CURVATURE COLLINEATIONS IN SOME COSMOLOGICAL MODELS

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In this paper symmetry properties of the Gödel universe and the steady state universe of Bondi and Gold have been studied. It is found that both of them admit proper curvature collineations. The collineation vectors have been determined.

1. INTRODUCTION

It has been observed that the existence of certain geometric symmetries definable in terms of Lie derivatives leads to conservation laws in the theory of general relativity. Katzin *et al.* (1969) have recently investigated a special type of symmetry property of Riemannian manifolds called curvature collineations (CC's) and have derived several results of geometrical and physical interest. A Riemannian space-time V_4 is said to admit a CC if there exists an infinitesimal transformation

$$\bar{x}^i = x^i + \xi^i (x)\delta t \qquad \text{...(1.1)}$$

for which

$$£R^h{}_{ijk} = 0 \qquad \text{...(1.2)}$$

where δt is a positive infinitesimal, £ denotes Lie derivative with respect to the vector field ξ^i and $R^h{}_{ijk}$ is Riemann curvature tensor. An important feature of the study of curvature collineations is that it indicates the types of elastic deformations of the space-time for which the gravitational properties are preserved. Several other symmetry properties of Riemannian manifolds have also been discussed from geometric as well as physical points of view, viz. motion, conformal motion, affine collineation, projective and conformal collineation. All these are defined in terms of Lie derivatives of certain geometric entities. A CC is called proper if the Riemannian four-fold does not admit any of the latter symmetries.

In a previous paper Singh and Ram (1973) have shown that the plane symmetric cosmological model admits proper CC only if it is conformally flat which serves as an illustration of the result obtained by Katzin *et al.* (1970). They have also shown that the Einstein static cosmological space-time admits proper CC. The object of this paper is to find CC's admitted by some other cosmological models, viz. Godel universe and the steady state universe of Bondi and Gold. In both the models the curvature collineations admitted are found to be proper.

2. Godel Cosmological model

Godel (1949) has shown that the following metric is compatible with an incoherent matter distribution

$$ds^2 = (dt+e^{\alpha x}dy)^2 - dx^2 - \tfrac{1}{2}e^{2\alpha x}dy^2 - dz^2 \tag{2.1}$$

where α is a constant with the dimension of an inverse length. It is well known that this model admits a five parameter group of motions.

The Lie derivative of Riemann curvature tensor can be written explicitly either in terms of partial or covariant differentiation.

$$£R^h_{ijk} \equiv R^h_{ijk;m}\,\xi^m - R^m_{ijk}\,\xi^h_{,m} + R^h_{mjk}\,\xi^m_{,i} + R^h_{imk}\xi^m_{;j} + R^h_{ijm}\xi^m_{,k}. \quad \ldots(2.2)$$

Throughout this paper a semicolon denotes covariant differentiation and comma denotes ordinary partial differentiation.

For the metric (2.1) the non-vanishing components of Christoffel symbol Γ^i_{jk} are

$$\left.\begin{array}{l} \Gamma^1_{24} = \frac{\alpha}{2}e^{\alpha x}, \quad \Gamma^1_{22} = \frac{\alpha}{2}e^{2\alpha x} \\ \Gamma^2_{14} = -\alpha e^{-\alpha x} \\ \Gamma^4_{14} = \alpha, \quad \Gamma^4_{14} = \frac{\alpha}{2}e^{\alpha x}. \end{array}\right\} \quad \ldots(2.3)$$

The non-zero components of Riemann curvature tensor R^h_{ijk} for (2.1) are

$$\left.\begin{array}{l} R^1_{214} = \frac{\alpha^2}{2}e^{\alpha x}, \quad R^1_{212} = \frac{3}{4}e^{2\alpha x} \\ R^2_{424} = \frac{\alpha^2}{2}, \quad R^2_{121} = \frac{\alpha^2}{2} \\ R^4_{141} = -\frac{\alpha^2}{2}, \quad R^4_{112} = \alpha^2 e^{\alpha x}, \quad R^4_{242} = \frac{\alpha^2}{4}e^{2\alpha x}. \end{array}\right\} \quad \ldots(2.4)$$

In view of the symmetry properties of eqn. (1.2) and the cyclic identity of the mixed curvature tensor, there are formally 80 equations. But since only a few components of the Riemann tensor are non-zero this number is reduced. It may be noted that the vanishing components of R^h_{ijk} also contribute to eqn. (1.2).

