EFFECTS OF CROSS FLOW IN UNSTEADY FLOW PAST A YAWED INFINITE CYLINDER

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The effects of cross flow in the laminar boundary layer on a yawed infinite cylinder are studied when the stream is in unsteady motion with power law perturbation in the chordwise direction and the cylinder is at rest. A general method is developed and two particular problems, one the flow past a yawed infinite wedge and the other flow past a yawed infinite circular cylinder are studied in detail. The skin friction coefficients and dimensionless displacement thicknesses in the chordwise and spanwise directions are calculated for small and large times and their variations with angle and perturbation parameters and also with time are analysed. The resultant skin friction and angle of deflection are also determined. The ranges of validity of solutions for small and large times are found graphically.

1. Introduction

Recently, Gupta (1972) studied the solution of unsteady three dimensional incompressible boundary layer equations on a yawed infinite cylinder when the cylinder is in an arbitrary motion with wall velocity $\varepsilon u_b(t)$ in the chordwise direction and the main stream velocity components are steady without swirl. He used the steady three dimensional incompressible boundary layers to solve the unsteady three dimensional incompressible boundary layers on a yawed cylinder.

The present paper is devoted to study the second type of problem in which the cylinder is at rest and the main stream velocity component in the chordwise direction is unsteady. That is, we find the solution of unsteady three dimensional incompressible boundary layer equations on a yawed infinite cylinder when the main stream velocity components in the chordwise and spanwise directions are :

$$U(x,t)=\sum_{b=0}^{\infty} c_{m+2b}\, x^{m+2b} + \varepsilon c_* x^l U_M(t),\ W=W_\infty \qquad \ldots (1.1)$$

and the cylinder being at rest (where x is the distance in the chordwise direction i.e. perpendicular to the leading edge on the surface of the cylinder, z the distance in the spanwise direction i.e. along the cylinder generator, t the time, $U_M(t)$ the arbitrary function of time; m, c_*, l the constants, ε a small reference constant parameter and U and W the main stream velocity components in the chordwise and spanwise directions). Since the cylinder is infinite, none of the flow quantities may vary along z. Following the technique and method of Gupta (1972), solutions are developed for small and large times. The chordwise and spanwise equations are solved simultaneously. A set of general differential equations are obtained from which we can deduce the solution for flow past a yawed infinite flat plate, flow past a yawed infinite wedge at zero angle of attack, flow past a yawed infinite circular cylinder, etc., by giving different values to the constants. In this paper the flow past a yawed infinite wedge and a yawed infinite circular cylinder are studied in detail. The skin friction components and displacement thicknesses in chordwise and spanwise directions are calculated for small and large times and the results are interpreted graphically. The resultant skin friction and angle of deflection are also determined.

In some cases, the joining of skin friction components for small and large times is not so smooth due to the slow convergence of small times solution. To make the solution converge more rapidly, following Meksyn (1961), we apply Euler's transformation.

2. Basic Equations

If the main stream is given by (1.1) and the flow is independent of the spanwise coordinate z, then the equations of motion for the three dimensional incompressible laminar boundary layer on a cylinder are :

$$\left.\begin{aligned} \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}&=\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}\\ \frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}&=\nu\frac{\partial^2 w}{\partial y^2}\\ \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}&=0 \end{aligned}\right\}\quad \dots(2.1)$$

with boundary conditions :

$$u=v=w=0 \text{ at } y=0,\ (x>0) \text{ and } u\to U,\ w\to W \text{ as } y\to\infty;$$

where x and z denote the coordinates in the wall surface, y denoting coordinate which is perpendicular to the wall, u, v, w are the velocity components in the x, y, z directions respectively and ν the kinematic viscocity.

Linearizing eqns. (2.1) by substituting (2.2) from Gupta (1972), and $U(x, t) = U_0(x) + \varepsilon U_1(x, t)$ and neglecting ε^2 terms, we get steady as well as equations for unsteady parts. The equations for unsteady parts u_1, v_1 and w_1 are :

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_0}{\partial x} + u_0 \frac{\partial u_1}{\partial x} + v_1 \frac{\partial u_0}{\partial y} + v_0 \frac{\partial u_1}{\partial y} = \frac{\partial U_1}{\partial t} + \frac{\partial (U_0 U_1)}{\partial x} + \nu \frac{\partial^2 u_1}{\partial y^2} \quad \text{... (2.2)}$$

$$\frac{\partial w_1}{\partial t} + u_1 \frac{\partial w_0}{\partial x} + u_0 \frac{\partial w_1}{\partial x} + v_1 \frac{\partial w_0}{\partial y} + v_0 \frac{\partial w_1}{\partial y} = \nu \frac{\partial^2 w_1}{\partial y^2} \quad \text{... (2.3)}$$

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0 \quad \text{... (2.4)}$$

with boundary conditions :

$$u_1 = v_1 = w_1 = 0 \text{ at } y = 0 \text{ and } u_1 \to U_1(x, t),\ w_1 \to 0 \text{ as } y \to \infty.$$

In preparation for the detailed analysis of unsteady boundary layer equations from Gupta (1972), the steady solution is quoted below :

$$U_0(x) = \sum_{b=0}^{\infty} c_{m+2b}\, x^{m+2b}, \quad \eta = \left[\frac{(m+1) c_m\, x^{m-1}}{2\nu} \right]^{1/2} y \quad \text{... (2.5)}$$

$$\psi_0(x, y) = \left(\frac{2\nu}{(m+1)\, c_m\, x^{m-1}} \right)^{1/2} \left[2 \sum_{b=0}^{\infty} (1 + b) x^{m+2b}\, c_{m+2b} \times f_{1+2b}(\eta) - c_m x^m f_1(\eta) \right] \quad \text{... (2.6)}$$

$$w_0 = W_\infty \left[\sum_{b=0}^{\infty} x^{n-1+2b}\ g_{2b}(\eta)\ \frac{c_{m+2b}}{c_m} \right] \quad \text{... (2.7)}$$

where $u_0 = \dfrac{\partial \psi_0}{\partial y}$, $v_0 = -\dfrac{\partial \psi_0}{\partial x}$, and c_{m+2b}, n, m are constants.

The above equations are general for flow past a yawed infinite cylinder and will reduce to particular problems of yawed infinite wedge and yawed infinite circular cylinder as given below.

(A) *Yawed infinite wedge*—The above analysis corresponds to the flow past a yawed infinite wedge at zero angle of attack if :

$$n = 1,\ U_0(x) = c_m x^m \text{ i.e. } c_m > 0 \text{ and } c_{m+2b} = 0 \text{ for } b > 1. \quad \text{... (2.8)}$$

The values of $f_1''(0)$ and $g_0'(0)$ for different values of β defined by $\beta = 2m/(m+1)$ are given in Table 1 of Gupta (1972).

(B) *Yawed infinite circular cylinder*—The above analysis corresponds to the flow past a yawed infinite circular cylinder of radius R if :

$$m = n = 1 \text{ and } c_{1+2b} = \frac{(-1)^b\, 2U_\infty}{(1+2b)!\, R^{1+2b}} \text{ from (1.2) (Gupta 1972).} \qquad \ldots (2.9)$$

The values of $f_1''(0)$, $f_3''(0)$, $g_5''(0)$ and $h_5''(0)$ are given in Table 2 of Gupta (1972) and f_5 have been expressed in terms of universal function as :

$$f_5(\eta) = g_5(\eta) + \frac{c^2_{m+2}}{c_m\, c_{m+4}}\, h_5(\eta). \qquad \ldots (2.10)$$

3. Solution for Large Times

The equations (2.2) and (2.4) constitute unsteady two dimensional equations. The equation (2.3) which is for the spanwise flow can be solved only if we know the solutions of (2.2) and (2.4). First we shall solve equation (2.2) by introducing a stream function ψ_1 such that (2.4) is satisfied.

