INCLUSION THEOREMS FOR THE SEQUENCE SPACES OF ALLEN

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For the sequence spaces E_r and F_r introduced by Allen, necessary and sufficient conditions on the matrix $A = (a_{nk})$ are found so that it should transform : (*i*) F_r into E_r, (*ii*) E_r into E_r, (*iii*) F_r into l and (*iv*) l into F_r.

1. INTRODUCTION

Inclusion theorems of matrix transformation deal with necessary and sufficient conditions on an infinite matrix to transform a given sequence space into a sequence space of required nature. We shall study theorems of this type in relation to sequence spaces introduced by Allen in his investigation on Kothe sequence spaces (see Cooke 1955). The following are the sequence spaces defined by Allen.

(*i*) E_r : The set of all sequences such that

$$| x_k | \leqslant Ak^r, \qquad k = 1, 2, 3, \ldots$$

(*ii*) F_r : The set of all sequences (x_k) such that

$$\sum | x_k | k^r \qquad k = 1, 2, 3, \ldots, \text{ where } r > 0$$

Allen has proved that the spaces E_r and F_r are perfect spaces in the sense of Kothe and Toeplitz. The sequence spaces of absolutely convergent series will be denoted by l. We shall find out necessary and sufficient condition on an infinite matrix so that it should transform

(1) F_r into E_r (Theorem 1)

(2) E_r into E_r (Theorem 2)

(3) F_r into l (Theorem 3)

(4) l into F_r (Theorem 4)

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2. Matrix Transformation of F_r into E_r

Let us consider the following matrix transformation

$$y_n = \sum_{k=1}^{\infty} a_{nk}\, x_k, \quad n = 1, 2, 3, \ldots \tag{2.1}$$

***Theorem* 1**—A necessary and sufficient condition that (y_n) should belong to E_r whenever (x_k) belongs to F_r is that

$$\frac{|a_{nk}|}{(nk)^r} \leqslant M, \text{ where } M \text{ is a constant} \tag{2.2}$$

***Proof*.** The condition is sufficient.

$$y_n \leqslant \sum_{k=1}^{\infty} |a_{nk}|\, |x_k| \quad n = 1, 2, 3, \ldots$$

$$= n^r \sum_{k=1}^{\infty} \frac{a_{nk}}{(nk)^r} |x_k|\, k^r .$$

Using the condition (2.2) and noting the fact that $(x_k) \in F_r$, we get $|y_n| \leqslant A\, n^r$ where A is a constant. This proves that $(y_n) \in E_r$. The condition is necessary.

Let us suppose that the condition (2.2) is not satisfied. Then there exist sequences n_p and k_p such that

$$\frac{|a_{n_p k_p}|}{(n_p\, k_p)^r} \to \infty \text{ as } p \to \infty . \tag{2.3}$$

Since the matrix $A = (a_{nk})$ is applicable to each member of F_r, we must have $(a_{nk}) \in E_r$ for each fixed n. This leads to the condition

$$|a_{nk}| \leqslant A_n\, k^r \text{ for each fixed } n \text{ and for all } k. \tag{2.4}$$

Let e_k be the sequence defined as follows. Let $x_k = 1$ and $x_j = 0$ for all $j \neq k$. Then $y_n = a_{nk}$. Since $y_n \in E_r$, we have

$$|a_{nk}| \leqslant B_k\, n^r \text{ for each fixed } k \text{ and for all } n. \tag{2.5}$$

In (2.4) and (2.5), we note that the following sequence (A_n) and (B_k) are unbounded as seen below.

If (A_n) is not unbounded, let it be bounded.

Since $\frac{|a_{nk}|}{(k^r)} \leqslant A_n \leqslant M$, we get $\frac{|a_{nk}|}{k^r} \leqslant M.$

Hence $\frac{|a_{nk}|}{n^r\, k^r} \leqslant \frac{M}{n^r}$. Since $r > 0$ we have $1/n^r \leqslant 1$ for all n. Using this fact, $\frac{|a_{nk}|}{(nk)^r} \leqslant M$ which contradicts our assumption (2.3) Hence the sequence (A_n) and similarly the sequence (B_k) are unbounded.

$$\frac{|a_{n_1 k_1}|}{(n_1 k_1)^r} > 2, \; A_{n_1} > 1, \text{ and } |a_{nk_1}| \leqslant B_{k_1} n^r \text{ for all } n. \tag{2.6}$$

Then choose $n_2 > n_1$ and $k_2 > k_1$ satisfying

$$\frac{|a_{n_2 k_2}|}{(n_2 k_2)^r} > 2(\tfrac{3}{2}+B_{K_1}) \; A_{n_1}, \; A_{n_1} < A_{n_2} \text{ and } |a_{nk_2}| \leqslant B_{k_2} n^r \text{ for all } n. \tag{2.7}$$

