ON A RANDOM TRIGONOMETRIC POLYNOMIAL

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For the random trigonometric polynomial

$$T(\theta) = \sum_{K=1}^{n} a_K \cos k\theta$$

when a_K are independent and uniformly distributed in $(-1, 1)$ we find the average number of real roots for large n.

§1. Consider the random trigonometric polynomial

$$T = T(\theta) = a_1 \cos \theta + a_2 \cos 2\theta + \ldots + a_n \cos n\theta \tag{1.1}$$

where a_i are independent and uniformly distributed in $(-1, 1)$. Let $N_n(T; \alpha, \beta)$ denote the average number of real roots of (1.1) falling with in (α, β), where multiple roots are counted only once. In this note we estimate the asymptotic value of $N_n(T; \alpha, \beta)$ for large values of n. At the end we state few results which can be proved under the same lines of the proof in section 2.

Dunnage (1966) discussed the case when a_i are independent normal random variables. Das (1968) took the polynomial of the from $\sum_{k=1}^{n} a_k b_k \cos k\theta$, where a_k are independent normal random variables and b_k are set of positive numbers and estimated $N_n(T; 0, 2\pi)$ for large n. Sambandham (1976) estimated $N_n(T; 0, 2\pi)$ for the polynomial $\sum_{k=1}^{n} a_k b_k \cos k\theta$ when a_k are dependent normal random variables and b_k are positive numbers. Kac (1943, 1949) took the polynomial of the form $\sum_{k=0}^{n-1} a_k t^k$ and estimated the average number of real roots when a_k are independent and normally distributed and uniformly distributed in $(-1, 1)$ respectively.

§2. We know that if a_i are independent random variables each having $\sigma(u)$ as its distribution function then the average number of roots of (1.1)

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falling within (α, β) is given by

$$N_n(T;\alpha,\beta) = (1/2\pi^2)\int_{-\infty}^{\infty}\int_{\alpha}^{\beta}\int_{-\infty}^{\infty}\eta^{-2}\Big[\prod_{k=p}^{n}\rho_{,}(\xi\cos k\theta)$$
$$-(1/2)\Big\{\prod_{k=1}^{n}\rho(\xi\cos k\theta+\eta k\sin k\theta)$$
$$+\prod_{k=1}^{n}\rho(\xi\cos k\theta-\eta k\sin k\theta)\Big\}\Big]d\eta\, d\theta\, d\xi$$

where

$$\rho(\xi) = \int_{-\infty}^{\infty}\exp(i\,\xi\,u)\,d\sigma(u)$$

is the characteristic function of the distribution function $\sigma(u)$.

Since we have assumed that a_i are uniformly distributed in $(-1, 1)$ we have

$$\rho(t) = [\sin t]/t.$$

Thus we may write

$$N_n(T;\alpha,\beta) = (1/2\pi^2)\int_{\alpha}^{\beta}\phi_n(\theta)\,d\theta$$

where

$$\phi_n(\theta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\eta^{-2}\Big[\prod_{k=1}^{n}\rho(\xi\cos k\theta)-(\tfrac{1}{2})\Big\{\prod_{k=1}^{n}\rho(\xi\cos k\theta+\eta k\sin k\theta)$$
$$+\prod_{k=1}^{n}\rho(\xi\cos k\theta-\eta k\sin k\theta)\Big\}\Big]d\xi\, d\eta.$$

Our main object of this note is to show that

$$N_n(T;0,2\pi) \sim 2n/\sqrt{3}. \tag{2.1}$$

Since the total average number of real roots is easily seen to be twice the average number of roots falling within $(0, \pi)$, it is sufficient to investigate $N_n(T;0,\pi)$. Hence it is enough if we show that

$$N_n(T;0,\pi) \sim n/\sqrt{3}. \tag{2.2}$$

For a suitable ϵ, with $0<\epsilon<\pi$, we have as $n\to\infty$, $\phi_n(\theta)$ approaches

$$\phi_\infty(\theta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\eta^{-2}\Big[\prod_{k=1}^{\infty}\rho(\xi\cos k\theta)-(1/2)\Big\{\prod_{k=1}^{\infty}\rho(\xi\cos k\theta+\eta k\sin k\theta)$$
$$+\prod_{k=1}^{\infty}\rho(\xi\cos k\theta-\eta k\sin k\theta)\Big\}\Big]d\xi\, d\eta.$$