Thus

$$u_1 = \frac{\partial\psi_1}{\partial y} \text{ and } v = -\frac{\partial\psi_1}{\partial x}. \qquad \ldots (3.1)$$

Let us assume the perturbation to be of the form:

$$U_1(x, t) = c_* x^l U_M(t) \qquad \ldots (3.2)$$

where c_* and l are constants Following Gupta (1972), and analogous to (2.6), we assume:

$$\psi_1(x, y, t) = c_*\left(\frac{2\nu}{(m+1)\,c_m}\right)^{1/2} \sum_{i=0}^{\infty}\sum_{b=0}^{\infty} \frac{c_{m+2b}}{c_m}\, \frac{d^i U_M}{dt_1^i}\, x^{\frac{1}{2}\{2l+4b+(1-m)(2i+1)\}}\, A_i^{(2b)}(\eta) \qquad \ldots (3.3)$$

where $t_1 = c_m t$, $c_m > 0$. Substituting (2.6), (3.1) to (3.3) in (2.2) and equating the coefficients of $\frac{d^i U_M}{dt_1^i}\, x^{\{l+2b+(1-m)(i-1)\}}$, we get

$$A_i'''^{(2b)}(\eta) + 2\sum_{a=0}^{b} \frac{c_{m+2a}}{c_m}\, \frac{c_{m+2b-2a}}{c_{m+2b}}\left(\frac{1+b-a}{1+m}\right)$$

$$(4b-4a+1)f_{1+2b-2a}(\eta)\,A_i''^{(2a)}-f_1(\eta)\,A_i''^{(2b)}$$

$$+2\left\{\frac{l+2b+(1-m)i+m}{m+1}\right\}f_1'(\eta)\,A_i'^{(2b)}(\eta)$$

$$-4\sum_{a=0}^{b}\left(\frac{1+b-a}{m+1}\right)(l+2b+(1-m)i+m)\;\frac{c_{m+2a}}{c_m}\;\frac{c_{m+2b-2a}}{c_{m+2b}}$$

$$\times f'_{1+2b-2a}A_i'^{(2a)}\eta+2\sum_{a=0}^{b}\frac{c_{m+2a}}{c_m}\;\frac{c_{m+2b-2a}}{c_{m+2b}}\left(\frac{1+b-a}{m+1}\right)$$

$$\times\{2l+4a+(1-m)(2i+1)\}f''_{1+2b-2a}(\eta)\,A_i^{(2a)}(\eta)$$

$$-\left\{\frac{2l+4b+(1-m)(2i+1)}{m+1}\right\}f_1''(\eta)\,A_i^{(2b)}(\eta)$$

$$+\left(\frac{2}{m+1}\right)\frac{c_m}{c_{m+2b}}\;\delta_{i1}\,\delta_{b0}+(l+m+2b)\delta_{i0}-\left(\frac{2}{m+1}\right)$$

$$\times A_{i-1}'^{(2b)}(\eta)=0. \qquad \ldots(3.4)$$

with boundary conditions

$$A_0^{(0)}(0)=A_0'^{(0)}(0)=0,\;A_0'^{(0)}(\infty)=1,\;A_i'^{(2b)}(0)=A_i'^{(2b)}=A_i'^{(2b)}(\infty)=0 \qquad \ldots(3.5)$$

for all other values of i and b, where

$$\delta_{ij}=0,\qquad i\neq j \text{ and } A_{-1}'^{(2b)}=0. \qquad \ldots(3.6)$$
$$=1,\qquad i=j.$$

Analogous to (2.11), we can express $A_i^{(4)}(\eta)$ in terms of the universal functions by assuming

$$A_i^{(4)}(\eta)=L_i^{(4)}(\eta)+\frac{c^2_{m+2}}{c_m\,c_{m+\cdot}}M_i^{(4)}(\eta). \qquad \ldots(3.7)$$

Substituting (3.7) in (3.4) and equating the terms containing the same coefficients, we get the differential equations for $L_i^{(4)}(\eta)$, $M_i^{(4)}(\eta)$. To solve eqn. (2.3) which is for the spanwise flow we assume

$$w_1=W_\infty\;c_*\sum_{i=0}^{\infty}\sum_{b=0}^{\infty}\frac{c_{m+2b}}{c_m{}^2}\;\frac{d^iU_M}{dt_1{}^i}\;x^{\{l+2b+(1-m)i+n-m-1\}}\;P_i^{(2b)}(\eta). \qquad \ldots(3.8)$$

Substituting (2.6), (2.7), (3.1) to (3.3) and (3.8) in equation (2.3) and equating the coefficients of $\frac{d^i U_M}{dt_1^i} x^{(l+2b+(1-m)i+n-2)}$, we get

$$P_i''^{(2b)}(\eta)+2\sum_{a=0}^{b}(1+b-a)\left(\frac{4b-4a+m+1}{m+1}\right)\frac{c_{m+2a}}{c_m}\frac{c_{m+2b-2a}}{c_{m+2b}}f_{1+2b-2a}(\eta)\,P_i'^{(2a)}(\eta)$$

$$-f_1(\eta)\,P_i'^{(2b)}(\eta)-4\sum_{a=0}^{b}\left(\frac{1+b-a}{1+m}\right)(l+2a+(1-m)i+n-m-1)$$

$$\frac{c_{m+2a}}{c_m}\frac{c_{m+2b-2a}}{c_{m+2b}}f'_{1+2b-2a}(\eta)\,P_i^{(2a)}(\eta)+2\left\{\frac{l+2b+(1-m)i+n-m-1}{1+m}\right\}$$

$$\times f_1'(\eta)P_i^{(2a)}(\eta)-2\sum_{a=0}^{b}\left\{\frac{2b-2a+n-1}{m+1}\right\}\frac{c_{m+2a}}{c_m}\frac{c_{m+2b-2a}}{c_{m+2b}}$$

$$\times g_{2b-2a}(\eta)\,A_i'^{(2a)}(\eta)+\sum_{a=0}^{b}\left\{\frac{2l+4a+(1-m)(2i+1)}{m+1}\right\}\frac{c_{m+2a}}{c_m}\frac{c_{m+2b-2a}}{c_{m+2b}}$$

$$\times g'_{2b-2a}(\eta)\;A_i^{(2a)}(\eta)-\left(\frac{2}{m+1}\right)P_{i-1}^{(2b)}(\eta)=0 \qquad \dots(3.9)$$

with boundary conditions

$$P_i^{(2b)}(0)=P_i^{(2b)}(\infty)=0,\text{ for all } i \text{ and } b \text{ and } P_{-1}^{(2b)}(\eta)=0. \qquad \dots(3.10)$$

Thus the unsteady three-dimensional boundary layer equations are reduced to ordinary differential equations. These equations have been integrated numerically and the values are given in Appendix 1.

