BASES IN A CERTAIN SPACE OF FUNCTIONS ANALYTIC IN THE HALF-PLANE

by P. K. KAMTHAN *and* S. K. SINGH GAUTAM, *Department of Mathematics, Indian Institute of Technology, Kanpur* 16

(Communicated by J. N. Kapur, F.N.A.)

(*Received* 17 *July* 1972; *after revision* 3 *December* 1972)

In this paper we have considered the space χ of all functions defined by the Dirichlet series and analytic in the half-plane. It is shown that the space χ equipped with a certain locally convex topology L becomes a Frechet space. We have characterized the form of continuous linear functionals on χ and a continuous linear operator T from χ to χ Further, we have also established the conditions under which a base in χ becomes a proper base and have given a characterization of proper bases in terms of semi-norms which generate the topology L on χ.

1. INTRODUCTION AND TERMINOLOGY

Let us assume throughout this note that $\{\lambda_n\}$ is a sequence of real numbers such that

$$0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n \to \infty \text{ as } n \to \infty; \qquad \ldots(1.1)$$

and

$$\limsup_{n\to\infty} \frac{\log n}{\lambda_n} = 0. \qquad \ldots(1.2)$$

We are concerned in this note with functions f represented by Dirichlet series. Let therefore $\sigma_c(f)$ and $\sigma_a(f)$ be respectively the abscissa of convergence and the abscissa of absolute convergence of f. Assume further that A is a given positive number and χ_A be the class of functions f where

$$f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}, \; s = \sigma + it \; (\sigma, \; t \text{ are reals}) \text{ and where further } a_n\text{'s are}$$

complex numbers ($a_n \equiv a_n(f)$) satisfying

$$\limsup_{n\to\infty} \frac{\log |a_n|}{\lambda_n} \leqslant -A \qquad \ldots(1.3)$$

and the sequence $\{\lambda_n\}$ satisfies (1.1) and (1.2). Clearly for each $f \in X$, $\sigma_c(f) = \sigma_a(f) \geqslant A$ (Markushevich 1965, p. 33). Observe that X_A includes all entire functions represented by Dirichlet series. Henceforth we shall drop the suffix A in X_A and will write X_A as X. Also we shall write $\mathbb{C}$ for the finite complex plane equipped with its usual topology. Then X forms a vector space with usual pointwise addition and scalar multiplication. In this paper we discuss the characterization of continuous linear functionals and proper bases in terms of semi-norms defined on X.

Now for each $f \in X$, define

$$\|f\|_\sigma = \sum_{n=1}^{\infty} |a_n| \, e^{\sigma\lambda_n}, \text{ for every } \sigma < A.$$

Clearly $\|f\|_\sigma$ exists on account of (1.3) and defines a norm on X, for each $\sigma < A$. We denote by $X(\sigma)$, the space X, equipped with the norm $\|...\|_\sigma$. Let L be the topology generated by the family of norms $\{\|f\|_\sigma;\ \sigma < A\}$. Then X is complete, metrizable locally convex space (see Lemma 1) and this topology is equivalent to the topology generated by the invariant metric d, where

$$d(f, g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{\|f-g\|_{\sigma_n}}{1+\|f-g\|_{\sigma_n}}, \text{(Fréchet combination), where } \{\sigma_i\} \text{ is}$$

a sequence, such that

$$\sigma_1 < \sigma_2 < \ldots < \sigma_n,\ \sigma_n \to A \text{ as } n \to \infty.$$

From now onwards we denote by X, the space X, equipped with the topology generated by metric d or by the topology generated by the norms $\{\|...\|_{\sigma_i};\ i = 1, 2, ...\}$.

A sequence $\{\alpha_n\} \subset X$ will be called 'linearly independent' if $\sum_{n=1}^{\infty} a_n\alpha_n = 0$ implies $a_n = 0$, $n \geqslant 1$, for all sequences $\{a_n\}$ of complex numbers for which $\sum_{n=1}^{\infty} a_n\alpha_n$ converges in X. The sequence $\{\alpha_n\} \subset X$, spans a subspace X_0 of X, if X_0 consists of all linear combinations $\sum_{n=1}^{\infty} a_n\alpha_n$, such that $\sum_{n=1}^{\infty} a_n\alpha_n$ converges in X. A sequence $\{\alpha_n\} \subset X$, which is linearly independent and spans a closed subspace X_0 of X, will be said to be a 'base' in X_0. If $e_n \in X$, $e_n(s) = e^{s\lambda_n}$, $n \geqslant 1$, then clearly $\{e_n\}$ is a base in X. A sequence $\{\alpha_n\} \subset X$ will be called a 'proper base' if it is a base and it satisfies the condition that for all sequences $\{a_n\}$ of complex numbers

$$\sum_{n=1}^{\infty} a_n\alpha_n \text{ converges in } X \text{ if and only if } \sum_{n=1}^{\infty} a_n e_n \text{ converges in } X.$$

