CREEP OF THICK-WALLED SPHERICAL VESSELS UNDER INTERNAL PRESSURE CONSIDERING LARGE STRAINS

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(Communicated by P. L. Bhatnagar, F.N.A.)

(*Received* 27 *August* 1973)

Creep analysis of a thick-walled spherical vessel made of a homogeneous and isotropic material and subjected to internal pressure has been presented. The strains considered are assumed to be large which necessitates the use of finite-strain theory for evaluating the expressions for stresses, creep strains and strain rates. The general theory developed has been applied to the solution of a specific problem using Norton's law of creep. A numerical example has also been given in the end. The analysis presented here may aid the designers in the prediction of correct creep strains, strain-rates and stresses in cases where large creep deformations of spherical vessels are permissible.

Introduction

The problem of creep deformation of thick-walled spherical pressure vessels has been studied previously by Johnson *et al.* (1961, 1963). These papers assume that the strains are small and that deformations can be referred to the original dimensions of the vessels rather than the instantaneous deformed values. While these assumptions are sufficiently accurate in the elastic and primary creep ranges, under secondary and tertiary creep the deformations may attain a value where small strains can no longer be assumed. With this in mind an attempt has been made in this paper to solve the problem of creep deformation of thick-walled spherical vessels under internal pressure considering large strains. The finite strain theory, developed in a paper by MacGregor *et al.* (1948), for the case of plastic flow of thick-walled tubes with large strains has been extended for application to the creep deformation of a thick-walled spherical pressure vessel. In the present paper various expressions for stresses, strains and creep rates, based on finite strain theory, have been obtained.

Consider a spherical vessel of internal and external radii a and b, respectively, subjected to internal pressure p.

Basic Equations

In the present analysis, following assumptions are made :

1. The material is homogeneous and isotropic.

2. The volume of the material remains constant (condition of incompressibility).
3. The ratios of the principal shear strain-rates to the principal shear stresses are equal.
4. The effective stress σ and effective strain (creep) rate are related by

$$\sigma = f(\dot{\epsilon}). \quad \ldots(1)$$

By symmetry the principal stresses in the two tangential directions are equal. We shall denote the stresses and creep rates in radial and tangential directions by subscripts r and t, respectively.

It may also be concluded from symmetry that the only displacement in this problem is radial and is such that concentric spherical surfaces remain concentric and spherical after deformation.

Considering the radial equilibrium in the deformed state, following equation of equilibrium is obtained

$$r' \frac{d\sigma_r}{dr'} = 2(\sigma_t - \sigma_r) \quad \ldots(2)$$

in which $r' = r + u$ is the radius to which an arbitrary element, originally at radius r, has moved and u is the displacement in the radial direction.

The natural tangential strain is defined by

$$\epsilon_t = \ln\left(1 + \frac{u}{r}\right). \quad \ldots(3)$$

and the natural radial strain is

$$\epsilon_r = \ln\left(1 + \frac{du}{dr}\right). \quad \ldots(4)$$

Eliminating u between eqns. (3) and (4) yields

$$r \frac{\partial \epsilon_t}{\partial r} = e^{\epsilon_r - \epsilon_t} - 1. \quad \ldots(5)$$

This is the equation of compatibility.

The effective stress σ and effective strain ϵ, by symmetry, can be written as

$$\sigma = \sigma_t - \sigma_r \quad \ldots(6)$$

and

$$\epsilon = \tfrac{2}{3}(\epsilon_t - \epsilon_r). \quad \ldots(7)$$

The condition of incompressibility yields

$$2\epsilon_t = -\epsilon_r \qquad \text{...(8)}$$

And therefore from eqns. (7), we obtain

$$\epsilon = 2\epsilon_t = -\epsilon_r. \qquad \text{...(9)}$$

Using eqns. (3), (4) and (5) in eqn. (2), the equation of equilibrium can be rewritten as

$$r\frac{\partial \sigma_r}{\partial r} = 2(\sigma_t - \sigma_r)\, e^{\varepsilon_r - \varepsilon_t}. \qquad \text{...(10)}$$

Substituting the value of $(\sigma_t - \sigma_r)$ from eqn. (6) and of ϵ_t and ϵ_r from eqn. (9), eqn. (10) becomes

$$r\frac{\partial \sigma_r}{\partial r} = 2e^{-(3/2)\varepsilon}\, \sigma. \qquad \text{...(11)}$$

Using eqn. (9) in eqn. (5) and integrating one obtains

$$\epsilon = \tfrac{2}{3}\ln\left[1 + \frac{K(t)}{r^3}\right] \qquad \text{...(12)}$$

where $K(t)$, a function of time, is the constant of integration with respect to r.