The unsteady parts of skin friction components in the chordwise and spanwise directions are

$$\tau_{1u}=\mu\left(\frac{\partial^2\psi}{\partial y^2}\right)_{y=0}=\mu\left(\frac{(m+1)\,c_m x^{m-1}}{2\nu}\right)c_*$$

$$\times\sum_{i=0}^{\infty}\sum_{b=0}^{\infty}\frac{c_{m+2b}}{c_m}x^{(l+2b+(1-m)i)}\,\frac{d^i U_M}{dt_1^i}\,A_i''^{(2b)}(0) \qquad \dots(3.11)$$

and

$$\tau_{1w} = \mu\left(\frac{\partial w_1}{\partial y}\right)_{y=0} = \mu c_* W_\infty\left(\frac{(m+1)c_m x^{m-1}}{2\nu}\right)$$

$$\times \sum_{i=0}^{\infty}\sum_{b=0}^{\infty} \frac{c_{m+2b}}{c^2_m} x^{(l+2b+(1-m)i+n-m-1)} \frac{d^i U_M}{dt_1{}^i} P_i{}'^{(2b)}(0). \qquad \ldots (3.12)$$

where μ being the coefficient of viscosity. The unsteady parts of the displacement thicknesses in the chordwise and spanwise directions are given by

$$\frac{\delta_u{}^* - \delta_{0u}{}^*}{\varepsilon} = \frac{1}{U_0^2}\left(\frac{2\nu}{(m+1)c_m x^{m-1}}\right)^{1/2}\Bigg[U_1\Bigg\{\sum_{b=0}^{\infty} c_{m+2b}\, x^{m+2b}$$

$$\times \underset{\eta\to\infty}{\text{Limit}}\left(2(1+b)\, f_{1+2b}(\eta) - \frac{U_0}{c_m x^m} A_0{}^{(2b)}(\eta)\right) - c_m x^m f_1(\infty)\Bigg\}$$

$$- U_0 c_m \sum_{b=0}^{\infty}\sum_{i=1}^{\infty} x^{(l+2b+(1-m)i)} \frac{c_{m+2b}}{c_m} A_i{}^{(2b)}(\infty)\frac{d^i U_M}{dt_1{}^i}\Bigg] \qquad \ldots (3.13)$$

and

$$\frac{\delta_w{}^* - \delta_{0w}{}^*}{\varepsilon} = -\left(\frac{2\nu}{(m+1)\, c_m x^{m-1}}\right)^{1/2} c_* x^l \sum_{i=0}^{\infty}\sum_{b=0}^{\infty}\Bigg[x^{(2b+(1-m)i+n-m-1)}$$

$$\times \frac{c_{m+2b}}{c_m{}^2} \frac{d^i U_M}{dt^i{}_1} \int_0^\infty P_i{}^{(2b)}(\eta)d\eta\Bigg] \qquad (3.14)$$

where $\delta_{0u}{}^*$ and $\delta_{0w}{}^*$ are the basic parts of the displacement thickness in the chordwise and spanwise directions. Thus for large times when the main stream is in unsteady motion in the chordwise direction, the skin friction and the displacement thickness in the chordwise direction are given by (3.11) and (3.13) whereas the skin friction and displacement thickness in the spanwise directions are given by (3.12) and (3.14). Now we shall consider below two particular problems: (i) a yawed infinite wedge and (ii) a yawed infinite circular cylinder.

(A) *Flow past a yawed infinite wedge*—In this case the steady main stream velocity in the chordwise direction and other constants are given by (2.8). The equations (3.11) to (3.14) become

$$C_Q=\frac{\tau_{1u}\sqrt{\dfrac{U_0x}{\nu}}}{\left(\dfrac{\mu}{\nu}U_0U_1\right)}=\left(\frac{1}{2-\beta}\right)^{1/2}\Big[A_0''^{(0)}(0)+\frac{x}{U_0(x)U_M(t)}A_1''^{(0)}(0)\times\left(\frac{dU_M}{dt}\right)+O\left(\frac{d^2U_M}{dt^2}\right)\Big] \quad \ldots(3.15)$$

$$D_Q=\frac{\tau_{1w}\sqrt{\dfrac{U_0x}{\nu}}}{\left(\dfrac{\mu}{\nu}W_\infty U_1\right)}=\left(\frac{1}{2-\beta}\right)^{1/2}\Big[P_0'^{(0)}(0)+\frac{x}{U_0U_M}P_1'^{(0)}(0)\frac{dU_M}{dt}+O\left(\frac{d^2U_M}{dt^2}\right)\Big] \quad \ldots(3.16)$$

$$\frac{\delta_u^*-\delta_{0u}^*}{\varepsilon}=\frac{U_1}{U_0}\left(\frac{\nu x}{U_0}\right)^{1/2}(2-\beta)^{1/2}\Big[\operatorname*{Limit}_{\eta\to\infty}(f-A_0^{(0)})-\frac{x}{U_0U_M}A_1^{(0)}(\infty)\frac{dU_M}{dt}+O\left(\frac{d^2U_M}{dt^2}\right)\Big] \quad \ldots(3.17)$$

$$\frac{\delta_w^*-\delta_{0w}^*}{\varepsilon}=-\frac{U_1}{U_0}\left(\frac{\nu x}{U_0}\right)(2-\beta)^{1/2}\Big[\int_0^\infty P_0^{(0)}(\eta)\,d\eta+\frac{x}{U_0U_M}\frac{dU_M}{dt}\int_0^\infty P_1^{(0)}(\eta)\,d\eta+O\left(\frac{d^2U_M}{dt^2}\right)\Big] \quad \ldots(3.18)$$

where $\operatorname*{Limit}_{\eta\to\infty}(f-A_0^{(0)})$, $A_0''^{(0)}(0)$, $P_0'^{(0)}(0)$, $A_1^{(0)}(\infty)$, $\int_0^\infty P_0^{(0)}(\eta)\,dn$ etc. are tabulated in appendix 1 for $l=0, 1, 2$ and for $\beta=0, 0.2, 0.4$. In particular when the stream is increased and set impulsively in uniform motion in chordwise direction at $t=0$ by velocity $\varepsilon\, c_* x^l$, then $U_M(t)$ becomes a unit function and eqns. (3.15) to (3.18) reduce to

$$C_Q=\left(\frac{1}{2-\beta}\right)^{1/2}A_0''^{(0)}(0),\; D_Q=\left(\frac{1}{2-\beta}\right)^{1/2}P_0'^{(0)}(0) \quad \ldots(3.19)$$

$$\frac{\delta_u^*-\delta_{0u}^*}{\varepsilon}=\frac{U_1}{U_0}\left(\frac{\nu x}{U_0}\right)^{1/2}(2-\beta)^{1/2}\operatorname*{Limit}_{\eta\to\infty}(f-A_0^{(0)}) \quad \ldots(3.20)$$

$$\frac{\delta_w{}^*-\delta_{0w}{}^*}{\varepsilon} = -\frac{U_1}{U_0}\left(\frac{\nu x}{U_0}\right)^{1/2} (2-\beta)^{1/2} \int_0^\infty P_0{}^{(0)}(\eta)\, d\eta. \quad \text{...(3.21)}$$

The unsteady part of the resultant skin friction τ_{1R} is defined by (3.25) in Gupta (1972) and in this case of impulsive motion after substituting the values of τ_{0u}, τ_{1u}, τ_{0w}, etc. it becomes.

