HOLOMORPHIC PROJECTIVE BIRECURRENT MANIFOLD

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The object of the paper is to study the properties of holomorphic projective curvature tensor $P(Z, T, W)$ in a Kähler manifold M_{2n} by imposing a basic condition on the H-Projective curvature tensor by

$$(D_X D_Y P)(Z, T, W) = a(X, Y) + P(Z, T, W) + (D_{D_Y X} P)(Z, T, W).$$

1. INTRODUCTION

We consider a $2n$ dimensional real manifold M_{2n} of differentiability class C^{r+1}. Let F be a vector valued linear function of M_{2n} such that

$$F(F(X)) + X = 0 \tag{1.1}$$

for an arbitrary vector field X.

Then F is said to give an almost complex structure to M_{2n} and M_{2n} is called an almost complex manifold.

Agreement (1.1)—All the equations which follow, hold for arbitrary vector fields X, Y, Z, T, W, .., etc.

Let the almost complex manifold M_{2n} be also endowed with the Hermitian metric g:

$$g(F(X), F(Y)) = g(X, Y), \tag{1.2}$$

then M_{2n} is called an almost Hermite manifold.

Let us put

$$\Omega(X, Y) = g(F(X), Y). \tag{1.3}$$

Then

$$\Omega(F(X), F(Y)) = -g(X, F(Y)) = \Omega(X, Y) \tag{1.4a}$$

$$-(F(X), \Omega Y) = g(X, Y) = \Omega(X, F(Y)) \tag{1.4b}$$

$$\Omega(X, Y) - \Omega(Y, X) = 0 \tag{1.4c}$$

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Suppose D is a Riemannian connexion in M_{2n} then

$$D_X Y - D_Y X = [X, Y], \tag{1.5a}$$

$$(D_X g)\,(Y, Z) = 0. \tag{1.5b}$$

An almost Hermite manifold for which

$$(D_X F)(Y) = 0 \tag{1.6a}$$

or

$$(D_X \Omega)(Y, Z) = 0 \tag{1.6b}$$

is satisfied, is called a Kähler manifold.

Let K be the curvature tensor of the Kähler manifold M_{2n} given by

$$K(X,Y,Z) = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z, \quad \ldots(1.7)$$

and let us define

$$'K(X, Y, Z, T) = g(K(X, Y, Z), T). \quad \ldots(1.8)$$

Then for a Kähler manifold, we have (Mishra 1970)

$$K(X, Y, F(Z)) = F(K(X, Y, Z)) \quad \ldots(1.9a)$$

and

$$'K(F(X), F(Y), Z, T) = {}'K(X, Y, Z, T) = {}'K(X, Y, F(Z), F(T)). \quad \ldots(1.9b)$$

Let Ric be the Ricci tensor of M_{2n} defined by (Mishra 1970)

$$\text{Ric}\,(Y,Z) = (C_1^1 K)\,(Y, Z), \quad \ldots(1.10)$$

where $C_1{}^1$ is the contraction operator.

Then in a Kähler manifold (Mishra 1970), we have

$$\text{Ric}\,(F(X), F(Y)) = \text{Ric}\,(X, Y) \quad \ldots(1.11a)$$

$$\text{Ric}\,(F(X), Y) + \text{Ric}\,(X, F(Y)) = 0 \quad \ldots(1.11b)$$

$$r(F(X)) = F(r(X)) \quad \ldots(1.12)$$

where

$$g(r(X), Y) \overset{\text{def}}{=} \text{Ric}\,(X, Y)$$

$$-\frac{1}{2}(C_3^1 F(K))(X, Y) \overset{\text{def}}{=} \text{Ric}\,(X, F(Y))$$

and

$(C_1^1 r) \overset{\text{def}}{=} R$ is the scalar curvature.

