ON THE CHARACTER OF MAGNETIC FIELD IN THE HYDROMAGNETIC GENERALIZED BE'NARD PROBLEM

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The present paper investigates the stability of a continuously stratified layer of viscous incompressible fluid of finite electrical conductivity statically confined between two horizontal and perfectly conducting boundaries of different but uniform temperature in the presence of a uniform vertical magnetic field acting opposite to the direction of gravity. The initial stratification which might be produced for example by a dissolved solute of negligible diffusivity is assumed to be of the exponential type namely $\rho=\rho_0\ e^{-\delta z}$ where δ is a constant which can be positive or negative and thus makes the initial stratification a monotone decreasing or increasing function of z respectively, ρ_0 is a positive constant and z the vertical coordinate while the temperature of the lower boundary is taken to be greater than that of the upper one, It is shown that marginal state, if exists, is definitely oscillatory. Further, for negative values of δ the system is unstable while for $\delta>0$ the Rayleigh number and the frequency of oscillation are calculated at the marginal state and the stabilizing charaeter of the magnetic field is established in situations where the magnetic Prandtl number is less than the thermal Prandtl number, a condition which is met by a large margin under most terrestrial conditions.

1. INTRODUCTION

The problem of the onset of thermal instability in a viscous incompressible layer statically confined between two horizontal boundaries and uniformly heated from below is known as the Bénard problem. The analysis of Rayleigh (1916), Jeffreys (1926), Low (1929), Pellew and Southwell (1940) and others have clearly shown that what decides the stability or otherwise of the above configuration is the numerical value of the non-dimensional parameter

$$R= \frac{g\,\alpha\,\beta\,d^4}{\kappa\nu},$$

called the Rayleigh number where g stands for the gravity, α the coefficient of volume expansion of the liquid, $\beta>0$ the uniform adverse temperature gradient maintained between the two boundaries, d the depth of the layer, κ the coefficient of thermometric conductivity and ν the coefficient of kinematic viscosity. Further, instability must set in when R exceeds a certain critical value R_c and when R just exceeds R_c a stationary pattern of motions must come to prevail. On account of its applications in the problems of meteorology and oceanography and various other fields of practical importance many authors in recent years have extended the Bénard model by taking into account various other factors which are relevant to certain real physical problems. Chandrasekhar (1952) has considered the effect of a uniform vertical magnetic field acting opposite to the direction of gravity on the Bénard problem and showed that magnetic field has a stabilizing character in this situation. It is interesting to note that in Chandrasekhar's model the marginal state could either be stationary or oscillatory for which sufficient conditions are obtained. In particular, for the case when the magnetic Prandtl number is less than the thermal Prandtl number it is the stationary pattern of motions which manifest at the marginal state. Banerjee (1972) investigated the effect of the initial non-homogeneity of the fluid by considering the Bénard problem wherein the liquid has an initial continuous density stratification given by $\rho=\rho_0\, e^{-\delta z}$, δ being a constant which can be positive, zero or negative (this problem will be referred to as the generalized Bénard problem in the subsequent discussion) and showed the stabilizing or destabilizing character of such an initial density distribution in the above respective cases. However, the character of the marginal state in this problem is shown to be definitely oscillatory when δ is positive while for negative values of δ there does not exist any marginal state and the system is unstable through non-oscillatory modes. Thus in some respect the effect of a magnetic field and an initially stable stratification (i. e. $\delta>0$) acting separately on the onset of thermal instability in layers of liquid heated from below are remarkably alike : they both inhibit the onset of instability. However, in some other respect they do have dissimilar tendencies. Thus while in the presence of the magnetic field instability could set in either as overstability or as stationary convection, it definitely sets in as overstability when an initially stable density distribution is present. On these accounts one must not suppose that acting together magnetic field and an initial density distribution will reinforce each other in every respect. On the contrary it becomes desirable to study the Bénard problem in the presence of both the magnetic field and the initial density stratification specially with a point of view to investigate the manner in which instability sets in and the character of the magnetic field on the generalized Bénard problem. The present paper is precisely in this direction. It is shown that the marginal state, if exists, is definitely oscillatory. Further, for negative values of δ the system is unstable through non-oscillatory modes. On the other hand, for $\delta > 0$ Rayleigh number and the frequency of oscillations are calculated at the marginal state and the stabilizing character of the magnetic field is established in situations where the magnetic Prandtl number is less than the thermal Prandtl number, a condition which is met by a large margin under most terrestrial conditions.