The vanishing components of Riemann tensor give

$$£R^1_{442} = 0 \Rightarrow \xi^1_{,2} = 0 \quad \ldots(2.5)$$

$$£R^4_{143} = 0 \Rightarrow \xi^1_{,3} = 0 \quad \ldots(2.6)$$

$$£R^4_{414} = 0 \Rightarrow \xi^1_{,4} + \xi^4_{,1} = 0 \quad \ldots(2.7)$$

$$£R^1_{114} = 0 \Rightarrow e^{\alpha x}\xi^2_{,1} + \xi^4_{,1} - \xi^1_{,4} = 0 \quad \ldots(2.8)$$

$$\pounds R^4_{243} = 0 \Rightarrow \xi^2{}_{,3} = 0 \quad \ldots(2.9)$$
$$\pounds R^4_{213} = 0 \Rightarrow \xi^4{}_{,3} + e^{2\alpha x}\xi^2{}_{,3} = 0 \quad \ldots(2.10)$$
$$\pounds R^3_{141} = 0 \Rightarrow \xi^3{}_{,4} = 0 \quad \ldots(2.11)$$
$$\pounds R^3_{414} = 0 \Rightarrow \xi^3{}_{,1} = 0 \quad \ldots(2.12)$$
$$\pounds R^4_{324} = 0 \Rightarrow \xi^3{}_{,2} = 0 \quad \ldots(2.13)$$
$$\pounds R^2_{224} = 0 \Rightarrow e^{2\alpha x}\xi^2{}_{,4} + 2\xi^4{}_{,2} = 0 \quad \ldots(2.14)$$
$$\pounds R^2_{342} = 0 \Rightarrow \xi^4{}_{,3} = 0 \quad \ldots(2.15)$$
$$\pounds R^2_{214} = 0 \Rightarrow \xi^2{}_{,1} = 0 \quad \ldots(2.16)$$

The non-vanishing components of Riemann tensor give

$$\pounds R^1_{441} = 0 \Rightarrow 2\xi^1{}_{,1} + e^{\alpha x}\xi^2{}_{,4} = 0 \quad (2.17)$$
$$\pounds R^1_{412} = 0 \Rightarrow e^{\alpha x}(\alpha\xi^1 - \xi^4{}_{,4} + 2\xi^1{}_{,1} + \xi^2{}_{,2}) + \xi^4{}_{,2} = 0 \quad \ldots(2.18)$$
$$\pounds R^2_{442} = 0 \Rightarrow \xi^4{}_{,4} = 0 \quad \ldots(2.19)$$
$$\pounds R^1_{414} = 0 \Rightarrow 2\xi^4{}_{,4} + e^{\alpha x}\xi^2{}_{,4} = 0. \quad \ldots(2.20)$$

The trival and redundant equations have been omitted.

From the above eqns. (2.5)–(2.20), it may be seen that

(*i*) $\xi^4{}_{,4} = 0,\ \xi^4{}_{,1} = 0,\ \xi^4{}_{,2} = 0,\ \xi^4{}_{,3} = 0.$

Therefore, we have

$$\xi^4 = a \qquad (a = \text{const.})$$

(*ii*) $\xi^1{}_{,2} = 0,\ \xi^1{}_{,3} = 0,\ \xi^1{}_{,1} = 0,\ \xi^1{}_{,4} = 0$

so that

$$\xi^1 = b. \qquad (b = \text{const.})$$

(*iii*) $\xi^2{}_{,4} = 0,\ \xi^2{}_{,3} = 0,\ \xi^2{}_{,1} = 0$ and $a\xi^1 + \xi^2{}_{,2} = 0.$

Integrating this equation we have

$$\xi^2 = cy + d. \qquad (c = -b\alpha,\ d = \text{const.}).$$

(*iv*) $\xi^3{}_{,1} = 0,\quad \xi^3{}_{,2} = 0,\quad \xi^3{}_{,4} = 0$ which means

that

$$\xi^3 = \xi^3(z).$$

Thus the CC vector admitted by Godel universe is

$$\left.\begin{aligned} \xi^1 &= b. \\ \xi^2 &= cy + d. \\ \xi^3 &= \xi^3(z). \\ \xi^4 &= a. \end{aligned}\right\} \quad \ldots(2.21)$$

In order to see whether the CC (2.21) admitted by Godel model is proper we apply the usual tests one by one (Aichelburg 1970).

1. *Motion* : $\pounds g_{ij} \equiv \xi_{j;i} + \xi_{i;j} = 0$

If we define

$$h_{ij} = \pounds\, g_{ij}$$

then it may easily be seen that $h_{12} \neq 0$ which means that $h_{ij} \neq 0$ in general. Therefore the CC (2.21) does not imply motion. Also it is not conformal motion.

2. *Affine Collineation* : $\pounds \Gamma^i{}_{jk} = 0$

It may be seen from (2.3) and (2.21) that $\xi_{1;22} = \frac{b\alpha^2}{4} e^{\alpha x} \neq 0$. Therefore the CC (2.21) is not in general affine collineation.