Since the convergence of $\phi_\infty(\theta)$ is uniform for $0\le\theta\le\epsilon$ we find

$$\lim_{n\to\infty} N_n(T;0,\epsilon) = (1/2\pi^2)\int_{0}^{\epsilon}\phi_\infty(\theta)\,d\theta$$

and this in turn implies

$$N_n\ (T\ ;0,\ \epsilon) = o\ (1). \tag{2.3}$$

Similar argument will show that

$$N_n\ (T\ ;\ \pi-\epsilon,\ \pi) = o(1). \tag{2.4}$$

Therefore it is sufficient if we prove

$$N_n\ (T\ ;\ \epsilon,\ \pi-\epsilon) \sim n/\sqrt{3}. \tag{2.5}$$

To prove this we introduce the following useful abbreviations. Let

$$A_n = \sum_{k=1}^{n} \cos^2 k\theta$$

$$B_n = \sum_{k=1}^{n} k \cos\ k\theta \sin k\theta$$

$$C_n = \sum_{k=1}^{n} k^2 \sin^2 k\theta$$

$$R_n = B_n/\sqrt{A_n\ C_n},$$

$$F_n\ (\eta) = \int_{-\infty}^{\infty} \prod_{k=1}^{n} \rho\ (\xi \cos\ k\theta/\sqrt{A_n} + \eta\ k \sin k\theta/\sqrt{C_n})\ d\xi$$

$$G_n\ (\eta) = \int_{0}^{\infty} \prod_{k=1}^{n} \rho\ (\xi\ \cos k\theta/\sqrt{A_n} + \eta\ k\ \sin k\theta/\sqrt{C_n})\ d\xi$$

From this we find

$$F_n^{\xi}\ (\eta) = G_n\ (\eta) + G_n\ (-\eta).$$

If we introduce the new variables $\xi' = \xi/\sqrt{A_n}$ and $\eta' = \eta/\sqrt{C_n}$ we obtain

$$\phi_n(\theta) = (C_n/A_n)^{1/2} \int_{-\infty}^{\infty} \eta^{-2}\ [F_n\ (0) - (1/2)(F_n\ (\eta) + F_n\ (-\eta)\ )]\ d\eta$$

$$= 2(C_n/A_n)^{1/2} \int_{0}^{\infty} \eta^{-2}\ [2G_n\ (0) - G_n\ (\eta) - G_n\ (-\eta)] d\eta.$$

If θ is such that $\epsilon \leqslant \theta \leq \pi - \epsilon$ then we can show that

$$A_n = n/2 + o(\sqrt{n})$$

$$B_n = o(n^{3/2})$$

$$C_n = n^{3/2} + o(n^{5/2})$$

$$R_n = o\ (n^{-1/2}).$$

For this θ we get

$$\int_{\epsilon}^{\pi-\epsilon} [(C_n/A_n)(1-R_n^2)]^{1/2} d\theta = (\pi - 2\epsilon)(n/\sqrt{3})\ (1 + o(n^{-1/2})). \tag{2.6}$$

Now let us take

$$\psi_n(\theta)=2\int_0^{\infty}\eta^{-2}[2G_n(0)-G_n(\eta)-G(-\eta)]d\eta$$

so that

$$N_n(T;\epsilon,\pi-\epsilon)=(1/2\pi^2)=\int_{\epsilon}^{\pi-\epsilon}(C_n/A_n)^{1/2}\psi_n(\theta)\,d\theta. \tag{2.7}$$

Since the results in Kac (1949) section 6, holds for our case also we point out only the proof of the following, that is, we prove for $\epsilon \leq \theta \leqslant \pi-\epsilon$ and $n > n_0$

$$|\psi_n(\theta)| < \gamma_1 \tag{2.8}$$

and

$$|\psi_n(\theta)-\pi\sqrt{1-R_n^2}| < \epsilon_1 \tag{2.9}$$

(here and in the following we denote the absolute constants by ν with subscripts) for small ϵ_1. Then

$$(1/2\pi^2\int_{\epsilon}^{\pi-\epsilon}(C_n/A_n)^{1/2}(\pi\sqrt{1-R_n^2}-\epsilon_1)\,d\theta \leq N_n(T;\epsilon,\pi-\epsilon)$$

$$\leq(1/2\pi^2)\int_{\epsilon}^{\pi-\epsilon}(C_n/A_n)^{1/2}(\pi\sqrt{1-R_n^2}+\epsilon_1)\,d\theta+o(1).$$

From this and making use of (2.6) we get (2.5) and hence (2.1). Hence it is enough if we prove (2.8) and (2.9).