2. The Physical Problem and Its Formulation

An infinite horizontal layer of an initially stratified viscous incompressible fluid of finite electrical conductivity is statically confined between two horizontal boundaries maintained at uniform but different temperatures with the temperature of the lower boundary being greater than that of the upper one in the presence of a uniform vertical magnetic field acting opposite to the direction of gravity. The problem is to investigate the stability of this initial stationary state.

Let the origin be taken on the lower boundary $z = 0$ with the positive direction of the z-axis along the vertically upward direction. Let $z = d\,(> 0)$ denote the upper boundary and T_0 and T_1 $(< T_0)$ respectively denote the uniform temperatures of the lower and upper boundaries. The xy plane then constitutes the horizontal plane $z = 0$.

The relevant equations of momentum, heat conduction, state (Banerjee (1972, 1973), magnetic induction, continuity and incompressibility which govern this problem are given by

$$\rho\left[\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}\right]=-\frac{\partial p}{\partial x_i}+\rho X_i+\mu\nabla^2 u_i+$$

$$+\frac{\mu_e}{4\pi}H_j\frac{\partial H_i}{\partial x_j}-\frac{\mu_e}{8\pi}\frac{\partial}{\partial x_i}(H_j H_j) \tag{1}$$

$$\frac{\partial T}{\partial t}+u_j\frac{\partial T}{\partial x_j}=\frac{\kappa_1}{\rho c_v}\frac{\partial^2 T}{\partial x_j\,\partial x_j}-\frac{p}{\rho c_v}\frac{\partial u_j}{\partial x_j}+\frac{\Phi}{\rho c_v} \tag{2}$$

$$\rho=\rho_0\left[e^{-\delta z}+\alpha(T_0-T)\right] \tag{3}$$

$$\frac{\partial H_i}{\partial t}+u_j\frac{\partial H_i}{\partial x_j}=H_j\frac{\partial u_i}{\partial x_j}+\eta\nabla^2 H_i \tag{4}$$

$$\frac{\partial H_j}{\partial x_j}=0 \tag{5}$$

$$\frac{\partial u_j}{\partial x_j}=0 \tag{6}$$

and

$$\frac{\partial \rho}{\partial t}+u_j \frac{\partial \rho}{\partial x_j}=0, \tag{7}$$

where x_j ($j = 1, 2, 3$) respectively denote the x, y and z coordinates u_j, X_i ($= \{0, 0, -g\}$), H_i ($i = 1, 2, 3$; $j = 1, 2, 3$) respectively denote the components of the velocity, external force and magnetic field in the x, y and z directions, T the temperature. ρ the density, p the presure, μ_e the magnetic permeability and α, μ, η and κ_1 (assumed constants) respectively stand for the coefficients of volume expansion, viscosity, magnetic diffusivity and heat conductivity, while c_v stands for specific heat of the fluid at constant volume. Further,

$$\Phi = 2\mu\, e_{ij}^2 - \tfrac{2}{3}\, \mu\, e_{jj}^2 \tag{8}$$

with

$$e_{ij} = \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right). \tag{9}$$

Clearly, the initial stationary state whose stability we wish to examine is then characterized by the following solutions for the velocity, temperature, density, pressure and magnetic field respectively :

$$\left.\begin{aligned} u_j &\equiv (0, 0, 0) \\ T &= T_0-\beta z \\ \rho &= \rho_0\,[e^{-\delta z}+\alpha\,(T_0-T)] \\ p &= p_0-g\rho_0\left[\frac{1}{\delta}\,(1-e^{-\delta z})+\frac{\alpha\beta z^2}{2}\right] \\ H_i &= (0, 0, H) \end{aligned}\right\} \tag{10}$$

where p_0 and ρ_0 are the pressure and density at the lower boundaty $z = 0$ and $\beta = \dfrac{T_0-T_1}{d}$ is the uniform maintained adverse temperature gradient.

