = \frac{n!\,(2n+\alpha+\delta+p+1)\,(n+\alpha+p+\delta+1)}{c^{\alpha-\sigma+1}\, \Gamma(\delta+\sigma+p+n+1)\, \Gamma(\alpha+p+n+1)} \times$$

$$\times \int_a^b y^{\alpha+p} \phi(y)\, P_n^{(\alpha+p,\delta)}\left(1-\frac{2y}{c}\right) dx. \tag{3.6}$$

To determine the unknown function $\phi(y)$, we substitute $A_n$ from (3.6) in (3.2) and interchange the order of summation and integration. In notation of (2.8), we have from (3.2) and (3.6) the following equation :

$$\int_a^b y^{\alpha+p} \phi(y)\, S(x,y)\, dy = f(x),\ a<x<b. \tag{3.7}$$

Using (2.11) we can write (3.7), in the notations of (2.9) and (2.10), as

$$\int_a^x \phi(y)\, S_y(x,y)\, dy + \int_x^b \phi(y)\, S_x(x,y)\, dy = \Gamma(\sigma)\, \Gamma(p+\sigma)\, x^{\alpha} f(x). \tag{3.8}$$

Interchanging the order of integrations in (3.8) and rearranging the terms we get

$$\int_a^x \eta(t)\,(x-t)^{\sigma-1}\, \Phi(t)\, dt = \Gamma(\sigma)\, \Gamma(\sigma+p)\, x^{\alpha} f(x) - \int_0^a \eta(r)\,(x-r)^{\sigma-1}\, Q(r)\, dr \tag{3.9}$$

where

$$\Phi(t) = \int_t^b \phi(y)\,(y-t)^{\sigma+p-1}\, dy, \quad a<t<b \tag{3.10}$$

$$Q(r) = \int_a^b \phi(y)\,(y-r)^{\sigma+p-1}\, dy, \quad 0<r<a \tag{3.11}$$

and $\eta(t)$ is given by (2.10).

Inverting the Abel integral equation (3.9) and using the result (2.12) we get

$$\eta(t)\,\Phi(t) = F(t) - \frac{\sin \sigma\pi}{\pi} \int_0^a \frac{(a-r)^{\sigma}\, \eta(r)\, Q(r)\, dr}{(t-r)\,(t-a)^{\sigma}}, \quad a<t<b \tag{3.12}$$

where

$$F(t) = \frac{\Gamma(p+\sigma)}{\Gamma(1-\sigma)} \frac{d}{dt} \int_a^t \frac{x^{\alpha} f(x)\, dx}{(t-x)^{\sigma}}. \tag{3.13}$$

To express $Q(r)$ in terms of $\Phi(t)$ we invert the Abel integral equation (3.10) :

$$\phi(y) = -\frac{\sin(\sigma+p)\pi}{\pi} \frac{d}{dy} \int_y^b \frac{\Phi(t)\, dt}{(t-y)^{\sigma+p}}, \quad a<y<b, \tag{3.14}$$

and substitute $\phi(y)$ from (3.14) in (3.11), then integration by parts yields

$$Q(r) = \frac{\sin(\sigma+p)\pi}{\pi} \Big[ (a-r)^{p+\sigma-1} \int_a^b \frac{\Phi(t)\, dt}{(t-y)^{\sigma+p}} + (p+\sigma-1) \int_a^b (y-r)^{p+\sigma-2}\, dy \int_y^b \frac{\Phi(t)\, dt}{(t-y)^{\sigma+p}} \Big]. \tag{3.15}$$

Interchanging the order of integration in the second term of (3.15) and evaluating the integral using (2.13) we have

$$Q(r) = \frac{\sin(\sigma+p)\pi}{\pi} (a-r)^{p+\sigma} \int_a^b \frac{\Phi(t)\, dt}{(t-r)\,(t-a)^{\sigma+p}}. \tag{3.16}$$

Substituting $Q(r)$ from (3.16) in (3.12) and interchanging the order of integrations we get

$$\eta(t)\,\phi(t) = F(t) - \int_a^b M(t,y)\,\Phi(y)\,dy, \quad a<t<b \tag{3.17}$$

where $\eta(t)$ and $F(t)$ are given by (2.10) and (3.13) respectively and

$$M(t,y) = \frac{\sin\sigma\pi\,\sin(\sigma+p)\pi}{\pi^2\,(t-a)^\sigma\,(y-a)^{\sigma+p}} \int_0^a \frac{(a-r)^{2\sigma+p}\,\eta(r)\,dr}{(t-r)\,(y-r)}\,. \tag{3.18}$$