3. *Projective Collineation* : $\pounds W^h{}_{ijk} = 0$ *and Conformal Collineation* $\pounds C^h{}_{ijk} = 0$.

A necessary condition for a CC to be a PC or conf. C is that $\xi^i{}_{;ijk} = 0$. But from (2.21) we see that $\xi^1{}_{;124} = \frac{3}{2}\, b\alpha^3 e^{\alpha x} \neq 0$.

Thus the CC vector ξ^i in (2 21) is proper.

3. Bondi and Gold Cosmological Model

Bondi and Gold metric of the steady state theory based on what is known as the perfect cosmological principle is given by (Adler *et al.* 1965)

$$ds^2 = dt^2 - \frac{R^2(t)}{R_0^{\,2}} (dx^2 + dy^2 + dz^2) \qquad \ldots(3.1)$$

where $R(t) = R_0\, e^{t/T}$, $R_0 = \text{const.}$, $T = \text{Hubble constant}$.

Thus $g_{ij} = \text{diag.} \left(-\frac{R^2}{R_0^2}, -\frac{R^2}{R_0^2}, -\frac{R^2}{R_0^2}, 1 \right)$.

The non-vanishing components of Christoffel three index symbol $\Gamma^i{}_{jk}$ are

$$\Gamma^1_{14} = \Gamma^2_{24} = \Gamma^3_{34} = \frac{1}{T}.$$

$$\Gamma^4_{11} = \Gamma^4_{22} = \Gamma^4_{33} = \frac{1}{T} e^{2t/T} \qquad \ldots(3.2)$$

The non-zero components of Riemann tensor are

$$\left.\begin{aligned}
&R^1_{212}=R^1_{313}=\frac{1}{T^2}e^{2t/T},\ R^1_{414}=-\frac{1}{T^2}\\
&R^2_{121}=R^2_{323}=\frac{1}{T^2}e^{2t/T},\ R^2_{424}=-\frac{1}{T^2}\\
&R^3_{131}=R^3_{232}=\frac{1}{T^2}e^{2t/T},\ R^3_{434}=-\frac{1}{T^2}\\
&R^4_{141}=R^4_{242}=R^4_{343}=\frac{1}{T^2}e^{2t/T}.
\end{aligned}\right\} \qquad \ldots(3.3)$$

The equation (1.2) by the use of (3.3) lead to the following :

$$\pounds R^1_{112}=0\Rightarrow \xi^1_{,2}+\xi^2_{,1}=0 \qquad \ldots(3.4)$$

$$\pounds R^4_{314}=0\Rightarrow \xi^1_{,3}+\xi^3_{,1}=0 \qquad \ldots(3.5)$$

$$\pounds R^1_{213}=0\Rightarrow \xi^2_{,3}+\xi^3_{,2}=0 \qquad \ldots(3.6)$$

$$\pounds R^1_{212}=0\Rightarrow \xi^4+T\xi^2_{,2}=0 \qquad \ldots(3.7)$$

$$\pounds R^2_{121}=0\Rightarrow \xi^4+T\xi^1_{,1}=0 \qquad \ldots(3.8)$$

$$\pounds R^1_{313}=0\Rightarrow \xi^4+T\xi^3_{,3}=0 \qquad \ldots(3.9)$$

$$\pounds R^1_{414}=0\Rightarrow \xi^4_{,4}=0 \qquad \ldots(3.10)$$

$$\pounds R^1_{114}=0\Rightarrow \xi^4_{,1}-e^{2t/T}\xi^1_{,4}=0 \qquad \ldots(3.11)$$

$$\pounds R^2_{224}=0\Rightarrow \xi^4_{,2}-e^{2t/T}\xi^2_{,4}=0 \qquad \ldots(3.12)$$

$$\pounds R^1_{413}=0\Rightarrow \xi^4_{,3}-e^{2t/T}\xi^3_{,4}=0. \qquad \ldots(3.13)$$

From eqns. (3.7), (3.8) and (3.9), it follows that $\xi^1_{,1}=\xi^2_{,2}=\xi^3_{,3}$. ...(3.14)

Also from eqns. (3.4), (3.5) and (3.6) we have

$$\xi^1_{,23}=0,\ \xi^2_{,13}=0,\ \xi^3_{,12}=0. \qquad \ldots(3.15)$$

Again with the help of (3.8) and (3.11) we get

$$\xi^4_{,11}=0.$$

Similarly

$$\xi^4_{,22}=\xi^4_{,33}=0.$$

Therefore ξ^4 can be written in the form

$$\xi^4=lx+my+nz.$$

where l, m and n are arbitrary constants.