Let the initial state described by eqns. (10) be slightly perturbed so that the perturbed state (denoted by primed symbols) is given by

$$\left.\begin{aligned} u_j' &= (u, v, w) \\ T' &= T_0-\beta z+\theta \\ \rho' &= \rho_0\left[e^{-\delta z}+\frac{\Delta\rho}{\rho_0}+\alpha\,(T_0-T-\theta)\right] \\ p' &= p+\Delta p \\ H' &= (h_x, h_y, h_z+H) \end{aligned}\right\} \tag{11}$$

where (u, v, w), θ, $\Delta\rho$, Δp and (h_x, h_y, h_z) are the respective perturbations in the velocity, temperature, density, pressure and magnetic fields.

Then the linearized perturbation equations (using Boussinesq approximation the and small-M-approximation (Banerjee 1972, 1973)) of momentum, heat conduction, magnetic induction, continuity and incompressibility become

$$\rho_0 \frac{\partial u}{\partial t} = - \frac{\partial}{\partial x} (\Delta p) + \mu \nabla^2 u + \frac{\mu_e H}{4\pi} \left[\frac{\partial h_x}{\partial z} - \frac{\partial h_z}{\partial x} \right] \tag{12}$$

$$\rho_0 \frac{\partial v}{\partial t} = - \frac{\partial}{\partial y} (\Delta p) + \mu \nabla^2 v + \frac{\mu_e H}{4\pi} \left[\frac{\partial h_y}{\partial z} - \frac{\partial h_z}{\partial y} \right] \tag{13}$$

$$\rho_0 \frac{\partial w}{\partial t} = - \frac{\partial}{\partial z} (\Delta p) + \mu \nabla^2 w + g \alpha \rho_0 \theta - g \Delta \rho \tag{14}$$

$$\frac{\partial \theta}{\partial t} = \beta w + \kappa \nabla^2 \theta, \quad \kappa = \frac{\kappa_1}{\rho_0 c_v} \tag{15}$$

$$\frac{\partial h_x}{\partial t} = H \frac{\partial u}{\partial z} + \eta \nabla^2 h_x \tag{16}$$

$$\frac{\partial h_y}{\partial t} = H \frac{\partial v}{\partial z} + \eta \nabla^2 h_y \tag{17}$$

$$\frac{\partial h_z}{\partial t} = H \frac{\partial w}{\partial z} + \eta \nabla^2 h_z \tag{18}$$

$$\frac{\partial h_x}{\partial x} + \frac{\partial h_y}{\partial y} + \frac{\partial h_z}{\partial z} = 0 \tag{19}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{20}$$

$$\frac{\partial}{\partial t} (\Delta \rho) - \rho_0 \, \delta w e^{-\delta z} = 0. \tag{21}$$

From equations (12) and (13), and (16) and (17) it follows that

$$\rho_0 \frac{\partial \zeta}{\partial t} = \mu \nabla^2 \zeta + \frac{\mu_e H}{4\pi} \frac{\partial \xi}{\partial z} , \tag{22}$$

$$\frac{\partial \xi}{\partial t} = \eta \nabla^2 \xi + H \frac{\partial \zeta}{\partial z} , \tag{23}$$

where ζ and $\frac{\xi}{4\pi}$ are respectively the components of vorticity and current density.

Analysing the perturbations in terms of normal modes by seeking solutions whose dependence on x, y and t is given by

$$\exp \left[i (k_x x + k_y y) + nt \right] \tag{24}$$

eqns. (12)—(23) become

$$\rho_0\, n\, u = -i\, k_x \Delta p + \mu \left(\frac{d^2}{dz^2} - k^2\right) u + \frac{\mu_e H}{4\pi}\left[\frac{dh_x}{dz} - ik_x\, h_z\right] \tag{25}$$

$$\rho_0\, n\, v = -i\, k_y \Delta p + \mu \left(\frac{d^2}{dz^2} - k^2\right) v + \frac{\mu_e H}{4\pi}\left[\frac{dh_y}{dz} - i\, k_y\, h_z\right] \tag{26}$$

$$\rho_0\, n\, w = -\frac{d}{dz}(\Delta p) + \mu \left(\frac{d^2}{dz^2} - k^2\right) w + g \propto \rho_0\, \theta - g \Delta \rho \tag{27}$$

$$n\,\theta = \beta w + \kappa \left(\frac{d^2}{dz^2} - k^2\right)\theta \tag{28}$$