$$\tau_{1R} = \mu\left(\frac{U_0}{\nu x}\right)^{1/2} \frac{U_1}{U_0}\left(\frac{1}{2-\beta}\right)^{1/2} \frac{[U_0^2 f_1''(0)\, A_0''^{(0)}(0) + W_\infty{}^2\, g'(0) P_0'^{(0)}(0)]}{[U_1^2 f''^2_1(0) + W_\infty{}^2\, g'^2_0(0)]^{1/2}} \quad \text{...(3.22)}$$

The angle of deflection which is equal to the angle between the resultant skin friction and the direction in the main stream is :

$$\gamma = \gamma_0 + \varepsilon\gamma_1 = \tan^{-1}\frac{\tau_w}{\tau_u} - \tan^{-1}\frac{W}{U},$$ where γ_0 is defined by (3.29) in Gupta (1972)

and γ_1 is :

$$\gamma_1 = \frac{1}{\tau_{0R}^2}(\tau_{0u}\tau_{1w} - \tau_{0w}\tau_{1u}) + \frac{W_\infty U_1}{U_0^2 + W_\infty{}^2} \quad \text{...(3.23)}$$

$$= \frac{U_1 W_\infty \{f_1''(0)\, P_0'^{(0)}(0) - g_0'(0)\, A_0''^{(0)}(0)\}}{\{U_0^2 f''^2_1(0) + W_\infty{}^2\, g_0'^2(0)\}} + \frac{W_\infty U_1}{U_0^2 + W_\infty{}^2}. \quad \text{...(3.24)}$$

From eqn. (3.19), we see that the skin friction coefficients depend on the parameters l and β. These increase with l and decrease with β in the impulsive motion ($U_M(t) = 1$). In accelerated motion ($U_M = 0\,(t)$) from (3.15), C_Q is larger than that in impulsive motion (when $U_M(t) = 1$) up to a certain value of l say l_0 (depending on β) and it exceeds when $l > l_0$. But the values of D_Q from (3.16) are always less in accelerated motion than that in impulsive motion for all l ($= 0, 1, 2$). The unsteady part of the deflection of the velocity vector can be understood from eqn. (3.24) by giving numerical values to the dimensionless quantities.

From eqns. (3.20) and (3.21), we understand that the magnitude of the displacement thicknesses in the chordwise and spanwise directions increase with an increase in l and decrease with an increase in β for an impulsive motion.

(B) *Flow past a yawed infinite circular cylinder*—In this case the steady main stream velocity in the chordwise direction and the other constants are given by (2.9) and (1.2) (Gupta 1972). In case when the main stream is increased and set impulsively in uniform motion in the chordwise direction at $t = 0$ with velocity $\varepsilon c_* x^l$, then eqns. (3.11) to (3.14) reduce to (using the values for c_{1+2b}'s given by 2.9)):

$$C_Q = \frac{\tau_{1u}\sqrt{\frac{c_1 x^2}{\nu}}}{\frac{\mu}{\nu}(c_1 x)U_1} = \Big[A_0''^{(0)}(0) - \frac{1}{3!}A_0''^{(2)}(0)\xi^2 + \frac{1}{5!}(L_0''^{(4)}(0) + \frac{10}{3}M_0''^{(4)}(0)\,\xi^4 + O(\xi^6)\Big], \qquad \ldots(3.25)$$

$$D_Q = \frac{\tau_{1w}\sqrt{\frac{c_1 x^2}{\nu}}}{\frac{\mu}{\nu}U_1 W_\infty} = \Big[P_0'^{(0)}(0) - \frac{1}{3!}P_0'^{(2)}(0)\xi^2 + O(\xi^4)\Big] \qquad \ldots(3.26)$$

$$\frac{\delta_u^* - \delta^*_{0u}}{\varepsilon} = \frac{1}{U_0^2}\left(\frac{\nu}{c_1}\right)U_1 c_1 x\Big[\operatorname*{Limit}_{\eta\to\infty}\left(f_1(\eta) - \frac{U_0}{c_1 x}A_0^{(0)}(\eta)\right) - \frac{1}{3!}\operatorname*{Limit}_{\eta\to\infty}\left(4f_3 - \frac{U_0}{c_1 x}A_0^{(0)}\right)^2 + \frac{1}{5!}\operatorname*{Limit}_{\eta\to\infty}\left(6f_5(\eta) - \frac{U_0}{c_1 x}A_0^{(4)}(\eta)\right)\xi^4 + O(\xi^6)\Big] \qquad \ldots(3.27)$$

$$\frac{\delta_w^* - \delta^*_{0w}}{\varepsilon} = -\left(\frac{\nu}{c_1}\right)^{1/2}\frac{U_1}{c_1 x}\Big[\int_0^\infty P_0^{(0)}(\eta)\,d\eta - \frac{1}{3!}\xi^2\int_0^\infty P_0^{(2)}(\eta)\,d\eta + O(\xi^4)\Big] \qquad \ldots(3.28)$$

where $A_0''^{(0)}(0)$, $P'^{(0)}(0)$, $\operatorname*{Limit}_{\eta\to\infty}(f_1 - A_0^{(0)})$, etc. are given in appendix 1.

The steady and unsteady parts of resultant skin friction are defined in (3.25) (Gupta 1972) and in case of yawed infinite circular cylinder the steady resultant skin friction is given by (3.34) (Gupta 1972). The unsteady resultant skin friction in case of impulsive motion (after substituting the values of τ_{0u}, τ_{1u} etc.) is written as :

$$\tau_{1R} = \frac{U_1\mu^2\left(\frac{c_1}{\nu}\right)}{\tau_{0R}}\Big[2U_\infty\Big\{\xi f_1''(0) - \frac{4}{3!}\xi^3 f_3''(0) + \ldots\Big\}\Big\{A_0''^{(0)}(0) - \frac{1}{3!}\xi^2 A_0''^{(2)}(0) + \ldots\Big\} + \frac{W^2_\infty}{c_1 x}\Big\{g_0'(0) - \frac{1}{3!}\xi^2 g_2'(0) + \ldots\Big\}\Big\{P_0'^{(0)}(0) - \frac{1}{3!}\xi^2 P'^{(2)}_0(0) + \ldots\Big\}\Big]. \qquad \ldots(3.29)$$

The unsteady angle of deflection is defined by (3.23) and in case of circular cylinder it is given by

$$\gamma_1 = \frac{\mu^2\left(\frac{c_1}{\nu}\right)W_\infty U_1}{\tau_{0R}^2}\left[\left\{P_0'^{(0)}(0) - \frac{1}{3!}\xi^2 P_0'^{(2)}(0) + \ldots\right\}\left\{f''_1(0) - \frac{4}{3!}\xi^2 f_3''(0) + \ldots\right\}\right.$$

$$+\left.\left\{A_0^{(0)}(0) - \frac{1}{3!}\xi^2 A''^{(2)}(0) + \ldots\right\}\left\{g_0'(0) - \frac{1}{3!}\xi^2 g_2'(0) + \ldots\right\}\right]$$

$$+\frac{U_1 W_\infty}{U_0^2 + W_\infty^2}\,. \qquad \ldots(3.30)$$

From the above analysis (after substituting numerical values from the appendix 1), we observe that C_Q given by (3 25) decreases whereas D_Q given by (3.26), increases with an increase in ξ for $l = 0, 1, 2$. For a given value of ξ, C_Q and D_Q, increases with an increase in l.