Equation (12), solved for r^3, reads

$$r^3 = \frac{K(t)}{e^{(3/2)\varepsilon} - 1}. \qquad \text{...(13)}$$

And at the inner radius a, eqn. (13) becomes

$$a^3 = \frac{K(t)}{e^{(3/2)\varepsilon_a} - 1} \qquad \text{...(14)}$$

where ϵ_a refers to the strain at the inner radius a.

Evaluating the value of constant K from eqn. (14) and substituting in eqn. (13), the strain at any radius r can be given as

$$\epsilon = \tfrac{2}{3}\ln\left[1 + \left(\frac{a}{r}\right)^3 (e^{(3/2)\varepsilon_a} - 1)\right]. \qquad \text{...(15)}$$

Differentiating eqn. (15) with respect to time t produces

$$\dot{\epsilon} = \frac{\left(\frac{a}{r}\right)^3 e^{(3/2)\varepsilon_a}}{1 + \left(\frac{a}{r}\right)^3 (e^{(3/2)\varepsilon_a} - 1)}\, \dot{\epsilon}_a. \qquad \text{...(16)}$$

This expresses the creep rate at any radius r as a function of given creep rate at the inner radius a.

Combining eqns. (13) to (15) we obtain a relation between creep rates and strains as

$$\dot{\epsilon} = \frac{1 - e^{-(3/2)\epsilon}}{1 - e^{-(3/2)\epsilon_a}} \dot{\epsilon}_a. \qquad \text{...(17)}$$

Differentiating eqn. (16) partially with respect to r and using eqn. (15), we obtain

$$\frac{\partial \dot{\epsilon}}{\partial r} = - \frac{3}{r} \dot{\epsilon} e^{-(3/2)\epsilon}. \qquad \text{...(18)}$$

Since $\sigma = f(\dot{\epsilon})$ and $\dot{\epsilon} = f(r, t)$, we can write

$$\frac{d\sigma_r}{d\dot{\epsilon}} = \frac{\partial \sigma_r / \partial r}{\partial \dot{\epsilon} / \partial r}. \qquad \text{...(19)}$$

With the help of eqns. (11), (18) and (19), one may obtain

$$\frac{d\sigma_r}{d\dot{\epsilon}} = - \tfrac{2}{3} \frac{\sigma}{\dot{\epsilon}}. \qquad \text{...(20)}$$

Integrating eqn. (20) and using the boundary conditions

$$\sigma_r = -p \text{ at } r = a; \quad \sigma_r = 0 \text{ at } r = b \qquad \text{...(21)}$$

the internal pressure p can be given in terms of creep rate as

$$p = \tfrac{2}{3} \int_{\dot{\epsilon}_b}^{\dot{\epsilon}_a} \frac{\sigma}{\dot{\epsilon}} \, d\dot{\epsilon} \qquad \text{...(22)}$$

where

$$\dot{\epsilon}_b = \frac{\left(\frac{a}{b}\right)^3 e^{(3/2)\epsilon_a}}{1 + \left(\frac{a}{b}\right)^3 (e^{(3/2)\epsilon_a} - 1)} \dot{\epsilon}_a \qquad \text{...(23)}$$

is the creep rate at the outer radius and is obtained from eqn. (15).

Stress and Strain Distribution

The effective stress σ can be obtained as a function of r from eqn. (1), provided the effective strain ϵ_a and creep rate $\dot{\epsilon}_a$, both at the inner radius a, are given.

The radial stress σ_r, at any radius r, can be obtained by the integration of eqn. (20), between suitable limits, as

$$\sigma_r = -p + \tfrac{2}{3} \int_{\dot{\epsilon}}^{\dot{\epsilon}_a} \frac{\sigma}{\dot{\epsilon}}\, d\dot{\epsilon}. \qquad \text{...(24)}$$

And, with the help of eqns. (6) and (24), the expressions for tangential stress at any radius r can be written as

$$\sigma_t = -p + \tfrac{2}{3} \int_{\dot{\epsilon}}^{\dot{\epsilon}_a} \frac{\sigma}{\dot{\epsilon}}\, d\dot{\epsilon} + f(\dot{\epsilon}). \qquad \text{...(25)}$$

Using eqns. (9) and (16) the creep rate $\dot{\epsilon}_t$, in the tangential direction, can be obtained as

$$\dot{\epsilon}_t = \tfrac{1}{2} \frac{\left(\frac{a}{r}\right)^3 e^{(3/2)\epsilon_a}}{1 + \left(\frac{a}{r}\right)^3 \left(e^{(3/2)\epsilon_a} - 1\right)} \dot{\epsilon}_a. \qquad \text{...(26)}$$

Solution Using Norton's Creep Law

The effective stress σ and creep rate $\dot{\epsilon}$ are assumed to be related by Norton's law

$$\dot{\epsilon} = B\left(\frac{\sigma}{\sigma_c}\right)^n \qquad \text{...(27)}$$

where B, n and σ_c are experimental constants and σ_c has the dimensions of σ.