$$nh_x = H\frac{du}{dz} + \eta\left(\frac{d^2}{dz^2} - k^2\right)h_x \tag{29}$$

$$nh_y = H\frac{dv}{dz} + \eta\left(\frac{d^2}{dz^2} - k^2\right)h_y \tag{30}$$

$$nh_z = H\frac{dw}{dz} + \eta\left(\frac{d^2}{dz^2} - k^2\right)h_z \tag{31}$$

$$i\, k_x\, h_x + i\, k_y\, h_y = -\frac{d}{dz}(h_z) \tag{32}$$

$$i\, k_x\, u + ik_y\, v = -\frac{dw}{dz} \tag{33}$$

$$n\,\Delta\rho = \delta\, \rho_0\, w\, e^{-\delta z} \tag{34}$$

$$\rho_0\, n\, \zeta = \mu\left(\frac{d^2}{dz^2} - k^2\right)\zeta + \frac{\mu_e H}{4\pi}\frac{d\xi}{dz} \tag{35}$$

$$n\,\xi = \eta\left(\frac{d^2}{dz^2} - k^2\right)\xi + H\frac{d\zeta}{dz}, \tag{36}$$

where $k=\sqrt{k_x^2+k_y^2}$ is the wavenumber of the perturbation, n is a constant which can be complex and $u, v, w, \Delta p, h_x, h_y, h_z, \theta, \Delta\rho, \zeta$ and ξ are now functions of z only. Now using the nondimensional quantities defined by

$$z_* = z/d\,,\ \sigma_* = \frac{nd^2}{\nu}\,;\ D_* = dD\,;\ a_* = kd\,;\ \theta_* = \theta$$

$$w_* = w\,;\ p_{1*} = \frac{\nu}{\kappa}\,;\ p_{2*} = \frac{\nu}{\eta}\,;\ M_* = d\delta\,;$$

where $\nu=\mu/\rho_0$, and dropping the asterisk for convenience in writing we derive the following equations from the system of eqn. (25)-(36);

$$(D^2 - a^2 - p_1\,\sigma)\theta = -\left(\frac{\beta d^2}{\kappa}\right)w \tag{37}$$

$$(D^2 - a^2 - p_2\sigma)\, h_z = -\left(\frac{Hd}{\eta}\right)Dw \tag{38}$$

$$(D^2-a^2)(D^2-a^2-\sigma)w+\frac{a^2 g\, Md}{n\, \nu}w+\left(\frac{\mu_e Hd}{4\pi\rho_0\nu}\right)D(D^2-a^2)h_z$$
$$=\frac{g \alpha d^2 a_2}{\nu}\theta \tag{39}$$

$$(D^2-a^2-p_2\sigma)\xi=-\left(\frac{Hd}{\eta}\right)D\zeta, \tag{40}$$

$$(D^2-a^2-\sigma)\zeta=-\left(\frac{\mu_e Hd}{4\pi\rho_0\nu}\right)D\xi. \tag{41}$$

We note that in deriving the equation (39) we have again made use of the small - M - approximation (as done by Banerjee 1972, 1973 ; i. e. $Re^{-Mz} \approx R_2$ and that equations (37)—(41) coincide with eqns. (119)—(123) on page 164 of Chandrasekhar (1961) when the fluid is initially homogeneous.

Solutions of eqns. (37)—(41) must be sought which satisfy the following boundary conditions, the boundaries being assumed free and perfect conducting :

$$0=w=D^2w=h_z=\theta=D\xi=D\zeta \tag{42}$$

on $z=0$ and 1. We observe that the equations and boundary conditions for ξ and ζ are formally independent of the others.

From eqns. (37)—(41), we obtain the following equation governing

$$W=\frac{\beta d^2}{\kappa}w \tag{43}$$

namely

$$\sigma(D^2-a^2)(D^2-a^2-\sigma)(D^2-a^2-p_2\sigma)(D^2-a^2-p_1\sigma)W+\frac{a^2 R_2}{p_1}\times$$
$$\times(D^2-a^2-p_1\sigma)W-\sigma Q D^2(D^2-a^2)(D^2-a^2-p_1\sigma)W$$
$$=\sigma R_1 a^2(D^2-a^2-p_2\sigma)W \tag{44}$$

where

$$\left.\begin{aligned} R_1' &= \frac{g \alpha \beta d^4}{\kappa \nu} \\ R_2 &= \frac{g\delta d^4}{\kappa\nu} \\ Q &= \frac{\mu_e H^2 d^2}{4\pi\rho_0\nu\eta} \end{aligned}\right\} \tag{45}$$

and

are respectively the Rayleigh number, the Rayleigh number corresponding to the initial nonhomogeneity and the Chandrasekhar number.

