APPROXIMATE ANALYTIC SOLUTION OF A STRONG SHOCK WITH RADIATION NEAR THE SURFACE OF THE STAR

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The authors have employed Chernyii's technique, in which the flow variables are expanded in series of powers of β, the density ratio across the strong shock, to study the propagation of a shock wave, originating in a stellar interior, when it approaches the surface of the star and assumes a self-similar character. The flow behind the shock is assumed to be spatially isothermal rather than adiabatic to simulate the conditions of large radiative transfer near the stellar surface. The solution in closed form up to second order term in β is obtained which is in good agreement with the exact solution. An analytic expression for the similarity exponent has also been obtained.

1. Introduction

In gas dynamics two types of self-similar processes, termed as self-similar motions of the first kind and the second kind have been considered (Zel'dovich and Raizer 1967). Taylor's (1950) explosion problem and one-dimensional centred rarefaction waves are examples of the flows of the first kind while emergence of a strong shock near the surface of a star (Sakurai 1960, Zel'dovich and Raizer 1967, Sachdev and Ashraf 1971) and converging cylindrical and spherical shocks (Guderly 1942) are examples of the flows of the second kind. In self-similar solution of the first kind the similarity exponent α, occurring in the law of shock propagation, is determined from the dimensional considerations or laws of conservation while in the solution of the second kind this exponent cannot be determined from these considerations in advance but is found from solving the differential equations which govern the flow. The solutions of self-similar motion of second kind are examples of solutions of differential equations which 'partially' forget their initial conditions.

In this paper, we consider the motion of a shock as it propagates in the outer layers of star far from its origin, near the centre. As the shock propagates in the outer layers of the star, it accelerates and the temperature behind increases. Besides, the mean free path of radiation, which is inversely proportional to density, becomes very large so that there is intense transfer of energy by radiation, leading to the levelling down of temperature gradient. Thus the flow behind the shock is rendered approximately isothermal. The spatial temperature gradient is zero behind the shock. The time-dependent temperature goes on changing as the shock propagates and this temperature is different from that ahead of the shock. Recently, Sachdev and Ashraf (1971) have solved this problem numerically.

In the present paper we have found an approximate analytic solution of this problem by employing Chernyii's (1957, 1960, 1961) technique, in which the flow variables are expressed in series of powers of β, the density ratio across the strong shock. We have found the solution in closed form up to second order terms in β. As remarked earlier, the similarity exponent, occurring in the law of shock propagation, is obtained by solving an eigenvalue problem for a single differential equation to which the similarity equations are reducible. We have found an analytic expression for α, the similarity exponent, from the considerations of singular points of the single differential equation and this gives values of α close to exact values. Comparison of our analytic solution with exact numerical solution of Sachdev and Ashraf (1971) shows an excellent aggreement. The error in the solution is $O(\beta^3)$ which is very small when γ is $O(1)$

2. Basic Equations and Boundary Conditions

We take origin of the coordinates on the surface of the star and positive X-axis points into the interior. We assume that the undisturbed density ρ_0 ahead of the shock is given by

$$\rho_0 = bx^{\delta} \qquad \ldots(2.1)$$

where b and δ are positive constants so that on the surface $\rho_0 = 0$. The time t is taken to be negative before the shock reaches the surface of the star and $t = 0$ is the instant at which the shock emerges at the surface. The shock position is assumed to be given by

$$X = A(-t)^{\alpha} \qquad \ldots(2.2)$$

where A and α are constants.

The basic equations governing the one-demensional isothermal flow behind the shock in terms of Lagrangian coordinate η and the time t are

$$\frac{\partial x}{\partial \eta} = \frac{1}{\rho} \qquad \ldots(2.3)$$

$$\frac{\partial^2 x}{\partial t^2} = -\frac{\partial p}{\partial \eta} \qquad \ldots(2.4)$$

$$\frac{\partial T}{\partial x} = 0 \qquad \ldots(2.5)$$

where x is the Eulerian coordinate and ρ, p and T are density, pressure and temperature behind the shock respectively. η is the Lagrangian coordinate defined by

$d\eta = \rho_0\, dx_0$, where x_0 is the value of x at the initial instant of time. We introduce the similarity variable μ defined by $\mu = \frac{\eta}{\eta_s}$ where η_s is the value of η at the shock.