4. Solution for Small Times

Following Sarma (1964) and solving eqns. (2.2), (2 4) and then eqn. (2.3) for small times, we get the skin friction coefficients C_Q and D_Q along chordwise and spanwise directions for a yawed infinite wedge and a yawed infinite circular cylinder as given below.

(A) *Flow past a yawed infinite wedge*— The skin friction coefficients C_Q and D_Q along chordwise and spanwise directions [making use of the values (2.8), (2.9)] for an impulsive motion i.e. when $U_M(t) = 1$ become (for detailed analysis see Gupta 1970).

$$C_Q = \frac{\tau_{1u}\sqrt{\frac{U_0 x}{\nu}}}{\left(\frac{\mu}{\nu}\right)U_0 U_1} = \left[\frac{1}{(\pi N)^{1/2}} + \frac{2(2l - \beta l + \beta)}{(2-\beta)\sqrt{\pi}} N^{1/2}\right.$$

$$+\frac{1}{16}\left\{\frac{(5-4l)(2-\beta) - 15\beta}{(2-\beta)^{3/2}}\right\} f_1''(0)\, N$$

$$-\beta\left\{\frac{(18l+39)(2-\beta) - 78\beta}{36(2-\beta)^2}\right\}\frac{N^{3/2}}{\sqrt{\pi}} +$$

$$+\left.\frac{1}{32}\left\{\frac{(2l - l\beta + \beta)(9-4l)(2-\beta)\ 19\beta}{(2-\beta)^{5/2}}\right\} f_1''(0)\, N^2 + O(N^{5/2})\right]$$

$$\ldots(4.1)$$

$$D_Q = \frac{\tau_{1w}\sqrt{\frac{U_0 x}{\nu}}}{\left(\frac{\mu}{\nu}\right) W_\infty U_1}\Bigg[\left\{\frac{(4l+\beta)(2-\beta)-3\beta}{8(2-\beta)^{3/2}}\right\} g_0'(0)\, N +$$

$$+ \frac{(2l - l\beta + \beta)(4l-1)(2-\beta)+\beta)}{(2-\beta)^{5/2}} g_0'(0)\, N^2 + O N^{5/2}\Bigg] \cdot \qquad \text{.. (4.2)}$$

where

$$N = (c_m x^m)\, t/x \text{ (a non-dimensional variable),} \qquad \text{...(4.3)}$$

and $f_1''(0)$, $g_0'(0)$ for different values of β are given in Table 1 of Gupta (1972).

The unsteady parts of the resultant skin friction and angle of deflection are defined in (3.25) (Gupta 1972) and by (3.23) and if we substitute the various values of τ_{iw}, τ_{1w} etc. (from small time solution), we can find their values. The skin friction coefficients C_Q and D_Q given by (4.1) and (4.2) are slow convergent series. To make them more convergent so that there is an agreement for small and large times solution following Meksyn (1961), we apply Euler's transformation to (4.1) (leaving first term) defind by

$$N^{1/2} = \frac{A}{1-A} \cdot \qquad \text{.. (4.4)}$$

and to (4.2) defined by

$$N = \frac{A_1}{1-A_1} \cdot \qquad \text{...(4.5)}$$

Substituting (4.4) in (4.1) and (4.5) in (4.2) and expanding in powers of A and A_1, we get

$$C_Q = \frac{1}{\sqrt{\pi}}\left(\frac{1-A}{A}\right) + \frac{2(2l-l\beta+\beta)}{(2-\beta)\sqrt{\pi}} A + \left[\frac{2(2l-l\beta+\beta)}{(2-\beta)\sqrt{\pi}} + \right.$$

$$\left. + \frac{1}{16(2-\beta)^{1/2}}\left(\frac{(5-4l)(2-\beta)-15\beta}{(2-\beta)}\right) f_1''(0) \right] A^2$$

$$+ \left[\frac{2(2l-l\beta+\beta)}{(2-\beta)\sqrt{\pi}} + \frac{1}{8(2-\beta)^{1/2}}\left(\frac{(5-4l)(2-\beta)-15\beta}{2-\beta}\right)\right.$$

$$\left. \times f_1''(0) - \frac{\beta(18l+39)(2-\beta)-78\beta}{36(2-\beta)^2\sqrt{\pi}}\right] A^3 + \left[\frac{2(2l-l\beta+\beta)}{(2-\beta)\sqrt{\pi}}\right.$$

$$+ \frac{3}{16(2-\beta)^{1/2}}\left(\frac{5-4l)(2-\beta)-15\beta}{2-\beta}\right) f_1''(0) -$$

(*equation continued on p.* 1059)

$$- \frac{\beta (18l + 39)(2-\beta) - 78\beta}{12 (2-\beta)^2 \sqrt{\pi}} + \frac{(2l - l\beta + \beta)(9 - 4l)(2-\beta)\, 19\beta}{32 (2-\beta)^{5/2}}$$

$$\left. \times f_1''(0) \right] A^4 + \ldots \quad \ldots (4.6)$$

$$D_Q = \left[\left\{ \frac{4l+3)(2-\beta) - 3\beta}{8(2-\beta)^{3/2}} \right\} g_0'(0)\, A_1 + \left[\frac{(4l+3)(2-\beta) - 3\beta}{8(2-\beta)^{3/2}} \right\} g_0'(0) \right.$$

$$\left. + \left\{ \frac{(2l - l\beta + \beta)\,[(4l-1)(2-\beta) + \beta]}{(2-\beta)^{5/2}} \right\} g_0'(0) \right] A_1^2 + \ldots \Big] . \quad \ldots (4.7)$$

From the above analysis, we observe that skin friction coefficient for small times depends on β, l and N (a non-dimensional variable). From equation (4.6), we note that chordwise skin friction coefficient C_Q is infinite at $A = 0$ and decrease with an increase in A $(= N^{1/2}/(1+N^{1/2}))$ for $\beta = 0, 0.2, 0.4$; l=0, 1, 2, and join with the steady values for large times solutions given by (3.19). These facts can be observed in Figs. 1 (a, b) for $\beta = 0.2, 0.4$ and $l = 0, 1, 2$.

From eqn. (4.7), we see that spanwise skin friction coefficient D_Q is zero at A_1 $(= N/(1 + N) = 0$. With an increase in A_1, D_Q increases and joins with the steady values of spanwise skin friction D_Q given in (3.19) for large times. These behaviours can be observed from Figs. 2 (a, b) for different values of β=0.2, 0.4; $l = 0, 1, 2$.