It is clear that w and W satisfy the same boundary conditions.

Equations (37)—(41) together with boundary conditions (42) present an eigen value problem for σ given values of the other parameters and a given state of the system is stable, marginal or unstable provided the real part σ_r of σ is negative, zero or positive respectively. Further, if $\sigma_r=0$ implies σ_i (the imaginary part of σ)$=0$ for every wave number 'a', then the principle of exchange of stabilities is valid ; otherwise, we will have overstability at least when instability sets in as certain modes.

3. Theorems

We now prove the following theorems :

Theorem 3.1—The principle of exchange of stabilities is not valid for the problem under discussion.

Proof : If possible let the priciple of exchange of stabilities be satisfied so that $\sigma=0$ is allowed by the governing equations and boundary conditions. We then have from equn (44)

$$(D^2-a^2)^2\, w=0. \tag{46}$$

Equation (46) together with the relavant boundary conditions on w, namely $w=0=D^2 w$ on $z=0$ and 1 yield $w\equiv 0$ as the only solution.

Using $w\equiv 0$, we have from eqn. (37)

$$(D^2-a^2)\,\theta=0. \tag{47}$$

The only solution of eqn. (47) which satisfy the boundary conditions $\theta=0$ on $z=0$ and 1 is $\theta\equiv 0$ Likewise, it clearly follows from eqn. (38) together with the appropriate boundary conditions that $h_z\equiv 0$. Further, eqns. (40) and (41) reduce to

$$(D^2-a^2)\,\xi = -\left(\frac{Hd}{\eta}\right)D\zeta \tag{48}$$

$$(D^2-a^2)\,\zeta = -\left(\frac{\mu_e\, Hd}{4\pi\,\rho_0 \nu}\right)D\xi\,. \tag{49}$$

From eqns. (48) and (49), we have

$$(D^2-a^2)\xi = \left(\frac{Hd}{\eta}\right)\left(\frac{\mu_e Hd}{4\pi\rho_0\nu}\right)D^2\xi\,. \tag{50}$$

Writing

$$(D^2-a^2)\,\xi = L \tag{51}$$

we have from eqn. (50)

$$(D^2-a^2)\,L=\left(\frac{Hd}{\eta}\right)\left(\frac{\mu_e Hd}{4\pi\rho_0\nu}\right)D^2\xi \tag{52}$$

and from t'e boundary conditions satisfied by ξ and ζ it follows that

$$L = 0 = D\xi \text{ at } z = 0 \text{ and } 1. \tag{53}$$

Now, multiplying eqn. (52) by $D^2\bar{\xi}$ ($\bar{\xi}$ being the complex conjugate of ξ) throughout, integerating the resulting equation over the vertical range of z and substituting for $\int_0^1 LD^2\bar{\xi}dz$ from eqn. (52) we have by making use of the boundary conditions (53), the following equation :

$$\int_0^1 [\,|D\xi|^2+a^2\,|\xi|^2]\,dz = \frac{-1}{\left(\frac{Hd}{\eta}\right)\left(\frac{\mu_e Hd}{4\pi\rho_0 \nu}\right)} \int_0^1 [\,|DL|^2+a^2\,|L|^2]\,dz. \tag{54}$$

Equation (54) clearly shows that $\xi \equiv 0$. Further, since ξ and ζ satisfy the same equations and boundary conditions, it follows that $\zeta \equiv 0$.

Now, $\zeta \equiv 0$ together with eqn. (33) imply that $u = v \equiv 0$ and $\xi \equiv 0$ together with equation (32) imply that $h_x = h_y = 0$. Thus $\sigma = 0$ corresponds to the trivial solution which is contrary to the starting assumption that the initial stationary state solution is nontrivially perturbed. This establishes the theorem.