Proceeding in this manner, let us suppose that n_{p-1} and k_{p-1} have already been chosen. Then we can find a $n_p > n_{p-1}$ and $k_p > k_{p-1}$ such that

$$\left.\begin{array}{l} |a_{n_p k_p}| > 2^{p-1} A_{n_{p-1}} \left(P + \frac{1}{2^{p-1}} + B_{k_1} + B_{k_2} + B_{k_3} + \cdots B_{k_{p-1}}\right) \\ A_{n_{p-1}} < A_{n_p} \text{ and } |a_{nk_p}| \leqslant B_{k_p} n^r \text{ for all } n. \end{array}\right\} \tag{2.8}$$

From the above $A_{n_1} \leqslant A_{n_2} \leqslant \ldots \leqslant A_{n_p} \leqslant \ldots.$

With help of the above, we shall construct a sequence $(x_k) \in F_r$ with the supplementary condition that $|x_k| \leq 1$ for every k and such that (y_n) has a subsequence for which $(|y_n|/n^r)$ is unbounded so that (y_n) does not belong to E_r. Let us define the sequence (x_k) as follows.

$$x_{k_1} = \frac{\text{Sgn } a_{n_1 k_1}}{k^r{}_1}$$

$$x_{k_q} = \frac{\text{Sgn } a_{n_p k_p}}{2^{p-1} A_{n_{p-1}} k_p^r}, \quad p = 2, 3, \ldots \tag{2.9}$$

and $\qquad x_k = 0 \text{ for } k \neq k_1, k_2, \; . \; k_p, \ldots \; .$

It is to check that $(x_k) \in F_r$.

$$y_{n_1} = a_{n_1 k_1} x_{k_1} + \sum_{p=2}^{\infty} a_{n_1 k_p} x_{k_p}$$

$$y_{n_1} \geqslant |a_{n_1 k_1}| \, |x_{k_1}| - |\sum_{p=2}^{\infty} a_{n_1 k_p} x_{k_p}|. \tag{2.10}$$

By making use of (2.6) and (2.9), we get

$$|a_{n_1 k_1} x_{k_1}| > 2 (n_1 k_1)^r \frac{1}{k_1^r} = 2 n_1^r. \tag{2.11}$$

By making use of (2.4) and (2.9) we have the following

$$\sum_{p=2}^{\infty} |a_{n_1 k_p}|\, |x_{k_p}| \leqslant A_{n_1} k_2^r \frac{1}{2A_{n_1} k_2^r} + A_{n_1} k_3^r \frac{1}{2^2 A_{n_2} k_3^r} + \ldots$$

$$\leqslant \frac{1}{2} + \frac{1}{2^2} + \ldots < 1.$$

Therefore we have from the above

$$\left| \sum_{p=2}^{8} a_{n_1 k_p} x_{k_p} \right| < 1 . \tag{2.12}$$

With the help of (2.11) and (2.12), we can rewrite (2.10) as,

$$|y_{n_1}| > 2n_1^r - 1 \text{ so that we have } \frac{|y_{n_1}|}{n_1^r} > 2 - \frac{1}{n_1^r}$$

since

$$\frac{1}{n_1^r} < 1 \text{ for } r > 0, \text{ we have } \frac{|y_{n_1}|}{n_1^r} > 1$$

$$y_{n_2} = a_{n_2 k_1} x_{k_1} + a_{n_2 k_2} x_{k_2} + \sum_{p=3}^{\infty} a_{n_2 k_p} x_{k_p} .$$

Using (2.9) in the above, we have

$$y_{n_2} = \frac{a_{n_2 k_2}}{2A_{n_1} k_2^r} + a_{n_2} x_{k_2} + \sum_{p=3}^{\infty} a_{n_2 k_p} x_{k_p}$$

$$|y_{n_2}| \geq \frac{|a_{n_2 k_2}|}{2A_{n_1} k_2^r} - |a_{n_2 k_1} x_{k_1}| - \sum_{q=3}^{\infty} |a_{n_2 k_p} x_{k_p}| \tag{2.13}$$

$$\frac{|a_{n_2 k_2}|}{2A_{n_1} k_2^r} > \left(\frac{5}{2} + B_{k_1} \right) n_2^r \tag{2.14}$$

and $|a_{n_2 k_2} x_{k_1}| \leq B_{k_1} n_2^r$ by using (2.7)

$$\sum_{p=3}^{\infty} |a_{n_2 k_p} x_{k_p}| \leqslant A_{n_2} k_3^r \frac{1}{2^2 A_{n_2} k_3^r} + A_{n_2} k_4^r \frac{1}{2^3 A_{n_2} k_4^r} + \ldots$$

$$\leq \frac{1}{2^2} \left(1 + \frac{1}{2} + \ldots \right) < \frac{1}{2} . \tag{2.15}$$