The equation (3.17) is Fredholm integral equation of the second kind for determining $\Phi$. The coefficients $A_n$ satisfying (3.1) to (3 3) can be found from (3.6) and (3.16). For $p=0$, $c=1$, the results given here and those obtained by Lowndes (1968*a*, p. 104) are identical. Further, the dual series equations (1) and (2) of Noble (1963) can be reduced to dual series equations of the same kind with $f=0$ and then their solution can be obtained from the solution of the triple series equations of the first kind, considered in the present paper (or in Lowndes 1968 *a*), by taking $b=c=1$, $p=0$.

## 4. Triple Series Equations of Second Kind

In this section we consider the triple series equations of the second kind

$$\sum_{n=0}^{\infty} \frac{B_n\,\Gamma(\alpha-\sigma+n+1)}{\Gamma(\alpha+n+1)} P_n^{(\alpha,\delta+p)}\left(1-\frac{2x}{c}\right) = g(x), \qquad 0<x<a \tag{4.1}$$

$$\sum_{n=0}^{\infty} \frac{B_n\,\Gamma(\delta+\sigma+p+n+1)}{c^{\sigma+p}\,\Gamma(\delta+n+1)} P_n^{(\alpha+p,\delta)}\left(1-\frac{2x}{c}\right) = 0, \qquad a<x<b \tag{4.2}$$

$$\sum_{n=0}^{\infty} \frac{B_n\,\Gamma(\alpha-\sigma+n+1)}{\Gamma(\alpha+n+1)} P_n^{(\alpha,\delta+p)}\left(1-\frac{2x}{c}\right) = h(x), \qquad b<x<c \tag{4.3}$$

where the parameters $\alpha$, $p$, $\sigma$ and $\delta$ satisfy the conditions stated in (3.4).

In order to solve the equations (4.1) to (4.3), we introduce two functions $\psi_1(x)$ and $\psi_2(x)$ defined by the equations

$$\sum_{n=0}^{\infty} \frac{B_n\,\Gamma(\delta+\sigma+p+n+1)}{c^{\sigma+p}\,\Gamma(\delta+n+1)} P_n^{(\alpha,\delta+p)}\left(1-\frac{2x}{c}\right) = \left(1-\frac{x}{c}\right)^{-\delta} \psi_1(x), \quad 0<x<a, \tag{4.4}$$

$$= \left(1-\frac{x}{c}\right)^{-\delta} \psi_2(x) \qquad b<x<c. \tag{4.5}$$

Using the orthogonality relation (2.6), we get from (4.4). (4.2) and (4.5), the coefficients $B_n$ in terms of the unknowns $\psi_1(x)$, $\psi_2(x)$:

$$B_n=\frac{n!\,(2n+\alpha+\delta+p+1)\,\Gamma(n+\alpha+\delta+p+1)}{c^{\alpha-\sigma+1}\,\Gamma(\delta+\sigma+p+n+1)\,\Gamma(\alpha+p+n+1)}\times$$

$$\times\left[\int_0^a y^{\alpha+p}\,\psi_1(y)\,P_n^{(\alpha+p,\delta)}\left(1-\frac{2y}{c}\right)dy+\right.$$

$$\left.+\int_b^c y^{\alpha+p}\,\psi_2(y)\,P_n^{(\alpha+p,\delta)}\left(1-\frac{2y}{c}\right)dy\right]. \tag{4.6}$$

Substituting $B_n$ from (4.6) in equations (4.1) and (4.3), interchanging order of integration and summation and using (2.11) we get, in the notations of (2.8) to (2.11), the equations

$$\int_0^x \psi_1(y)\,S_y(x,y)\,dy+\int_x^a \psi_1(y)S_x(x,y)dy+\int_b^c \psi_2(y)S_x(x,y)\,dy$$

$$=\Gamma(\sigma)\,\Gamma(\sigma+p)\,x^{\alpha}\,g(x),\qquad 0<x<a, \tag{4.7}$$

$$\int_0^a \psi_1(y)\,S_y(x,y)\,dy+\int_0^x \psi_2(y)\,S_y(x,y)\,dy+\int_b^c \psi_2(y)\,S_x(x,y)\,dy$$