Equation (27) can be rewritten as

$$\sigma = \frac{\sigma_c}{B^{1/n}} \dot{\epsilon}^{1/n}. \qquad \text{...(28)}$$

Substitution of value of σ from eqn. (28) into eqn. (22) and its integration yields an expression for internal pressure p

$$p = \frac{2\sigma_c n}{3B^{1/n}} \left(\dot{\epsilon}_a^{1/n} - \dot{\epsilon}_b^{1/n}\right) \qquad \text{...(29)}$$

where $\dot{\epsilon}_a$ and $\dot{\epsilon}_b$ are defined by eqns. (16) and (23), respectively.

Rearrangement of eqn. (29) with the help of eqn. (23) yields an expression for the creep rate $\dot{\epsilon}_a$ at $r = a$ as a function of the strain ϵ_a at $r = a$, which can be given as

$$\dot{\epsilon}_a = \frac{3^n p^n B}{2^n \sigma_c^n n^n} \left[1 - \left\{\frac{\left(\frac{a}{b}\right)^3 e^{(3/2)\epsilon_a}}{1 + \left(\frac{a}{b}\right)^3 \left(e^{(3/2)\epsilon_a} - 1\right)}\right\}^{1/n}\right]^{-n} \cdot \qquad \text{...(30)}$$

Integrating eqn. (30) and simplifying, the strain ϵ_a can be obtained as a function of time t, given by

$$\int_0^{\varepsilon_a} \left[1 - \left\{ \frac{\left(\frac{a}{b}\right)^3 e^{(3/2)\varepsilon_a}}{1 + \left(\frac{a}{b}\right)^3 \left(e^{(3/2)\varepsilon_a} - 1\right)} \right\}^{1/n} \right]^n d\epsilon_a = \frac{3^n p^n B}{2^n \sigma_c^n n^n} t. \qquad \ldots(31)$$

Numerical Example and Discussion

For numerical computations following values of the constants were taken

$$B = 5.10^{-12} \text{ 1/day}, \ \sigma_c = 1000 \text{ psi}, \ n = 6, \ \frac{b}{a} = 5, \ p = 40000 \text{ psi}.$$

The values of the constants are the same as used by Rimrott (1959) for the case of creep deformation of thick-walled cylinders under internal pressure.

The integrals appearing in eqn. (31) were evaluated numerically. The numerical calculations were carried out on an I.B.M. 1620 Computer.

The creep rate $\dot{\epsilon}_a$ at the internal radius a has been evaluated numerically from eqn. (30) and plotted against the strain ϵ_a at internal radius in Fig. 1. It is found to increase with increasing strain.

Fig. 2 shows a plot of strain ϵ_a at the inner radius a versus time t as obtained from eqn. (31). The strain increases non-linearly with time as the deformation under creep continues.

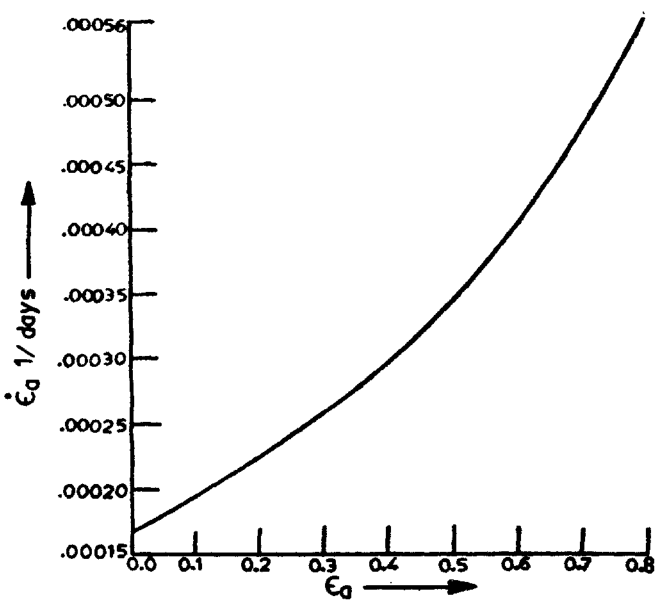

FIG. 1. Creep rate versus strain, both at internal radius.