Theorem 3.2—For $R_2 > 0$, the Rayleigh number and the frequency of oscillations at the marginal state are respectively given by

$$R_1 = \frac{R_2\,(p_1+p_2)}{p_1} - (\pi^2+a^2)\left[\frac{p_1+p_2+p_1p_2}{a^2}\right]\left[\frac{-B-\sqrt{B^2-4AC}}{2A}\right] + \frac{Q\pi^2\,(\pi^2+a^2)}{a^2}+\frac{(\pi^2+a^2)^3}{a^2} \tag{55}$$

$$\sigma_i^2 = \frac{-B-\sqrt{B^2-4AC}}{2A} \tag{56}$$

where

$$A = -\,p_1\,p_2^2\,(\pi^2+a^2)\,(1+p_1) < 0 \tag{57}$$

$$B = p_2^2\,a^2\,R_2+p_1\pi^2\,(p_2-p_1)\,(\pi^2+a^2)\,Q-p_1\,(1+p_1)\,(\pi^2+a^2)^3 \tag{58}$$

$$C = R_2\,a^2\,(\pi^2+a^2) > 0. \tag{59}$$

Proof : From Theorem 3.1 the marginal state is characterized by $\sigma = i\,\sigma_i$ where σ_i is real. Now following Chandrasekhar (1961) (i. e. neglecting surface current density) we can show that for the proper solution for W belonging to the lowest mode is

$$W = C_1\ \sin\ \pi z \tag{60}$$

where C_1 is a constant.

Substituting this solution for W in eqn. (44) we obtain the characteristic equation

$$\sigma(\pi^2+a^2)\,(\pi^2+a^2+\sigma)\,(\pi^2+a^2+p_1\sigma)\,(\pi^2+a^2+p_2\sigma)+\sigma Q\pi^2(\pi^2+a^2)\,(\pi^2+a^2+p_1\sigma)$$
$$=\sigma R_1 a^2\,(\pi^2+a^2+p_2\sigma) - \frac{a^2 R_2}{p_1}(\pi^2+a^2+p_1\sigma)\,(\pi^2+a^2+p^2\sigma)\,. \tag{61}$$

Equating separately the real and imaginary parts of eqn. (61) and using the fact that $\sigma = i\sigma_i$ we obtain

$$p_1 p_2(\pi^2+a^2)\sigma_i^4+[p_2R_1a^2-p_2R_2a^2-(p_1+p_2+1)(\pi^2+a^2)^3-p_1Q\pi^2(\pi^2+a^2]\sigma_i^2 + \frac{R_2a^2}{p_1}(\pi^2+a^2)^2=0 \tag{62}$$

$$(\pi^2+a^2)^4-p_1\sigma_i^2(\pi^2+a^2)^2-p_1 p_2\sigma_i^2(\pi^2a^2)^2-p_2\sigma_i^2(\pi^2+a^2)+Q\pi^2(\pi^2+a^2)^2 = R_1a^2(\pi^2+a^2)-\frac{R_2a^2}{p_1}(\pi^2+a^2)(p_1+p_2) \tag{63}$$

From eqn. (63) we obtain

$$R_1=\frac{R_2}{p_1}(p_1+p_2)-\sigma_i^2\Big[\frac{p_1+p_2+p_1p_2}{a^2}\Big](\pi^2+a^2)+\frac{Q\pi^2(\pi^2+a^2)}{a^2}+\frac{(\pi^2+a^2)^3}{a^2}. \tag{64}$$

Now substituting for R_1 from eqn. (64) in eqn. (62) it follows that

$$A\sigma_i^4+B\sigma_i^2+C=0. \tag{65}$$

Solving the above equation for σ_i^2 and retaining the positive root (since $\sigma_i^2>0$ essentially), we have

$$\sigma_i^2=\frac{-B-\sqrt{B^2-4AC}}{2A}. \tag{66}$$

Making use of eqn. (66), eqn. (64) gives

$$R_1=\frac{R_2(p_1+p_2)}{p_1}-(\pi^2+a^2)\Big[\frac{(p_1+p_2+p_1p_2)}{a^2}\Big]\Big[\frac{-B-\sqrt{B^2-4AC}}{2A}\Big]+ + \frac{Q\pi^2(\pi^2+a^2)}{a^2}+\frac{(\pi^2+a^2)^3}{a^2}. \tag{67}$$

Equations (67) and (66) give the values of the Rayleigh number and the frequency of oscillations at the marginal state. This proves the theorem.