2. Topological Structure of X

Below we throw some light on the topological structure of the space X in terms of its completeness and the characterization of continuous linear functionals on X etc.

Lemma 1—The space X is complete with respect to the metric d and hence it is a Fréchet space.

Proof · Let $\{f_p\}, f_p(s) = \sum_{n=1}^{\infty} a_n{}^{(p)} e^{s\lambda_n}$, be a Cauchy sequence in X. Hence it is a Cauchy sequence in $X(\sigma)$ for each real σ. Let $\varepsilon > 0$ be given. Then there exists a positive integer N, $N(\varepsilon, \sigma)$, such that

$$\sum_{n=1}^{\infty} \left| a_n{}^{(p)} - a_n{}^{(q)} \right| e^{\sigma\lambda_n} < \varepsilon, \text{ for all } p, q \geqslant N. \qquad \ldots(2.1)$$

Therefore $\{a_n{}^{(m)}\}$ is a Cauchy sequence for each $n \geqslant 1$ and so it converges to some element in $\mathbb{C}$, say a_n. Let $q \to \infty$ in (2.1), we get

$$\sum_{n=1}^{\infty} \left| a_n{}^{(p)} - a_n \right| e^{\sigma\lambda_n} \leqslant \varepsilon, \text{ for all } p \geqslant N. \qquad \ldots(2.2)$$

To complete the proof we have yet to show that $f = \sum_{n=1}^{\infty} a_n e_n \in X$. We can choose a σ_i, such that $A < \sigma_i + \varepsilon$. From (2.2) we have

$$\sum_{n=1}^{\infty} \left| a_n{}^{(p)} - a_n \right| e^{\sigma_i\lambda_n} \leqslant \varepsilon, \text{ for all } p \geqslant N_1 \qquad \ldots(2.2a)$$

where $N_1 = N_1(\varepsilon, \sigma_i)$. Keep p fixed in (2.2a) and observe in view of (1.3) that

$| a_n{}^{(p)} | \leqslant e^{(-A+\varepsilon)\lambda_n}$, for all $p \geqslant N_2$ where $N_2 = N_2(\varepsilon, p)$. Then
$| a_n | \leqslant | a_n{}^{(p)} - a_n | + | a_n{}^{(p)} |$
$\Rightarrow | a_n | \leqslant \varepsilon\, e^{\sigma_i\lambda_n} + e^{(-A+\varepsilon)\lambda_n}$

for all $n \geqslant M = \max (N_1, N_2)$. Thus

$$\limsup_{n\to\infty} \frac{\log |a_n|}{\lambda_n} \leqslant -A.$$

Thus $\sum_{n=1}^{\infty} a_n e_n \in X$. Hence $f_p \to f$ in X, where $f \in X$ and this completes the proof of the result.

Lemma 2 (*i*)—A continuous linear functional f on $X(\sigma)$ is of the form

$f(\alpha) = \sum_{n=1}^{\infty} c_n a_n$, $\alpha = \sum_{n=1}^{\infty} a_n e_n$ if and only if $|c_n|/e^{\sigma\lambda_n}$ is bounded for all $n \geqslant 1$.

(*ii*) A continuous linear functional f on X is of the form $f(\alpha) = \sum_{n=1}^{\infty} a_n c_n$,

$\alpha = \sum_{n=1}^{\infty} a_n e_n$ if and only if $\{|c_n|/e^{\sigma\lambda_n}\}$ is bounded for some $\sigma < A$.