A straightforward calculation from the above equations leads to the structure of the CC vector ξ^i as

$$\left.\begin{aligned}
\xi^1 &= \frac{l}{2T}(y^2+z^2) - \frac{lT}{2}e^{-2t/T} - \frac{1}{T}\left(\frac{lx^2}{2}+myx+nzx\right)\\
\xi^2 &= \frac{m}{2T}(z^2+x^2) - \frac{mT}{2}e^{-2t/T} - \frac{1}{T}\left(lxy+\frac{m}{2}y^2+nzy\right)\\
\xi^3 &= \frac{n}{2T}(x^2+y^2) - \frac{nT}{2}e^{-2t/T} - \frac{1}{T}\left(lxz+myz+\frac{nz^2}{2}\right).\\
\xi^4 &= lx+my+nz.
\end{aligned}\right\} \quad \ldots(3.16)$$

Applying the method of section (2), it can be shown that the CC vector (3.16) admitted by Bondi and Gold model is proper.

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(*See Appendix I on pages* 1029-30)

APPENDIX I

For unsteady flow, the connected differential equations for the functions $A_0^{(0)}(\eta)$, $A_1^{(0)}(\eta)$, $A_0^{(2)}(\eta)$, $P_0^{(0)}(\eta)$, $L_0^{(4)}(\eta)$, $M_0^{(1)}(\eta)$ etc., associated with the problems of wedge and circular cylinder can be obtained from (3.4), (3.5) and (3.9) and their numerically integrated values are given below :

Solution associated with wedge problem

when $l = 0$

β	$A_0''^{(0)}(0)$	$\underset{\eta\to\infty}{\mathrm{Lt}}(f_1 - A_0^{(0)})$	$A_1''^{(0)}(0)$	$A_1^{(0)}(\infty)$	$P_0'^{(0)}(0)$	$P_1'^{(0)}(0)$	$\int_0^\infty P_0^{(0)}(\eta)\,d\eta$
0	0.7044	−0 6084	1.2000	0.7275	0.2348	−0.0382	0.6083
0.2	0 8221	−0.2828	0.9720	0.5031	0.2018	−0.6084	0.4588
0.4	0.9372	−0.1481	0 7946	0.3679	0.1748	−0.0795	0.3660

when $l = 1.0$

β	$A_0''^{(0)}(0)$	$\underset{\eta\to\infty}{\mathrm{Lt}}(f_1 - A_0^{(0)})$	$A''^{(0)}_1(0)$	$A_1^{(0)}(\infty)$	$P_0'^{(0)}(0)$	$P_1'^{(0)}(0)$	$\int_0^\infty P_0^{(0)}(\eta)\,d\eta$
0.0	2.8866	−3 1654	−0.6304	−2.0446	0 8812	−1.0863	2.0260
0.2	2.4035	−1.6554	0.0869	−0.5428	0.6038	−0.5014	1.2612
0.4	2.1740	−1.0426	0.2667	−0.1800	0.4816	−0.3296	0.9533

when $l = 2.0$

β	$A_0''^{(0)}(0)$	$\underset{\eta\to\infty}{\mathrm{Lt}}(f_1 - A_0^{(0)})$	$A''^{(0)}_1(0)$	$A_1^{(0)}(\infty)$	$P_0'^{(0)}(0)$	$P_1'^{(0)}(0)$	$\int_0^\infty P_0^{(0)}(\eta)\,d\eta$
0.0	4.4935	−4.6038	−1.3572	−2.8523	1.3863	−1.5172	2.9937
0.2	3 6344	−2.4719	−0.3209	−0.8945	0.9157	−0.6862	1.8077
0.4	3.1765	−1.6063	−0 0032	−0.3876	0.7099	−0.4286	1.3394

Solution associated with circular cylinder problem

l	$A_0''^{(0)}(0)$	$A_0''^{(2)}(0)$	$L_0''^{(4)}(0)$	$M_0''^{(4)}(0)$	$P_0'^{(0)}(0)$	$P_0'^{(2)}(0)$	$A_0^{(0)}(\infty)$
0	1.2326	1.5064	2.208	−0.7139	0.0	−0.2171	4.7517
1	1.8489	1.4493	1.9029	−0.3575	0.2853	−0.2603	5.0761
2	2.3962	1.4297	1.6631	−0.0667	0.4195	−0.3553	5.1190

l	$A_0^{(2)}(\infty)$	$L_0^{(4)}(\infty)$	$M_0^{(4)}(\infty)$	$\int_0^\infty P_0^{(0)}(\eta)\,d\eta$	$\int_0^\infty P_0^{(2)}(\eta)\,d\eta$
0	0.6725	0.3681	−0.8699	0.0	−0.4434
1	−0.0521	−0.4855	0.6953	0.5131	−1.3523
2	−0.5648	−1.0147	1.9941	0.7421	−1.8563