Theorem 3.3— For $R_2>0$, $\frac{\partial R_1}{\partial Q}>0$ for all $Q>0$ provided $p_2<p_1$ holds. (68)

Proof: Differentiating equation (64) w r. t. Q partially and remembering that the frequency of oscillations depends on Q in general, we have

$$\frac{\partial R_1}{\partial Q}=-\Big[\frac{(p_1+p_2+p_1p_2)}{a^2}\Big](\pi^2+a^2)\frac{\partial\sigma_i^2}{\partial Q}+\frac{\pi^2(\pi^2+a^2)}{a^2}. \tag{69}$$

Further, differentiating equation (66) partially w. r. t. Q, it follows that

$$2A\frac{\partial\sigma_i^2}{\partial Q}=-p_1\pi^2(p_2-p_1)(\pi^2+a^2)- -\tfrac{1}{2}(B^2-4AC)^{-\frac{1}{2}}[2p_1^2\pi^4(p_2-p_1)^2(\pi^2+a^2)^2Q+ +2\{p_2^2a^2\pi^2R_2-p_1\pi^2(p_1+1)(\pi^2+a^2)^3\}p_1(p_2-p_1)(\pi^2+a^2)] \tag{70}$$

Now, substituting for $\frac{\partial \sigma_i^2}{\partial Q}$ from eqn. (70) in eqn. (69), we get

$$\frac{\partial R_1}{\partial Q} = \frac{\pi^2(p_1-p_2)(p_1+p_2+p_1 p_2)(\pi^2+a^2)}{2a^2 p_2^2 (1+p_1)}\left[1+\frac{B}{\sqrt{B^2-4AC}}\right]+ + \frac{\pi^2(\pi^2+a^2)}{a^2} . \tag{71}$$

But, since $\frac{B}{\sqrt{B^2-4AC}}<1$ irrespective of whether B is positive or negative (this is because when $R_2>0$, A and C have different signs), we have from eqn. (71) under the condition $p_2 < p_1$

$$\frac{\partial R_1}{\partial Q}>0 \text{ for all } Q>0.$$

Hence the theorem.

Theorem 3.4— For $R_2<0$, no marginal state exists provided $p_2<p$ holds good.

Proof: Let the marginal state exist so that under the given conditions $\sigma_r=0$ is allowed by the governing equations and boundary conditions. Clearly σ_i cannot be zero for otherwise principle of exchange of stabilities will be valid which contradicts Theorem 3.1 which is true irrespective of whether R_2 is positive or negative. We then have from eqn. (66) (which still holds good)

$$\sigma_i^2 = \frac{-B\pm\sqrt{B^2-4AC}}{2A}, \tag{72}$$

where, we must now take $R_2<0$ in the expressions for A, B and C as given by eqns, (57), (58) and (59) respectively. Thus, under the conditions of the theorem it is clear that

$$\left.\begin{array}{l} A< 0 \\ B< 0 \\ C< 0 \end{array}\right\} \tag{73}$$

Equation (72) does not allow any positive root for σ_i^2 when inequalities (73) are satisfied. However, σ_i^2 is essentialiy posititive since σ_i is real. Therefore our starting assumption, namely that $\sigma_r=0$ is possible is not correct and this establishes the theorem. It implies that under the above conditons an arbitrary mode is either damped or amplified. We discuss this point in the subsequent theorems.

Theorem 3.5—For $R_2<0$ and $p_2<p_1$, non-oscillatory modes exist and are unstable.

Proof: For $R_2<0$ and $p_2<p_1$, the only modes that can exist are modes for which $\sigma_r\neq 0$ and $\sigma_i=0$. By putting $\sigma=\sigma_r$ in equation (61) we derive the characteristic equation for such modes as

$$\sigma_r(\pi^2+a^2)(\pi^2+a^2+\sigma_r)(\pi^2+a^2+p_1\sigma_r)(\pi^2+a^2+p_2\sigma_r)+$$
$$+\sigma_r Q\pi^2(\pi^2+a^2)(\pi^2+a^2+p_1\sigma_r)$$
$$=\sigma_r R_1 a^2(\pi^2+a^2+p_2\sigma_r)-\frac{a^2R_2}{p_1}(\pi^2+a^2+p_1\sigma_r)(\pi^2+a^2+p_2\sigma_r) \qquad (74)$$

where, now R_2 is negative.