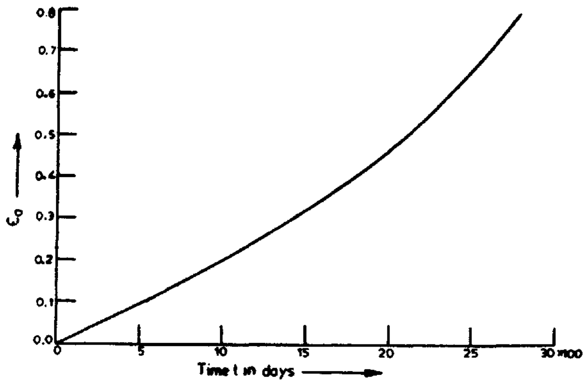

FIG. 2. Strain at internal radius versus time.

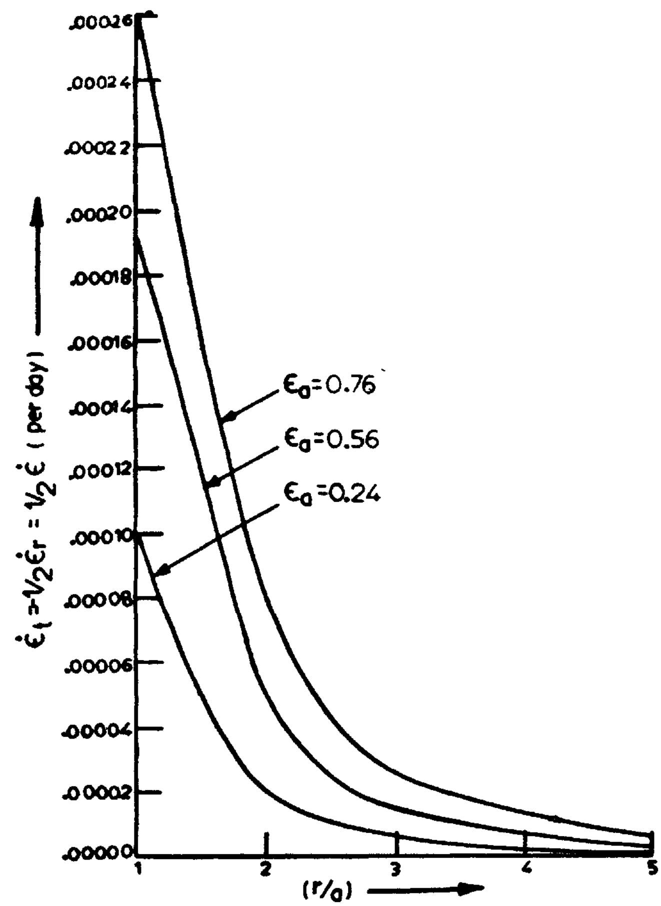

FIG. 3. Tangential strain rate $\dot{\epsilon}_t$ at various radii of spherical vessel.

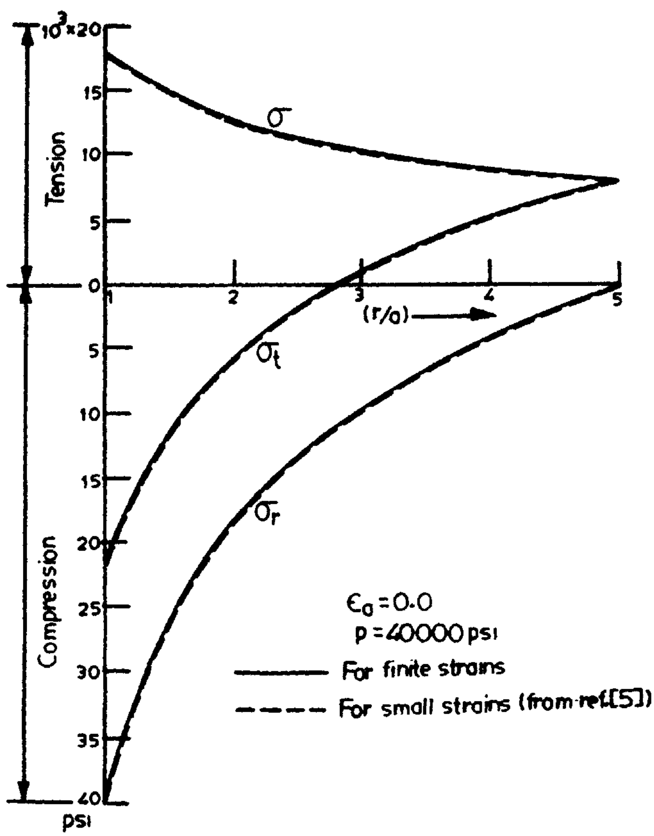

FIG. 4. Stress distribution at various radii of vessel for $\epsilon_a = 0.0$.