§3. We now establish (2.8) and (2.9).

Since $|\rho(u)| \leq 1$ for $\epsilon \leq \theta \leq \pi$ we can find an absolute constant γ_2 such that $|G_n(\eta)| < \gamma_2$. This leads to the proof of (2.8).

To show (2.9) we proceed as follows. We can find γ_3 such that for $\epsilon \leq \theta \leq \pi$, $n > n_0$ and $|\eta| < \epsilon$ such that

$$[|2G_n(0)-G_n(\eta)-G_n(-\eta)|]/\eta^2 < \gamma_3.$$

Also if

$$H(\eta)=\int_0^{\infty}\exp[-1/6)(\xi^2+2R_n\xi\eta+\eta^2)]\,d\xi,$$

for η ranging between ϵ and $1/\epsilon$ then we know that from Kac (1949) for suitable ϵ_1

$$|G_n(\eta)-H_n(\eta)| < \epsilon_1.$$

This shows that for $n > n_0$ and θ in the range $\epsilon \leq \theta \leq \pi-\epsilon$

$$\left|2\int_{\epsilon}^{1/\epsilon}\eta^{-2}[2G_n(0)-G_a(\eta)-G_n(-\eta)]\,d\eta\right.$$

$$\left.-2\int_{\epsilon}^{1/\epsilon}\eta^{-2}[2H_n(0)-H_n(\eta)-H_n(-\eta)]\,d\eta\right| < \epsilon_2$$

for suitable ϵ_2. By elementary computation we find

$$2 \int_0^\infty \eta^{-2} \left[2H_n(0)-H_n(\eta)-H_n(-\eta)\right] d\eta = \pi\sqrt{1-R_n^2}.$$

This result leads us to

$$\left| \pi\sqrt{1-R_n^2} - 2\int_\epsilon^{1/\epsilon} \eta^{-2} \left[2H_n(0)-H_n(\eta)-H_n(-\eta)\right] d\eta \right| = o(\epsilon)$$

uniformly for $0 < \theta \leq \pi$. Combining these results we find

$$|\psi_n(\theta)-\pi\sqrt{1-R_n^2}| < o(\epsilon)+o(1)+\nu_4.$$

This leads to the proof of (2.9).

§4. Without proof we state the following results.

If a_k are sequence of mutually independent uniformly distributed and b_k are positive constants then the average number of real roots of $T=\sum_{k=1}^{n} a_k b_k \cos k\theta$ when the multiple roots are counted only once are

(*i*) for $b_k = k^m, m \geq 0$

$N_n(T; 0, 2\pi) = (2m+1 / 2m+3)^{1/2} 2n+o(n^{1/2})$.

(*ii*) for $b_k = k^m, 0 > m > -1/2$

$N_n(T; 0, 2\pi) = (2m+1 / 2m+3)^{1/2} 2n+ o(n^{1/2-m})$

(*iii*) for $b_k = k^{-1/2}$

$N_n(T; 0, 2\pi) = 2n(2\log n)^{-1/2}+ o(n/\log n)$

(*iv*) for $b_k = k^{-m_1}$, $1/2 < m_1 < 3/2$

$$\left[\frac{n^{3-2m_1}}{Q(6-4m_1)}\right]^{1/2}(1+o(1)) < N_n(T; 0, 2\pi)$$

$$< \left[\frac{n^{3-2m_1}}{P(6-4m_1)}\right]^{1/2}(1+o(1))$$

where

$$P = (7/16) \sum_{k=1}^{\frac{n-2}{4}} (4k-2)^{-2m_1} \text{ and } Q = \sum_{k=1}^{n} k^{-2m_1}.$$

(*v*) for $b_k \leq k^{-3/2}$

$N_n(T; 0, 2\pi) = o(\log n)$.

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