The projective curvature tensor $*W$, the conformal curvature tensor V, the conharmonic curvature tensor L and the concircular curvature tensor C are given by (Mishra 1965)

$$*W(Z, T, W) = K(Z, T, W) - \frac{1}{(2n-1)}\Big[\mathrm{Ric}(T,W)Z - \mathrm{Ric}(Z,W)T\Big] \quad ...(1.13)$$

$$V(Z, T, W) = K(Z, T, W) - \frac{1}{2(n-1)}\Big[\mathrm{Ric}(T, W)Z - \mathrm{Ric}(Z, W)T + r(Z)g(T, W)$$

$$-r(T)g(Z, W) + \frac{R}{2(2n-1)(n-1)}\Big[g(T, W)Z - g(Z, W)T\Big] \quad ...(1.14)$$

$$L(Z,T,W) = K(Z,T,W) - \frac{1}{2(n-1)}\Big[\mathrm{Ric}(T,W,)Z - \mathrm{Ric}(Z,W)T$$

$$+r(Z)\, g(T,W) - r(T)\, g(Z,W)\Big] \quad ...(1.15)$$

$$C(Z, T, W) = K(Z, T, W) - \frac{R}{2n(2n-1)}\Big[\, g(T,W)Z - g(Z,W)T\,\Big], \text{ respectively} \quad ...(1.16)$$

The H-projective tensor of the Kähler manifold is given by the relation (Mishra 1970).

$$P(Z, T, W) = K(Z, T, W) - \frac{1}{2(n+1)}\Big[\mathrm{Ric}(T, W)Z - \mathrm{Ric}(Z, W)T$$

$$-\mathrm{Ric}(T, F(W))F(Z) + \mathrm{Ric}(Z, F(W))F(T) + 2\mathrm{Ric}(Z, F(T))F(W)\Big] \quad .. (1.17)$$

Definition—A manifold M_{2n} will be called (i) K-birecurrent if

$$(D_Y D_X K)(Z, T, W) = a(X, Y)\, K(Z, T, W) + (D_{D_Y X} K)\,(Z, T, W) \quad .. (1.18)$$

(ii) Ricci birecurrent if

$$(D_Y D_X \mathrm{Ric})(Z, T) = a(X, Y)\, \mathrm{Ric}(Z,T) + (D_{D_Y X}\mathrm{Ric})(Z, T) \quad . (1.19)$$

which yields

$$(D_Y D_X r)(Z) = a(X, Y)\, r(Z) + (D_{D_Y X} r)(Z) \quad ...(1.20)$$

(iii) projective birecurrent if

$$(D_Y D_X {}^*W)\,(Z, T, W) = a(X, Y)\, {}^*W(Z, T, W) + (D_{D_Y X}{}^*W)\,(Z, T, W) \quad .. (1.21)$$

(iv) conformal birecurrent if

$$(D_Y D_X V)(Z, T, W) = a(X, Y) V(Z, T, W) + (D_{D_Y X} V)(Z, T, W) \quad \ldots(1.22)$$

(v) conharmonic birecurrent if

$$(D_Y D_X L)(Z, T, W) = a(X, Y)\, L\, (Z, T, W) + (D_{D_Y X} L)(Z, T, W) \quad \ldots(1.23)$$

(vi) concircular birecurrent if

$$(D_Y D_X C)(Z, T, W) = a(X, Y)\, C(Z, T, W) + (D_{D_Y X} C)(Z, T, W) \quad \ldots(1.24)$$

(vii) H-projective birecurrent if

$$(D_Y D_X P)(Z, T, W) = a(X, Y)\, P(Z, T, W) + (D_{D_Y X} P)(Z, T, W) \quad \ldots(1.25)$$

where $a(X, Y)$ is a non-vanishing C^∞ function.

2. *H*-Projective Curvature Tensor

Theorem 2.1—For a Kähler manifold if any two of the following properties hold for the same recurrence parameter, the third also holds :

(a) It is a birecurrent manifold,

(b) It is a *H*-projective birecurrent manifold,

(c) It is a Ricci birecurrent manifold.

Proof : From (1.17), we have in consequence of (1.6*a*) that

$$(D_X P)(Z, T, W) = (D_X K)(Z, T, W) - \frac{1}{2(n+1)} \Big[(D_X \operatorname{Ric})(T, W) Z - (D_X \operatorname{Ric})(Z, W) T - (D_X \operatorname{Ric})(T, F(W)) F(Z) + (D_X \operatorname{Ric})(Z, F(W)) F(T) + 2 (D_X \operatorname{Ric})(Z, F(T)) F(W) \Big]. \quad \ldots(2.1)$$