Hence we have from the above

$$\sum_{p=3}^{\infty} |a_{n_2 k_p} x_{k_p}| < \frac{1}{2} .$$

By making use of (2.14) and (2.15) in (2.13) we get

$$|y_{n_2}| > \left(\frac{5}{2} + B_{k_1} \right) n_2^r - B_{k_1} n_2^r - \frac{1}{2}$$

Therefore

$$\frac{|y_{n_2}|}{n_2^r} > \frac{5}{2} - \frac{1}{2} = 2, \text{ since } \frac{1}{n^r} < 1 \text{ for } r > 0.$$

Thus we have proved

$$\frac{|y_{n_2}|}{n_2^r} > 2.$$

Proceeding exactly in the same manner, we can find a n_p such that $\frac{|y_{n_p}|}{n_y^r} > p$. That is $\frac{y_n}{n^r}$ tends to infinity through a subsequence of values of n. Hence (y_n) does not belong to E_r, even though $(x_k) \in F_r$. This contradiction proves that the condition is necessary.

3. Matrix Transformation of E_r into E_r

This result is contained in the following theorem.

Theorem 2—A necessary and sufficient condition that (y_n) should belong to E_r, whenever (x_k) belongs to E_r is that

$$\theta_n = \frac{1}{n^r} \sum_{k=1}^{\infty} |a_{nk}| \, k^r \leqslant M, \tag{3.1}$$

where M is a constant.

Proof: The condition is sufficient.

We have

$$|y_n| \leqslant \sum_{k=1}^{\infty} |a_{nk}| \, |x_k|.$$

Since

$$(x_k) \in E_r, \; |x_k| \leqslant A \, k^r,$$

where A is a constant.

By the given condition

$$\sum_{k=1}^{\infty} |a_{nk}| \, k^r \leqslant M \, n^r.$$

Using these two we get $|y_n| \leqslant AM\, n^r$ so that $(y_n) \in E_r$.

The condition is necessary.

If the condition is not satisfied, we shall find a subsequence (n_p) such that

$$\theta_{n_p} \to \infty \quad \text{as } p \to \infty. \tag{3.2}$$

Since A is applicable to each E_r, (a_{nk}), $k = 1, 2, \ldots$ for each fixed n must belong to F_r so that we get

$$\sum_{k=1}^{\infty} |a_{nk}| \, k^r < \infty. \tag{3.3}$$

Let (e^k) be the sequence defined as $x_k = k^r$ and $x_j = 0$ for $j \neq k$. $(e^k) \in E_r$ and hence its transform $y_n = a_{nk}\, k^r \in E_r$ for each fixed k.

Using this, we have the following condition.

$$\frac{|\, a_{nk} \,|\, k^r}{n^r} \leqslant A_k \qquad \text{for each fixed } k. \tag{3.4}$$

We shall construct a sequence (x_k) such that (x_k/k^r) is bounded with the supplementary condition $|\, x_k/k^r \,| \leqslant 1$, and show that the corresponding (y_n) does not belong to E_r.

By (3.2), we can choose a n_1 satisfying the condition,

$$\theta_{n_1} > 1 \quad \text{which is} \quad \sum_{k=1}^{\infty} |\, a_{n_1 k} \,|\, k^r > n_1^r. \tag{3.5}$$

Having chosen n_1, we can find a k_1 by (3.3) such that

$$\sum_{k=k_1+1}^{\infty} |\, a_{n_1 k} \,|\, k^r < \frac{\epsilon}{2} \tag{3.6}$$

$$\sum_{k=1}^{k_1} |\, a_{n_1 k} \,|\, k^r = \sum_{k=1}^{\infty} |\, a_{n_1 k} \,|\, k^r - \sum_{k_1+1}^{\infty} |\, a_{n_1 k} \,|\, k^r .$$

By using (3.5) and (3.6) in the above, we get

$$\sum_{k=1}^{k_1} |\, a_{n_1 k} \,|\, k^r \geqslant n_1^r - \frac{\epsilon}{2} . \tag{3.7}$$

Let us define $x = k^r$ Sgn $a_{n_1 k}$ $\quad 1 \leqslant k \leqslant k_1$

$$y_{n_1} = \sum_{k=1}^{k_1} |\, a_{n_1 k} \,|\, k^r + \sum_{k_1+1}^{\infty} a_{n_1 k}\, x_k$$

$$|\, y_{n_1} \,| \geqslant \sum_{k=1}^{k_1} |\, a_{n_1 k} \,|\, k^r - \sum_{k_1+1}^{\infty} |\, a_{n_1 k} \,|\, k^r$$

By using (3.6) and (3.7), we have from the above,

$$|\, y_{n_1} \,| \geqslant n_1^r - \frac{\epsilon}{2} - \frac{\epsilon}{2} = n_1^r - \epsilon \text{ which implies that } \frac{|\, y_{n_1} \,|}{n_1} > 1 - \epsilon$$

By (3.4), $|\, a_{nk} \,|\, k^r \leqslant A_k\, n^r$ foı each fixed k and for all n.