The tangential creep rate $\dot{\epsilon}_t$ at various radii of the vessel was calculated from eqn. (26) and has been shown in Fig. 3 for three values of ϵ_a viz. 0.24, 0.56 and 0.76. The tangential creep rate is found to decrease rapidly with increasing radius for all the three values of ϵ_a.

Figures 4 and 5 show the stress distribution in the wall of the vessel, as obtained from eqns. (6), (24) and (25), at two stages of deformation. Figure 4 exhibits the stress distribution just at the onset of creep (i.e. $\epsilon_a = 0$) and Fig. 5 shows the stress distribution after there is a considerable deformation (i.e. $\epsilon_a = 0.62$). The stresses obtained from the solution given in Finnie and Heller (1959) based on the assumption of infinitesimal (small) strains, have also been plotted in these figures for comparison.

The most significant results of the present investigations are observed from Figs. 4 and 5. The figures show a negligibly small difference between the two stress distributions based on the assumptions of finite and infinitesimal strains for $\epsilon_a = 0$ i.e. for the case of no deformation, as it should be. However, after a considerable deformation under creep (i.e. for $\epsilon_a = 0.62$) the difference between the radial and tangential

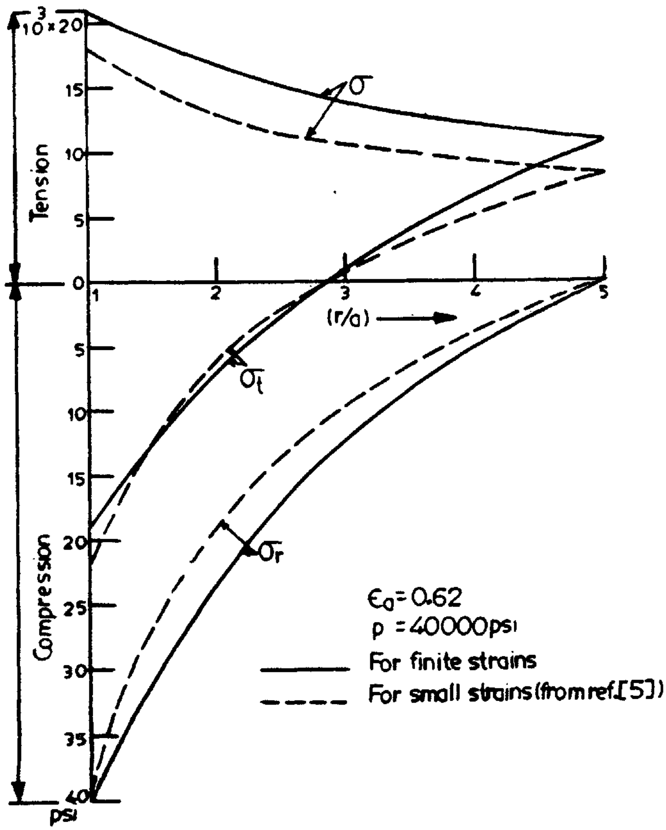

FIG. 5. Stress distribution at various radii of vessel for $\varepsilon_a = 0.62$.

stresses and so also for the effective stress for the two assumptions, is very large. It is to be expected that as the strain at the inner boundary increases from its initial value, the radial, tangential and effective stresses all increase continuously and the difference in the predictions from the two theories goes on increasing.

As the creep rate is usually a power function of effective stress, it can be concluded that it will increase at a much faster rate. Thus, assuming large strains we see that the creep rate of the thick-walled spherical vessel increases rapidly even though the creep rate of the same material when subjected to constant true stress in simple tension is constant. This is an important effect which is overlooked in the analyses making use of the infinitesimal strain theory. Therefore, the predictions based on the results from small or infinitesimal strain analyses would be on unsafe side from a design point of view.

ACKNOWLEDGEMENT

One of the authors (V. K. A.) gratefully acknowledges the financial assistance by the University Grants Commission, India.

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