(B) *Flow past a yawed infinite circular cylinder*—The skin friction coefficients C_Q and D_Q along cordwise and spanwise directions for an impulsive motion (making use of the values of (2.9) become (for detailed analysis see Gupta 1970) :

$$C_Q = \frac{\tau_{1w}\sqrt{\dfrac{c_1 x^2}{\nu}}}{\left(\dfrac{\mu}{\nu} (c_1 x)\, U_1 \right)} = \frac{1}{(\pi N)^{1/2}} + 2 \left\{ (l+1) - \frac{1}{6} (l+3)\, \xi^2 \right.$$

$$\left. + \frac{1}{120} (l+5)\, \xi^4 + O(\xi^6) \right\} \frac{N^{1/2}}{\sqrt{\pi}} - \left\{ \left(\frac{2l+5}{8} \right) f_1''(0) \right.$$

$$\left. - \frac{1}{12} (2l+15) f_3''(0)\, \xi^2 + (6l+75) \frac{1}{480} f_5''(0)\, \xi^4 + O(\xi^6) \right\} N$$

$$- 4 \left\{ \left(\frac{6l-13}{48} \right) - \left(\frac{3l-19.5}{36} \right) \xi^2 + \frac{1}{1080} (18l - 195)\xi^4 + O(\xi^6) \right\} \frac{N^{3/2}}{\sqrt{\pi}}$$

$$- \left\{ \frac{(l+1)(2l+5)}{\text{[illegible]}} f_1''(0) - \frac{1}{24} (l+1)(2l+15) f_3''(0) \right.$$

$$+ (l+3) \left(\frac{2l+9}{4} \right) f_1''(0)\, \xi^2 + \frac{1}{240} \left(\frac{(2l+13)(l+5)}{8} f_1''(0) \right.$$

$$\left. + (l+3)(2l+19) \frac{5}{3} f_3''(0) + (l+1) \left(\frac{6l+75}{4} \right) f_5''(0) \right) \xi^4$$

$$\left. + O(\xi^6) \right\} N^2 + O(N^{5/2}). \quad \ldots (4.8)$$

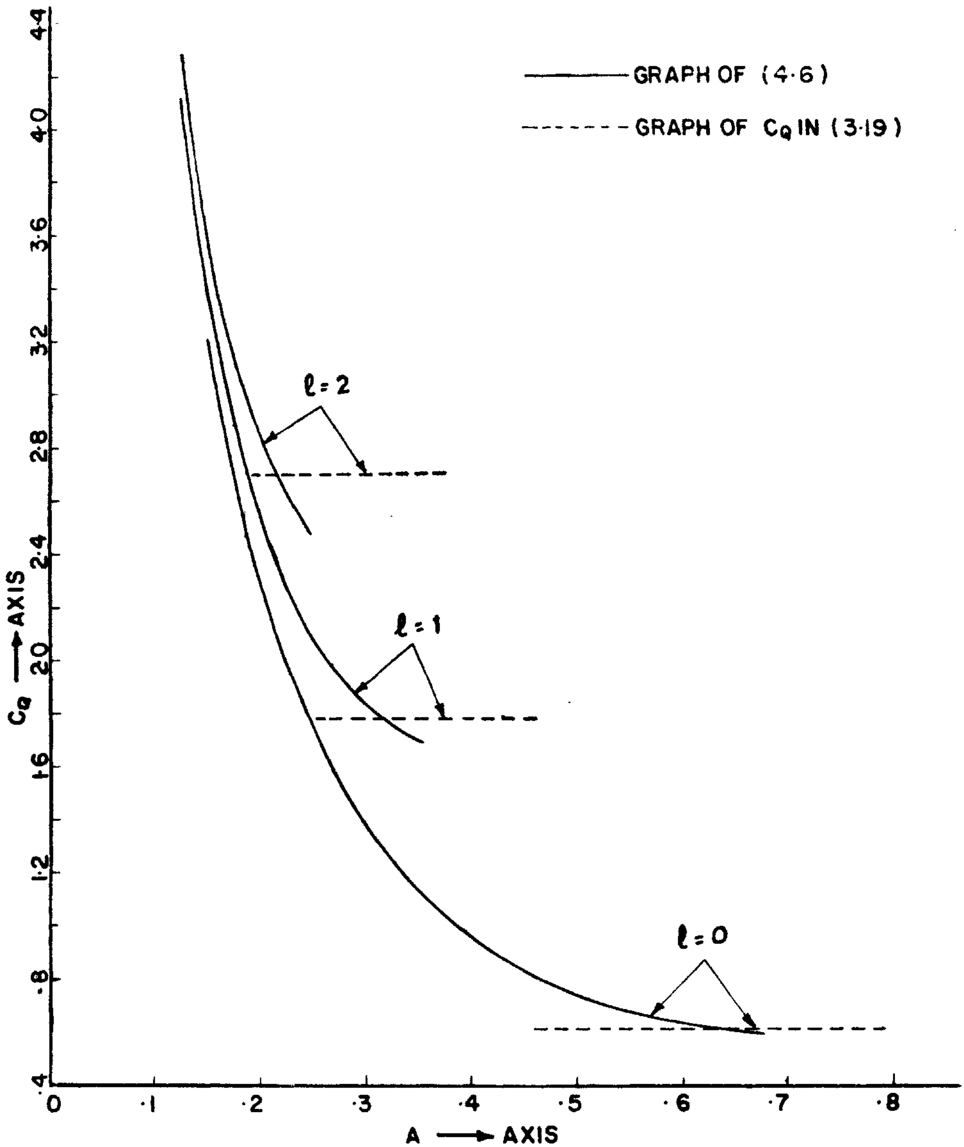

FIG. 1*a*. Variation of C_Q for small and large times when $\beta=0.2$ (wedge problem)