Hence

$$\sum_{k=1}^{k_1} |\, a_{nk} \,|\, k^r \leq n^r \{A_1 + A_2 + A_3 + \ldots + A_{k_1}\}.$$

Hence

$$\sum_{k=1}^{k_1} | a_{nk} | k^r \leq n^r C_{k_1} \tag{3.8}$$

where C_{k_1} is the sum of the k_1 terms.

Now choose $n_2 > n_1$, satisfying the following condition.

$$\theta_{n_2} > (2+2C_{k_1}) \tag{3.9}$$

which is

$$\sum_{k=1}^{\infty} | a_{nk} | k^r > n_2^r (2+2\, C_{k_1}) .$$

Having choosen $n_2 > n_1$, choose $k_2 > k_1$ such that

$$\sum_{k_2+1}^{\infty} | a_{n_2} k^r | < \frac{\epsilon}{2} \tag{3.10}$$

$$\sum_{k_1+1}^{k_2} | a_{n_2 k} | k^r = \sum_{k=r}^{\infty} | a_{n_2 k} | k^r - \sum_{k=1}^{k_1} | a_{n_2 k} | k^r - \sum_{k_1+1}^{\infty} | a_{n_2 k} | k^r .$$

With the help of (3:8), (3.9) and (3.10), we have from the above,

$$\sum_{k_1+1}^{k_2} | a_{n_2 k} | k^r \geqslant n_2^r (2+2\, C_{k_1}) - C_{k_1} n_2^r - \frac{\epsilon}{2} . \tag{3.11}$$

Define

$$x_k = k^r \text{ Sgn } a_{n_2 k}, \quad k_1+1 < k \leq k_2$$

$$y_{n_2} = \sum_{k=1}^{k_1} a_{n_2 k}\, x_k + \sum_{k_1+1}^{k_2} a_{n_2 k}\, x_k + \sum_{k_2+1}^{\infty} a_{n_2 k}\, x_k .$$

By using (3.8), (3.10) and (3.11), we get

$$| y_{n_2} | \geqslant n_2^r (2+2C_{k_1}) - C_{k_1} n_2^r - \frac{\epsilon}{2} - C_{k_1} n_2^r - \frac{\epsilon}{2}$$

$$| y_{n_2} | \geqslant 2\, n_2^r - \epsilon$$

which implies that

$$\frac{| y_{n_2} |}{n_2^r} > 2-\epsilon .$$

Proceeding in this manner, we can choose $n_p > n_{p-1}$ and $k_p > k_{p-1}$ such that

$$\theta_{n_p} > \{p+2\, (C_1+C_2+\ldots+C_{k_{p-1}})\}$$

and

$$\sum_{k=k_p+1}^{\infty} | a_{n_p k} | k^r < \frac{\epsilon}{2} .$$

Define

$$x_k = k^r \text{ Sgn } a_{n_p k} \text{ for } k_{p-1} < k \leq k_p.$$

Continuing as in the case of y_{n_1} and y_{n_2}, we can show that

$$\frac{| y_{n_p} |}{n_p^r} > p - \epsilon.$$

Since ϵ is arbitrary, (y_n) tends to infinity through a subsequence (n_p). Hence (y_n) does not belong to E_r.

4. Theorems

It was proved by Knopp and Lorentz (1949) that a necessary and sufficient condition that for a matrix $A = (a_{nk})$ to transform l into l is that there exists a constant M such that

$$\sum_{h=1}^{\infty} | a_{nk} | \leq M \text{ for } k = 1, 2, 3, \ldots \tag{4.1}$$

We note that the mapping (x_k) to $(x_k k^r)$ is a one-to-one correspondence between F_r and l. Hence the matrix (a_{nk}) maps F_r into l if, and only if, the matrix (a_{nk}/k^r) maps l into l. Similarly the matrix (b_{nk}) maps l into F_r if, and only if, the matrix $(b_{nk} n^r)$ is a mapping l into l. These observations together with the condition (4.1) lead to the following two theorems.

Theorem 3—A necessary and sufficient condition that $A = (a_{nk})$ should transform F_r into l is $\sum_{h=1}^{\infty} | a_{nk} | \leq k^r M$ where M is a constant.

Theorem 4 A necessary and sufficient condition that $A = (a_{nk})$ should transform l into F_r is $\sum_{n=1}^{\infty} | a_{nk} | n^r \leqslant M$, where M is a constant.

The other inclusion theorems on matrix transformations between the sequence spaces of Allen and the well-known sequence spaces will be given in another paper.

References

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