Again differentiating (2.1) with respect to D_Y and using (1.6*a*), we have

$$(D_Y D_X P)(P, T, W) = (D_Y D_X K)(Z, T, W) - \frac{1}{2(n+1)} \Big[(D_Y D_X \operatorname{Ric})(T, W) Z - (D_Y D_X \operatorname{Ric})(Z, W) T - (D_Y D_X \operatorname{Ric})(T, F(W)) F(Z) + (D_Y D_X \operatorname{Ric})(Z, F(W)) F(T) + 2 (D_Y D_X \operatorname{Ric})(Z, F(T)) F(W) \Big] \quad \ldots(2.2)$$

Let the manifold be birecurrent and H-projective birecurrent then using definitions (i) and (vii), we have

$$(D_{D_Y X} P)(Z, T, W)-(D_{D_Y X} K)(Z, T, W)+a(X, Y)(P(Z, T, W)-K(Z, T, W))$$
$$= -\frac{1}{2(n+1)}\Big[(D_Y D_X \text{ Ric})(T, W)Z-(D_Y D_X \text{ Ric})(Z, W)T$$
$$-(D_Y D_X \text{ Ric})(T, F(W))F(Z)+(D_Y D_X \text{Ric})(Z, F(W))F(T)$$
$$+2(D_Y D_X \text{Ric})(Z, F(T))F(W)\Big]. \quad ..(2.3)$$

Substituting $P(Z, T, W)-K(Z, T, W)$ from (1.17) and

$(D_{D_Y X} P)(Z, T, W)-(D_{D_Y X} K)(Z, T, W)$

from (2.1) in (2.3), we have

$$\Big[(D_Y D_X \text{ Ric})(T, W)-(D_{D_Y X} \text{ Ric})(T, W)-a(X, Y)\text{Ric}(T, W)\Big]Z$$
$$-\Big[(D_Y D_X \text{ Ric})(Z, W)-D_{D_Y X} \text{ Ric})(Z, W)-a(X, Y)\text{Ric}(Z, W)\Big]T$$
$$-\Big[(D_Y D_X \text{ Ric})(T, F(W))-D_{D_Y X} \text{ Ric})(T, F(W))-a(X, Y)\text{ Ric}(T, F(W))\Big]F(Z)$$
$$+\Big[(D_Y D_X \text{ Ric})(Z, F(W))-(D_{D_Y X} \text{ Ric})(Z, F(W))-a(X, Y)\text{Ric}(Z, F(W))\Big]F(T)$$
$$+\Big[(D_Y D_X \text{ Ric})(Z, F(T))-(D_{D_Y X} \text{ Ric})(Z, F(T))-a(X, Y)\text{ Ric}(Z, F(T))\Big]$$
$$\times F(W)=0 \quad ...(2.4)$$

Since the equations hold for arbitrary vector fields X, Y, Z, T, W, . etc. Hence we get

$$(D_Y D_X \text{Ric})(Z, T)-(D_{D_Y X} \text{Ric})(Z, T) = a(X, Y)\text{ Ric}(Z, T)$$

which implies that the Kähler manifold is a Ricci birecurrent manifold.

Now, we shall show that (b) and (c) ⇒ (a).

Let the manifold M_{2n} be H-projective birecurrent and Ricci birecurrent manifold for the same recurrence parameter. Using (1.19) and (1.25) in (2.2), we have

$$(D_Y D_X K)(Z, T, W) = (D_{D_Y X} P)(Z, T, W)+\frac{1}{2(n+1)}\Big[(D_{D_Y X}\text{Ric})(T, W)Z$$
$$-(D_{D_Y X}\text{Ric})(Z, W)T+(D_{D_Y X}\text{Ric})(Z, F(W))F(T)$$
$$-(D_{D_Y X}\text{Ric})(T, F(W))F(Z)$$
$$+2(D_{D_Y X}\text{Ric})(Z, F(T))F(W)\Big]+$$

(*equation continued on p.* 1036)

$$+ a(X,Y)\Big[P(Z, T, W)+\frac{1}{2(n+1)}\Big\{\mathrm{Ric}(T, W)Z-\mathrm{Ric}(Z, W)T$$

$$-\mathrm{Ric}(T, F(W))F(Z) +\mathrm{Ric}(Z, F(W))F(T)+2\mathrm{Ric}(Z, F(T))F(W)\Big\}\Big]. \quad \ldots(2.5)$$

Making use of (1.17) and (2.1) in (2.5) we get

$$(D_Y D_X K)(Z, T, W) = (D_{D_Y X}K)(Z, T, W)+a(X, Y)K(Z, T, W) \quad \ldots(2.6)$$

which implies that the Kähler manifold is a birecurrent manifold.