PROOF : Let f be a continuous linear functional on $X(\sigma)$. Then

$$f(\alpha) = \sum_{n=1}^{\infty} a_n e_n. \quad \alpha = \sum_{n=1}^{\infty} a_n e_n, \text{ where further } c_n = f(e_n).$$

Hence there exists a constant k such that

$$|f(\alpha)| \leqslant k \|\alpha\|_\sigma, \text{ for all } \alpha \in X.$$

Take $\alpha = e_n = e^{s\lambda_n} \in X$, this implies that

$$|c_n| \leqslant k\, e^{\sigma\lambda_n},\ n \geqslant 1.$$

Conversely, let α be as before and consider $f(\alpha) = \sum_{n=1}^{\infty} a_n c_n$, where $\left\{ |c_n| / e^{\sigma\lambda_n} \right\}$ is bounded. $f(\alpha)$ does exist, since

$$\left|\sum_{n=1}^{\infty} a_n c_n\right| \leqslant \sum_{n=1}^{\infty} |a_n c_n| \leqslant k \sum_{n=1}^{\infty} |a_n|\, e^{\sigma \lambda_n} < +\infty;$$

$$= k\|\alpha\|_\sigma.$$

Hence f is a continuous linear functional on $\chi(\sigma)$. This proves the first part of the lemma.

The proof of the second part follows from (i) and the fact that, a functional f on χ is continuous if and only if it is continuous with respect to $\chi(\sigma)$, for each $\sigma < A$ (this is true for locally convex spaces, see Wilansky 1964, p. 147).

The following result will be of much use in our investigation :

Theorem 1—A necessary and sufficient condition that there exists a continuous linear transformation $T : \chi \to \chi$ with $T e_n = \alpha_n$, $n = 1, 2, \ldots$ is that for each $\sigma < A$

$$\limsup_{n\to\infty} \frac{\log \|\alpha_n\|_\sigma}{\lambda_n} < A. \qquad (2.3)$$

Proof : Let there exist a continuous linear transformation T from χ into χ with $Te_n = \alpha_n$, $n = 1, 2, \ldots$. Then for a given σ, there exists a σ_1, such that $(\sigma, \sigma_1 < A)$

$$\|T e_n\|_\sigma \leqslant k\, \|e_n\|_{\sigma_1}$$

$$= k\, e^{\sigma_1 \lambda_n}, \text{ for } n \geqslant 1$$

$$\Rightarrow \frac{\log \|T e_n\|_\sigma}{\lambda_n} \leqslant \frac{\log k}{\lambda_n} + \sigma_1, \text{ for } n \geqslant 1$$

$$\Rightarrow \limsup_{n\to\infty} \frac{\log \|T e_n\|_\sigma}{\lambda_n} \leqslant \sigma_1 < A.$$

Conversely, let (2.3) hold and let $\alpha = \sum_{n=1}^{\infty} a_n e_n \in \chi$. Then there exists an $\varepsilon > 0$, such that

$$\frac{\log \|\alpha_n\|_\sigma}{\lambda_n} \leqslant A - \varepsilon, \text{ for all } n \geqslant N_1(\varepsilon)$$

$$\|\alpha_n\|_\sigma \leqslant e^{(A-\varepsilon)\lambda_n}$$

Choose $\delta > 0$, such that $\delta < \varepsilon$, then

$$|a_n| \leqslant e^{(-A+\delta)\lambda_n} \quad (\text{for all } n \geqslant N_2(\delta)).$$

Hence

$$|a_n|\,\|\alpha_n\|_\sigma \leqslant e^{(-A+\delta)\lambda_n}\, e^{(A-\varepsilon)\lambda_n}, \text{ for all } n \geqslant \max(N_1, N_2) = e^{(\delta-\varepsilon)\lambda_n}$$

$$\Rightarrow \sum_{n=1}^{\infty} |a_n|\,\|\alpha_n\|_\sigma \text{ is convergent}$$

and as σ is arbitrarily less than A, we find that $\sum_{n=1}^{\infty} a_n\alpha_n$ is convergent in X. Hence there exists a transformation $T : X \to X$, such that $T(\alpha) = \sum_{n=1}^{\infty} a_n\alpha_n$, for each $\alpha \in X$.

Then T is linear, $Te_n = \alpha_n$, $n = 1, 2, \ldots$, and given $\sigma < A$, there exists $\delta > 0$, such that

$$\frac{\log \|\alpha_n\|_\sigma}{\lambda_n} \leqslant A - \delta, \text{ for all } n \geqslant N.$$

$$\Rightarrow \|\alpha_n\|_\sigma \leqslant e^{(A-\delta)\lambda_n}, \text{ for all } n \geqslant N$$

$$\Rightarrow \|\alpha_n\|_\sigma \leqslant k\, e^{(A-\delta)\lambda_n}, \text{ for all } n \geqslant 1.$$

Hence

$$\|T\alpha\| \leqslant k \sum_{n=1}^{\infty} |a_n|\, e^{(A-\delta)\lambda_n} = k\, \|\alpha\|_{A-\delta}.$$

Hence T is continuous.