$$=\Gamma(\sigma)\,\Gamma(\sigma+p)\,x^{\alpha}\,h(x),\qquad b<x<c. \tag{4.8}$$

Interchanging the order of integrations and rearranging the terms in (4.7) and (4.8) we find

$$\int_0^x \frac{\eta(t)}{(x-t)^{1-\sigma}}[\Psi_1(t)+R(t)]\,dt=\Gamma(\sigma)\,\Gamma(p+\sigma)\,x^{\alpha}\,g(x),\qquad 0<x<a, \tag{4.9}$$

$$\int_b^x \frac{\eta(t)}{(x-t)^{1-\sigma}}\,\Psi_2(t)\,dt+\int_0^b\frac{\eta(t)\,R(t)}{(x-t)^{1-\sigma}}\,dt+\int_0^a\frac{\eta(t)}{(x-t)^{1-\sigma}}\,\Psi_1(t)\,dt$$

$$=\Gamma(\sigma)\,\Gamma(\sigma+p)\,x^{\alpha}\,h(x),\quad b<x<c \tag{4.10}$$

where $\eta(t)$ is given by (2.10) and

$$\Psi_1(t)=\int_t^a \psi_1(y)\,(y-t)^{\sigma+p-1}\,dy,\qquad 0<t<a \tag{4.11}$$

$$\Psi_2(t)=\int_t^c \psi_2(y)\,(y-t)^{\sigma+p-1}\,dy,\qquad b<t<c, \tag{4.12}$$

$$R(t)=\int^c \psi_2(y)\,(y-t)^{\sigma+p-1}\,dy,\qquad 0<t<b. \tag{4.13}$$

Inverting the Abel integral equation (4.9) we get

$$\eta(t)\,\psi(t) = G(t) - \eta(t)\,R(t), \qquad 0<t<a \tag{4.14}$$

where

$$G(t) = \frac{\Gamma(\sigma+p)}{\Gamma(1-\sigma)}\frac{d}{dt}\int_0^t \frac{x^{\alpha} g(x)}{(t-x)^{\sigma}}\,dx. \tag{4.15}$$

Using (4.14) we can write equation (4.10) as

$$\int_b^x \frac{\eta(t)\,\psi_2(t)}{(x-t)^{1-\sigma}}\,dt = \Gamma(\sigma)\,\Gamma(\sigma+p)\,\lambda^{\alpha}\,h(x) - \int_0^a \frac{G(r)\,dr}{(x-r)^{1-\sigma}} - \int_a^b \frac{\eta(r)\,R(r)}{(x-r)^{1-\sigma}}\,dr, \qquad b<x<c. \tag{4.16}$$

Solving the Abel integral equation (4.16) and evaluating the integrals. using (2.12) we have

$$\eta(t)\,\psi_2(t) = T(t) - \frac{\sin\sigma\pi}{\pi(t-b)^{\sigma}}\int_a^b \frac{\eta(r)\,R(r)\,(b-r)^{\sigma}}{(t-r)}\,dr, \quad b<t<c \tag{4.17}$$

$$T(t) = \frac{\Gamma(\sigma+p)}{\Gamma(1-\sigma)}\frac{d}{dt}\int_b^t \frac{x^{\alpha}\,h(x)}{(t-x)^{\sigma}}\,dx \quad \frac{\sin\sigma\pi}{\pi(t-b)^{\sigma}}\int_0^a \frac{(b-r)^{\sigma}\,G(r)}{(t-r)}\,dr. \tag{4.18}$$

The equations (4.11) and (4.12) may also be inverted to yield

$$\psi_1(y) = -\frac{\sin(\sigma+p)\pi}{\pi}\frac{d}{dy}\int_y^a \frac{\psi_1(t)\,dt}{(t-y)^{1-\sigma-p}}, \quad 0<y<a \tag{4.19}$$

$$\psi_2(y) = -\frac{\sin(\sigma+p)\pi}{\pi}\frac{d}{dy}\int_y^c \frac{\psi_2(t)\,dt}{(t-y)^{1-\sigma-p}}, \quad b<y<c. \tag{4.20}$$

We need precisely the same analysis to express $R(r)$ in terms of $\psi_2(t)$ as employed in deriving (3.15) from (3.11) and 3-14. From (4.13) and (4.20) we have

$$R(r) = \frac{\sin(\sigma+p)\pi}{\pi}(b-r)^{\sigma+p}\int_b^c \frac{\psi_2(t)\,dt}{(t-r)\,(t-b)^{\sigma+p}}. \tag{4.21}$$