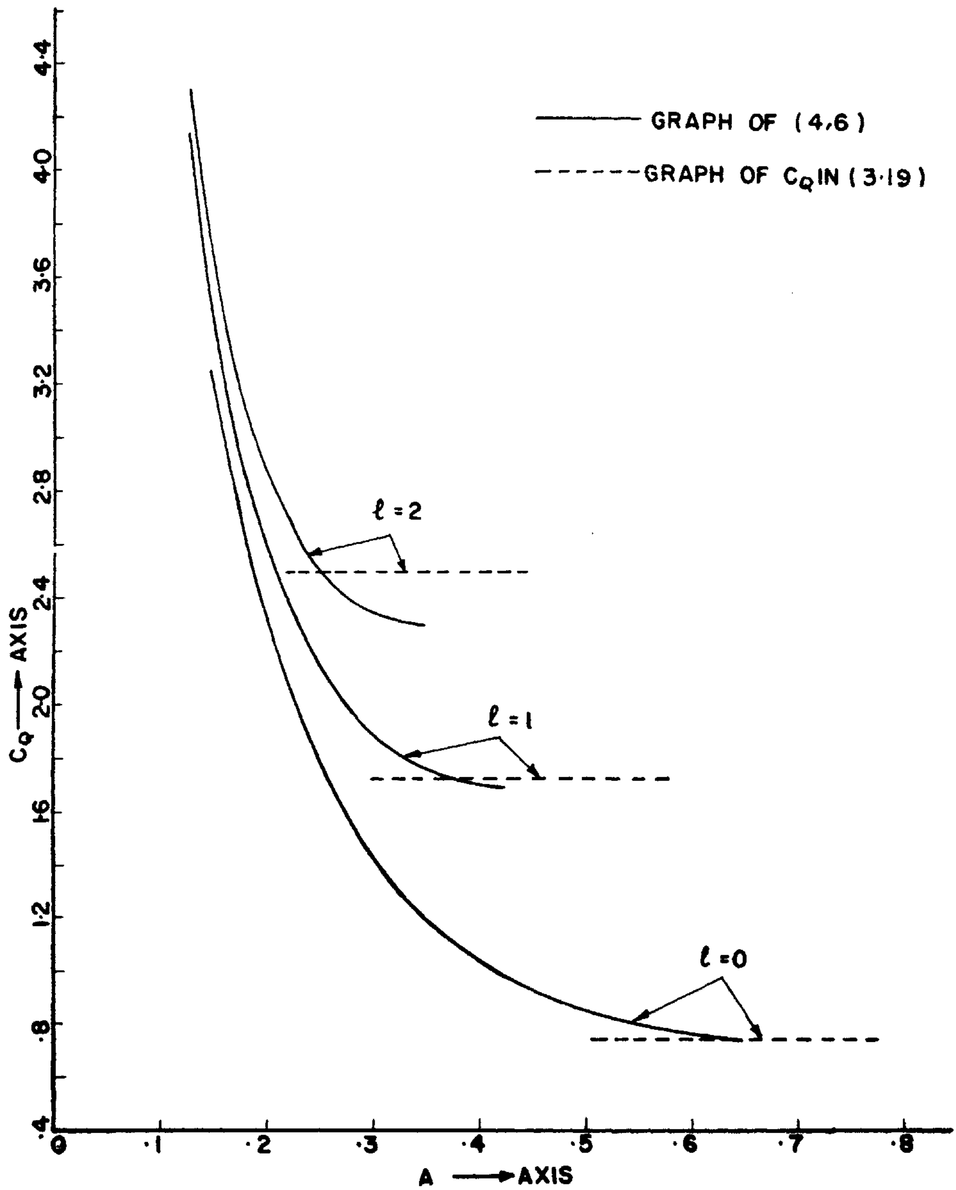

FIG. 1*b*. Variation of C_Q for large and small times when β=0.4 (wedge problem).

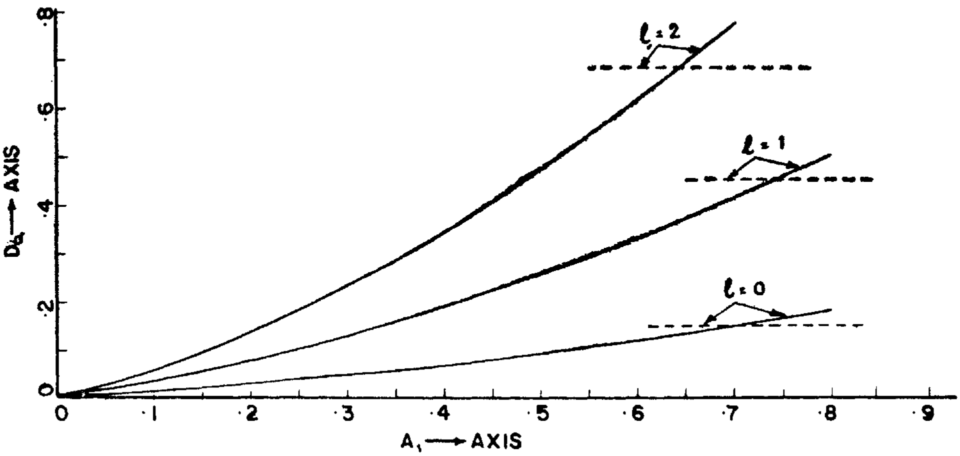

FIG. 2a. Variation of D_Q when β=0.2 (wedge problem) [——Graph of (4.7);– – –Graph of (3.19)].

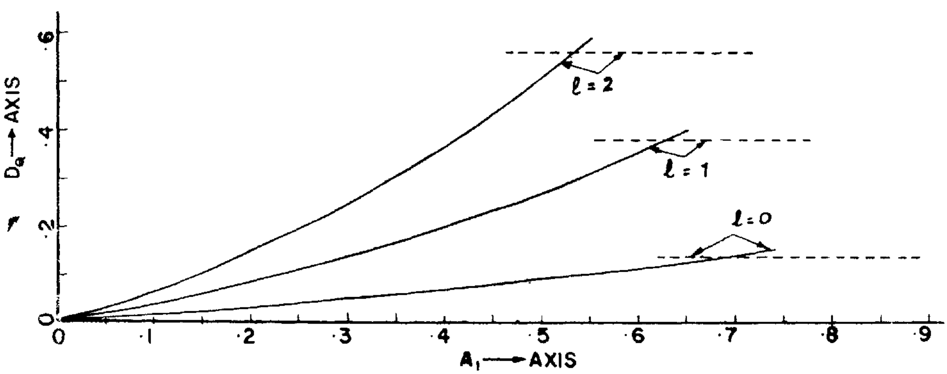

FIG. 2b. Variation of D_Q when β=0.4.

$$D_Q = \frac{\tau_{1w}\sqrt{\dfrac{c_1 x^2}{\nu}}}{\left(\dfrac{\mu}{\nu} W_\infty \; U_1\right)} = \Bigg[\Big\{ (0.28525)l - (0.04343)\,(l-3)\,\xi^2 + \ldots \Big\} N$$

$$+ \Big\{0.14262\, l\,(l+1) - \frac{1}{24}\Big(0.5212(l+1)(l-3)$$

$$+ \;0.5705\,(l+3)\,(l+2)\Big)\xi^2 + \ldots\Big\} N^2 + O\,(N^{5/2})\Bigg], \qquad \ldots (4.9)$$

where $N = c_1 t$.