3. Proper Bases and Their Characterizations

To prove our main result on the characterization of proper bases in X, we first of all prove two important results in the form of lemmas.

Lemma 3—Let $\{\alpha_n\} \subset X$. Then the following three properties are equivalent

(A_1) $\limsup_{n\to\infty} \frac{\log \|\alpha_n\|_\sigma}{\lambda_n} < A$, for all $\sigma < A$.

(B_1) For all sequences $\{a_n\}$ of complex numbers, convergence of $\sum_{n=1}^{\infty} a_n e_n$ implies the convergence of $\sum_{n=1}^{\infty} a_n \alpha_n$ in X.

(C_1) For all sequences $\{a_n\}$ of complex numbers, convergence of $\sum_{n=1}^{\infty} a_n e_n$ implies that $a_n \alpha_n$ tends to zero in X.

PROOF : It is clear that (B_1) implies (C_1) and we have already proved $(A_1) \Rightarrow (B_1)$ in the proof of sufficiency part of Theorem (1). We need therefore only to prove that $(C_1) \Rightarrow (A_1)$.

Assume that (C_1) is true and (A_1) is false. This implies that, for some $\sigma' < A$,

$$\limsup_{n\to\infty} \frac{\log \|\alpha_n\|_{\sigma'}}{\lambda_n} \geqslant A.$$

Hence there exists a sequence $\{n_k\}$ of positive integers, such that

$$\frac{\log \|\alpha_{n_k}\|_{\sigma'}}{\lambda_{n_k}} \geqslant A - \frac{1}{k}, \text{ for all } k = 1, 2, \ldots .$$

Define $\{a_n\}$ by

$$a_n = \begin{cases} e^{-(A - \frac{1}{k})\lambda_{n_k}}, \text{ for } k = 1, 2, \ldots \\ 0, n \neq n_k \end{cases}$$

So we have

$$|a_{n_k}| e^{\sigma\lambda n_k} = e^{-(A - \frac{1}{k})\lambda_{n_k}} e^{\sigma\lambda n_k} = e^{-(\sigma A + \frac{1}{k})\lambda_{n_k}}.$$

There exists a k, large enough such that $A - (\sigma + (1/k)) > 0$.

$$\Rightarrow \sum_{k=1}^{\infty} |a_{n_k}| e^{\sigma\lambda_{nk}} \text{ converges in } X, \text{ for all } \sigma < A.$$

But

$$|a_{n_k}| \, \|\alpha_{n_k}\|_{\sigma'} \geqslant e^{-(A-\frac{1}{k})\lambda_{n_k}} e^{(A-\frac{1}{k})\lambda_{n_k}} = 1.$$

$\Rightarrow a_{n_k} \alpha_{n_k}$ does not tend to zero in X, a contradiction and this contradicts (C_1). So (C_1) implies (A_1).

LEMMA 4—The following three conditions are equivalent for any sequence $\{\alpha_n\} \subset X$.

$$(\alpha) \quad \lim_{\sigma \to A} \left\{ \liminf_{n\to\infty} \frac{\log \|\alpha_n\|_\sigma}{\lambda_n} \right\} \geqslant A.$$

(β) For all sequences $\{a_n\}$ of complex numbers convergence of $\sum_{n=1}^{\infty} a_n \alpha_n$ in X implies the convergence of $\sum_{n=1}^{\infty} a_n e_n$ in X.

(γ) For all sequences $\{a_n\}$ of complex numbers $a_n\alpha_n$ tends to zero in X implies the convergence of $\sum_{n=1}^{\infty} a_n e_n$ in X.

PROOF: It is clear that (γ) implies (β). We shall prove that (β) implies (α) and (α) implies (γ).