Substituting $R(r)$ from (4.21) in (4.17) and interchanging the order of integrations we obtain

$$\eta(t)\,\psi_2(t) = T(t) - \int_b^c N(t,y)\,\psi_2(y)\,dy, \quad b<t<c \tag{4.22}$$

where $\eta(t)$, $T(t)$ are given by the equations (2.10). (4.18), (4.15) and

$$N(t, y) = \frac{\sin \sigma\pi \sin(\sigma+p)\pi}{\pi^2 (t-b)^\sigma (y-b)^{\sigma+p}} \int_a^b \frac{(b-r)^{(2\sigma+p)} \eta(r)\, dr}{(t-r)(y-r)}. \tag{4.23}$$

The equation (4.22) is a Fredholm integral equation of the secord kind from which $\psi_2$ can be determined ; $\psi_1$, $\psi_2$ are then found from (4.21), (4.14), (4.19), and (4.20). The coefficients $B_n$ satisfying the triple series equations (4.1) to (4.3) under the conditions (3.4) can be obtained from (4.6). For $p = 0$, $c = 1$, the results obtained here and those obtained by Lowndes (1968*a* ; p. 106) are identical.

## 5. Particular Cases

If we put $c = \delta$ in equations (3.1) to (3.3) and let $\delta \to \infty$, then using (2.3) we find that they reduce to the triple series equations

$$\sum_{n=0}^{\infty} A_n L_n^{(\alpha+p)}(x) = 0, \; 0<x<a, \; b<x<\infty \tag{5.1}$$

$$\sum_{n=0}^{\infty} A_n \frac{\Gamma(\alpha-\sigma+n+1)}{\Gamma(\alpha+n+1)} L_n^{(\alpha)}(x) = f(x), \; a<x<b. \tag{5.2}$$

The solution of (5.1), (5.2) which we obtain by applying the limit process (which we are by now quite familar with) is given by

$$A_n = \frac{n!}{\Gamma(\alpha+p+n+1)} \int_a^b y^{\alpha+p} \phi(y) L_n^{(\alpha+p)}(y)\, dy$$

where $\phi$ is to be determined from (3.14) and the Fredholm equation (3.17) with $M(t, y)$ and $F(t)$ as given in (3.18) and (3.13) respectively and

$$\eta(t) = t^{\alpha-\sigma} e^{t}. \tag{5.3}$$

The solution is valid under the conditions (*i*), (*ii*) and (*iv*) of (3.4) since the condition (*iii*) is not a genuine condition in this case.

Similarly, putting $c = \delta$ in equation (4.1) to (4.3) and taking the limit as $\delta \to \infty$ we find, in view of (2.3), that they become

$$\sum_{n=0}^{\infty} B_n \frac{\Gamma(\alpha-\sigma+n+1)}{\Gamma(\alpha+n+1)} L_n^{(\alpha)}(x) = g(x), \quad 0<x<a \tag{5.4}$$

$$= h(x), \quad b<x<\infty \tag{5.5}$$

$$\sum_{n=0}^{\infty} B_n L_n^{(\alpha+p)}(x) = 0 \quad , \quad a<x<b. \tag{5.6}$$

The solution of the triple series equations (5.4) to (5.6) which we obtain by applying the limit process to the solution of the triple series equations (4.1) to (4.3) is given by

$$B_n = \frac{n!}{\Gamma(\alpha+p+n+1)} \left[ \int_0^a y^{\alpha+p} \psi_1(y) L_n^{(\alpha+p)}(y)\, dy + \int_a^\infty y^{\alpha+p} \psi_2(y) L_n^{(\alpha+p)}(y)\, dy \right]$$

where $\psi_1$ and $\psi_2$ are to be found, in terms of $\psi_1$ and $\psi_2$, from (4.11) and (4.12). $\psi_2$ is to be determined from the Fredholm integral equation (4.22) in which $N$, $T$ and $\eta$ are given by (4.23), (4.18), (4 15) and (5.3). Then $\psi_1$ can be obtained from (4.21) and (4.14) taking the value of $\eta$ given in (5.6). The solution is valid under the conditions (*i*), (*ii*) and (*iv*) of (3.4).

The triple series equations (5.1), (5.2) and (5.6) with $p = 0$ are essentially those considered by Lowndes (1969). In this case we find, after making appropriate changes in the parameters, that the solutions are in complete agreement.

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