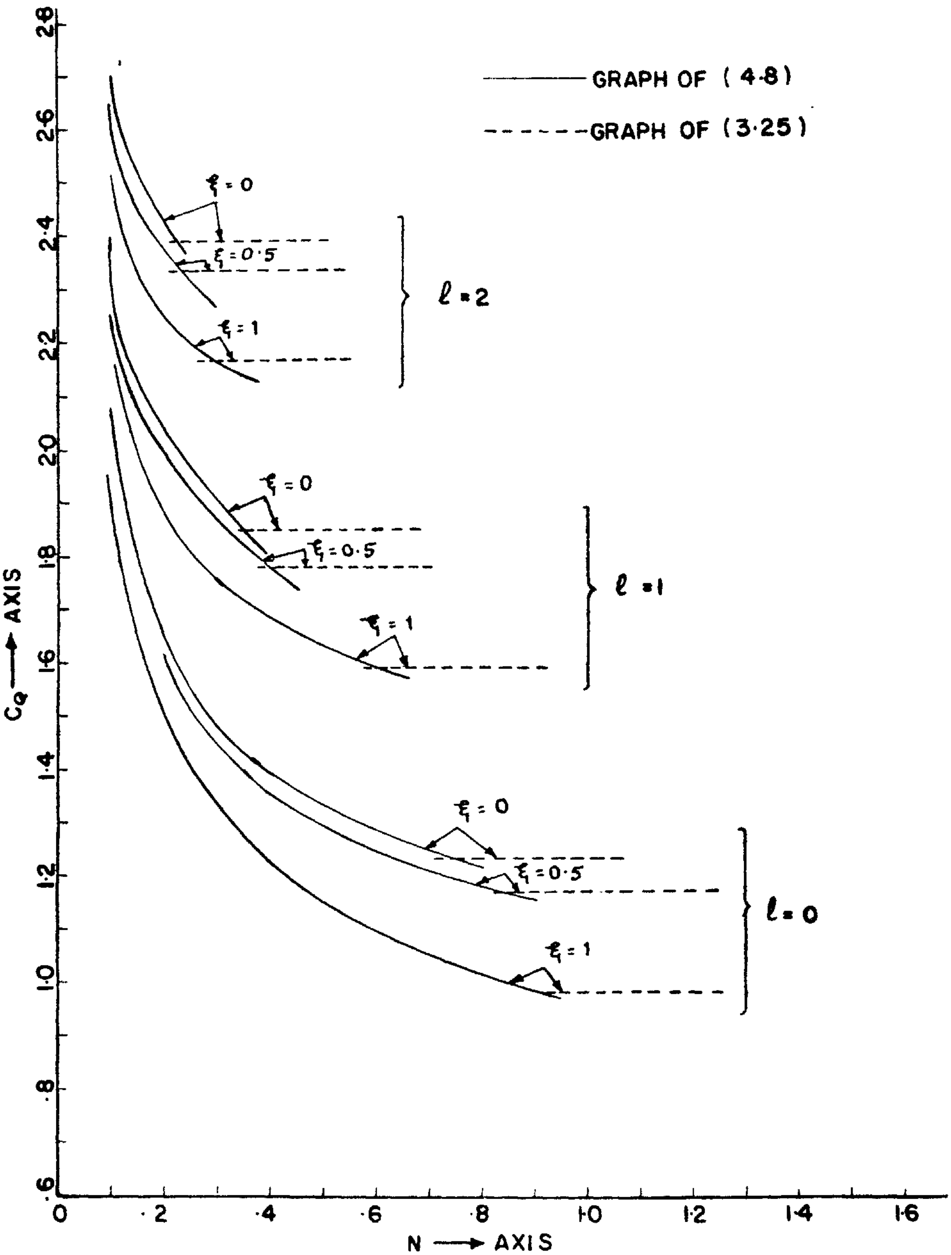

FIG. 3. Variation of C_Q for small and large times (circular cylinder problem).

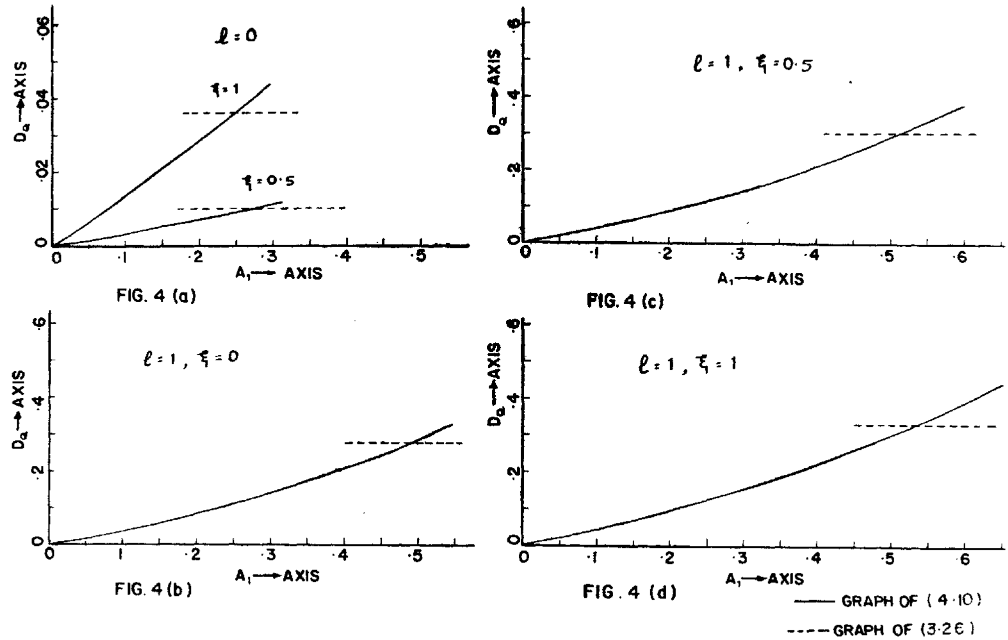

Fig. 4. Variation of D_Q for small and large times (circular cylinder problem).

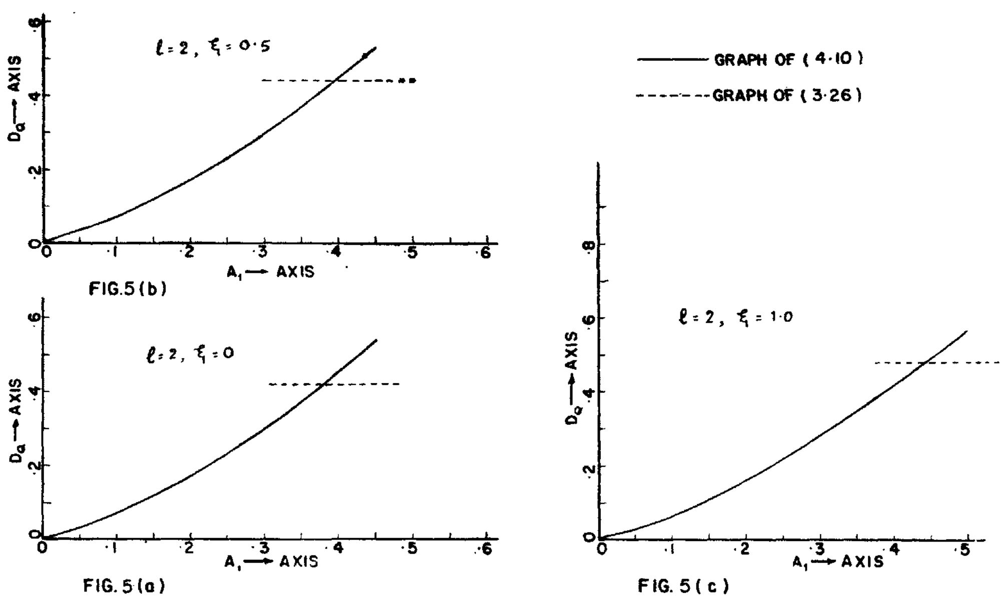

Fig. 5. Variation of D_Q for small and large times (circular cylinder problem).

The unsteady parts of the resultant skin friction and angle of deflection can also be found (for small times) if we substitute the values of τ_{0u}, τ_{1u} etc. in (3.25) (Gupta 1972) and in (3.23).

The skin friction coefficient D_Q given by (4.9) is a slow convergent series. To make it rapid so that there is an agreement of skin friction coefficient for small and large times, we apply Euler's transformation defined by (4.5). Substituting (4.5) in (4.9) and expanding in powers of A_1, we have

$$D_Q = \left\{(0.28525)l - 0.04343\,(l-3)\,\xi^2 + \ldots\right\} A_1 + \left[\left\{0.28525l - 0.043(l-3)\xi^2 + \ldots\right\} + \left\{0.1426l\,(l+1) - \frac{1}{24}(0.5212(l+1)(l-3) + 0.5705(l+2)(l+3))\,\xi^2 + \ldots\right\}\right] A_1^2 + \ldots \quad \ldots(4.10)$$

From eqn. (4.8) we observe that C_Q is infinite at $N=0$ and decreases with an increase in N and shows a smooth joining with the steady values of large times solution given by (3.25). These facts can be seen from Fig. 3 for different values of $l = 0, 1, 2$ and $\xi = 0, 0.5, 1.0$. D_Q given by (4.10) is zero at $A_1 = 0$ and increases with an increase in A_1 ($=N/(1+N)$) and joins with the steady values of large times solution given by (3.26). These facts can be observed from Figs. 4 (a, b, c, d) and 5(a, b, c).

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