Equation (74) is an equation of degree four in σ_r with real coefficients and hence allows four roots for σ_r which may be real or complex. If we denote the roots by σ_{r1}, σ_{r2}, σ_{r3} and σ_{r4}, we have from eqn. (74)

$$\sigma_{r1}\,\sigma_{r2}\,\sigma_{r3}\,\sigma_{r4}<0 \qquad (75)$$

since R_2 is negative.

Inequality (75) shows that one of the roots must be negative and one must be positive. Thus under the conditions of the theorem, for given values of the parameters, we do have non-oscillatory modes, either two or four such that at least one is amplified in time and makes the system unstable. This establishes the theorem.

Theorem 3.6—For $R_2<0$ and $p_2<p_1$, the system is unstable.

Proof: This follows easily from theorem 3.5 since nonoscillatory modes exist and are amplified. Thus irrespective of the existence of oscillatory modes and their character there are always certain modes (the non-oscillatory ones) which are unstable. The system itself is then unstable according to the linear stability theory.

4. Concluding Remarks

The investigations presented in this paper are based on the rather plausible hypothesis that in reality a fluid is necessarily initially non-homogeneous and one may not be justified in neglecting this initial non-homogeneity, however small, everywhere in the equations of motion. On the contrary, even the slightest amount of this initial nonhomogeneity may turn out to be quite significant for the problem under consideration and hence the behaviour of a homogeneous fluid should be deduced from that of a non-homogeneous one in the limit of very small non-homogeneity. The importance of this initial non-homogeneity of the fluid has been clearly demonstrated by Banerjee (1971, 1972, 1973, 1973*a*) in the context of the Bénard problem and the Taylor problem of rotating cylinders and therefore it is desirable to analyse the hydromagnetic Bénard problem when the fluid is taken to be initially non-homogeneous. This will also provide a support to the reliability to the results of Chandrasekhar's model. The above point of view of looking at the investigations, presented in the paper appears more fundamental than the one given at the outset, namely, extending the results of the generalized Bénard model in the framework of magnetohydrodynamics and investigating the role of the magnetic field. The result contained in Theorem 3.1, in a sense is decisive. Thus while in the presence of the magnetic field alone instability could set in either as stationary convection or as overstability, it definitely sets in as overstability when an initially stable density stratification is also present. Thus, overstability as the mode of the onset of instability is very likely in Chandrasekhar's model. This result is not surprising at all as it is well known that stable stratification permits internal waves. Theorem 3.2 gives the Rayleigh number and the frequency of oscillations that characterize the marginal state. Putting $R_2=0$, $p_2=0$ and $Q=0$ we recover the results of Rayleigh (1916) and Pellow and Southwell (1940), $R_2=0$ those of Chandrasekhar (1952), and $p_2=0$, $Q=0$ those of Banerjee (1972). Theorem 3.3 shows the stabilizing character of the magnetic field under the condition $p_2 < p_1$ which as already mentioned is met by a large margin under most terrestrial conditions. Thus the character of the magnetic field in the present situation remains what it was in Chandrasekhar's model. The difference lies in the manner in which instability sets in. The character of the marginal state is dominated by the properties of initial stratification while the magnetic field helps in postponing the onset of instability by raising the Rayleigh number at the marginal state. When the initial stratification is monotónically increasing with respect to the vertical coordinate, i.e. $R_2 < 0$ Theorem 3.4 shows that there is no marginal state, the problem thus again being dominated by the initial stratification as the same result holds true for the generalized Bénard model also. This result is a little surprising though could possibly be attributed to the neglect of mass diffusion in the governing equations. The extension of the present model by taking mass diffusion into account will be the subject matter of a latter communication. Theorem 3.5 shows that non-oscillatory modes do exist and are unstable and thus make the system unstable according to the linear stability theory. The investigation of the character of the oscillatory modes, if there exists any, is thus of secondary importance. The system remains unstable irrespective of their existence and character as concluded in Theorem 3.6. We note that this result is analogous to the corresponding result for the generalized Bénard problem (Banerjee 1972) wherein the non-oscillatory modes are amplified irrespective of the character of the applied uniform temperature gradient. Thus, for $R_2 < 0$, even the application of a uniform vertical magnetic field cannot stabilize the system. The character of the non-oscillatory modes appear to be completely guided by the character of the initial density stratification.

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