Similarly, we can show that (a) and (c) $\Rightarrow$ (b), which proves the statement.

Theorem 2.2—For a Kähler manifold if any two of the following properties hold for the same recurrence parameter, the third also holds :

(a) It is a projective birecurrent manifold,
(b) It is a H-projective birecurrent manifold,
(c) It is a Ricci birecurrent manifold.

PROOF : From (1.13) and (1.17), we have

$$P(Z, T, W)-W(Z, T, W)= \frac{3}{2(2n-1)(n+1)}\Big[\mathrm{Ric}(T, W)Z - \mathrm{Ric}(Z, W)T\Big]$$

$$+\frac{1}{2(n+1)}\Big[\mathrm{Ric}(T, F(W))F(Z)-\mathrm{Ric}(Z, F(W))F(T)-2\mathrm{Ric}(Z, F(T)F(W)\Big]$$

Remaining part of the proof follows from the pattern of the proof of Theorem (2.1)

Theorem 2.3—For a Kähler manifold, we have

$$P(Z, T, W)-V(Z, T, W)-\frac{1}{(n^2-1)}\Big[\mathrm{Ric}(T, W)Z-\mathrm{Ric}(Z, W)T\Big]$$

$$+\frac{R}{2(2n-1)(n-1)}\Big[g(T, W)Z-g(Z, W)T\Big]-\frac{1}{2(n-1)}\Big[r(Z)\,g(T, W)$$

$$r(T)\,g(Z, W)\Big]+\frac{1}{2(n+1)}\Big[\mathrm{Ric}(T, F(W))\,F(Z)$$

$$-\mathrm{Ric}(Z,F(W))\,F(T)-2\mathrm{Ric}(Z, F(T))F(W)\Big].$$

Hence, if any two of the following properties hold for the same recurrence parameter, the third also holds :

(a) It is a conformally birecurrent manifold,
(b) It is a H-projective birecurrent manifold,
(c) It is a Ricci birecurrent manifold.

Theorem 2.4—For a Kähler manifold, we have

$$P(Z, T, W) - L(Z, T, W) = \frac{1}{(n^2-1)}\Big[\mathrm{Ric}(T, W)Z - \mathrm{Ric}(Z, W)T\Big]$$

$$+ \frac{1}{2(n-1)}\Big[r(Z)\, g(T, W) - r(T)\, g(Z, W)\Big] + \frac{1}{2(n+1)}\Big[\mathrm{Ric}(T, F(W))\, F(Z)$$

$$- \mathrm{Ric}(Z, F(W))\, F(T) - 2\mathrm{Ric}(Z, F(T))\, F(W)\Big]$$

Hence if any two of the following properties hold for the same recurrence parameter, the third also holds :

(a) It is a conharmonic birecurrent manifold,

(b) It is a *H*-projective birecurrent manifold,

(c) It is a Ricci birecurrent manifold.

Theorem 2.5—For a Kähler manifold, we have

$$P(Z, T, W) - C(Z, T, W) = \frac{R}{2n(2n-1)}\Big[g(T, W)Z - g(Z, W)T\Big]$$

$$- \frac{1}{2(n+1)}\Big[\mathrm{Ric}(T, W)Z - \mathrm{Ric}(Z, W)T - \mathrm{Ric}(T, F(W))F(Z)$$

$$+ \mathrm{Ric}(Z, F(W))F(T) + 2\mathrm{Ric}(Z, F(T))\, F(W)\Big].$$

Hence, if any two of the following properties hold for the same recurrence parameter, the third also holds :

(a) It is a concircular birecurrent manifold,

(b) It is a *H*-projective birecurrent manifold,

(e) It is a Ricci birecurrent manifold.

The proofs of Theorem 2.3 to 2.5 are similar to the proof of Theorem 2.1.

REFERENCES

Mishra, R. S. (1965). A Course in Tensor with Applications to Riemannian Geometry. Pothishala Private Ltd., Allahabad.

————(1970). Some properties of *H*-projective curvature tensor in a Kähler manifold. *Indian J. pure appl. Math.*, 1, 336-40.