First we suppose that (β) holds but (α) does not hold. Therefore

$$\lim_{\sigma \to A} \left\{ \liminf_{n\to\infty} \frac{\log \|\alpha_n\|_\sigma}{\lambda_n} \right\} < A.$$

Since, $\|\ldots\|_\sigma$ increases as σ increases, this implies that for each $\sigma < A$,

$$\liminf_{n\to\infty} \frac{\log \|\alpha_n\|_\sigma}{\lambda_n} < A, \text{ for all } \sigma < A.$$

If η be a small positive number, there exists an increasing sequence $\{n_r\}$, such that

$$\frac{\log \|\alpha_{n_r}\|_\sigma}{\lambda_{n_r}} \leqslant A - \eta$$

$$\Rightarrow \|\alpha_{n_r}\|_\sigma \leqslant e^{(A-\eta)\lambda_{n_r}} .$$

Choose $\delta < \eta$, and define $\{a_n\}$ by

$$a_n = \begin{cases} e^{-(A-\delta)\lambda_{n_r}}, & r = 1, 2, \ldots \\ 0, & n \neq n_r \end{cases}$$

Then for every $\sigma < A$,

$$\sum_{n=1}^{\infty} |a_n| \, \|\alpha_n\|_\sigma = \sum_{r=1}^{\infty} |a_{n_r}| \, \|\alpha_{n_r}\|_\sigma$$

$$\leqslant \sum_{r=1}^{\infty} e^{(A-\eta)\lambda_{n_r}} e^{-(A-\delta)\lambda_{n_r}}$$

$$= \sum_{r=1}^{\infty} e^{(\delta-\eta)\lambda_{n_r}}$$

and the last series is convergent since $\delta < \eta$. Hence for this sequence $\{a_n\}$, $\sum_{n=1}^{\infty} a_n \alpha_n$ converges in $X(\sigma)$, for each $\sigma < A$, and hence converges in X. But, we have

$$\sum_{n=1}^{\infty} |a_n| e^{\sigma\lambda_n} = \sum_{r=1}^{\infty} |a_{n_r}| \, e^{\sigma\lambda_{n_r}} \sum_{r=1}^{\infty} e^{-(A-\delta)\lambda_{n_r}} e^{\sigma\lambda_{n_r}}$$

$$= \sum_{r=1}^{\infty} e^{(\sigma+\delta-A)\lambda_{n_r}}.$$

Given δ choose $\sigma < A$ such that $\sigma + \delta > A$, and last series is divergent for this σ.

Hence $\sum_{n=1}^{\infty} a_n e_n$ does not converge in X and this contradicts (β). Hence (β) implies (α).

To prove that (α) implies (γ), we assume that (α) is true but (γ) is not true. Hence there exists a sequence $\{a_n\}$ of complex numbers for which $a_n \alpha_n$ tends to zero in X, but $\sum_{n=1}^{\infty} a_n e_n$ does not converge in X. This implies that

$$\limsup_{n\to\infty} \frac{\log |a_n|}{\lambda_n} > -A.$$

Hence given $\delta > 0$, there exists a sequence $\{n_k\}$ of positive integers, such that

$$\frac{\log |a_{n_k}|}{\lambda_{n_k}} \geqslant e^{(A-\delta)\lambda_{n_k}}.$$

Now choose a positive number σ, such that, $\delta > 2\eta$. (α) being true, we can find a number $\sigma = \sigma(\eta)$, such that

$$\liminf_{n\to\infty} \frac{\log \|\alpha_n\|}{\lambda_n} \geqslant A - \eta.$$

Hence there exists $N = N(\eta)$, such that

$$\frac{\log \|\alpha_n\|_\sigma}{\lambda_n} \geqslant A - 2\eta, \text{ for all } n \geqslant N.$$

Therefore,

$$|a_{n_k}| \|\alpha_{n_k}\|_\sigma \geqslant e^{(-A+\delta)\lambda_{n_k}} e^{(A-2\eta)\lambda_{n_k}}, \; n_1 \geqslant N$$

$$= e^{(\delta-2\eta)\lambda_{n_k}} \to +\infty$$

as $k \to \infty$, since $\delta > 2\eta$. This shows that $a_n\alpha_n$ does not tend to zero in X and this is a contradiction. Therefore we conclude that (α) implies (γ).

Now we state our main result which follows from Lemma 3 and Lemma 4.

Theorem 2—A base in a closed subspace X_0 of X is proper if and only if condition (A_1) and (α) are satisfied.

Acknowledgement

The authors are grateful to the referee for pointing out certain oversights in our original version.

References

Markushevich, A. I. (1965). Theory of Functions of a Complex Variable, Vol. II. Prentice-Hall, Inc., New Jersey.

Wilansky, Albert (1964). Functional Analysis. Blais Dell Publishing Company, U S.A.