Following Laumboch and Probstein (1970) and Zel'dovich and Raizer (1967) we assume that the heat flux across the optically thin layer is continuous so that the classical shock conditions hold. So for a strong shock we have the boundary conditions at the shock as

$$\left.\begin{aligned} u_s &= (1-\beta)\,\dot{X} \\ \rho_s &= \rho_0/\beta \\ p_s &= (1-\beta)\,\rho_0 \dot{X}^2 \end{aligned}\right\} \qquad \ldots(2.6)$$

where β is the density ratio across the shock and $\dot{X}$ is the shock velocity.

Equations (2.3), (2.4) and (2.5) may also be written as

$$\frac{\partial u}{\partial \eta} = -\frac{1}{\rho^2}\frac{\partial \rho}{\partial t} \qquad \ldots(2.7)$$

$$\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial \eta} \qquad \ldots(2.8)$$

$$\frac{\partial T}{\partial \eta} = 0 \qquad \ldots(2.9)$$

where u is the particle velocity behind the shock. Following Laumboch and Probstein (1970) we seek the solution in the similarity form as

$$x = X\xi(\mu) \qquad \ldots(2.10)$$

$$u = (1-\beta)\,\dot{X}\nu(\mu) \qquad \ldots(2.11)$$

$$\rho = \frac{\rho_0(X)\, g(\mu)}{\beta} \qquad \ldots(2.12)$$

$$p = (1-\beta)\,\rho_0(X)\,\dot{X}^2\,\pi(\mu) \qquad \ldots(2.13)$$

At the shock, where $\mu = 1$, the boundary conditions (2.6) for reduced functions $\nu(\mu)$, $g(\mu)$ and $\pi(\mu)$ become

$$\nu(1) = g(1) = \pi(1) = 1. \qquad \ldots(2.14)$$

From eqn. (2.9) and the gas law we obtain

$$\frac{d}{d\mu}\left(\frac{\pi}{g}\right) = 0. \qquad \ldots(2.15)$$

Integrating (2.15) with boundary condition (2.14), we get

$$g(\mu) = \pi(\mu) \qquad \ldots(2.16)$$

We substitute expressions (2.10) to (2.13) in eqns. (2.3), (2.7) and (2.8) and obtain

$$\frac{d\xi}{d\mu} = \frac{\beta}{1+\delta}\,\frac{1}{g} \qquad \text{...(2.17)}$$

$$\beta\delta + (1-\beta)\,(1+\delta)\,g\,\frac{dv}{d\mu} = \beta(1+\delta)\,\mu\,\frac{d\ln g}{d\mu} \qquad \text{...(2.18)}$$

$$\alpha^{-1}(1-\alpha)\,v + (1+\delta)\,\mu\,\frac{dv}{d\mu} = (1+\delta)\,g\,\frac{d\ln g}{d\mu}. \qquad \text{...(2.19)}$$

In the above we have used eqn. (2.16).

Following Chernyii (1961) we expand the flow variables in series of powers of β, the density ratio across the strong shock

$$\xi = \xi^{(0)} + \beta\xi^{(1)} + \beta^2\xi^{(2)} + \ldots \qquad \text{...(2.20)}$$

$$v = v^{(0)} + \beta v^{(1)} + \beta^2 v^{(2)} + \ldots \qquad \text{...(2.21)}$$

$$g = g^{(0)} + \beta g^{(1)} + \beta^2 g^{(2)} + \ldots \qquad \text{...(2.22)}$$

The boundary conditions (2.14) may be expressed as

$$\left.\begin{aligned} \xi^{(0)} &= 1,\ \xi^{(1)} = \xi^{(2)} = \quad = 0 \\ v^{(0)} &= 1,\ v^{(1)} = v^{(2)} = \quad = 0 \\ g^{(0)} &= 1,\ g^{(1)} = g^{(2)} = \quad = 0. \end{aligned}\right\} \qquad \text{...(2.23)}$$

We substitute expansions (2.20)–(2.22) into eqns. (2.17), (2.18) and (2.19) and obtain the following system of differential equations up to second order terms in β

$$\frac{d\xi^{(0)}}{d\mu} = 0 \qquad \text{...(2.24)}$$

$$\frac{d\xi^{(1)}}{d\mu} = \frac{1}{(1+\delta)\,g^{(0)}} \qquad \text{...(2.25)}$$

$$\frac{d\xi^{(2)}}{d\mu} = -\frac{g^{(1)}}{(1+\delta)\,g^{2(0)}} \qquad \text{...(2.26)}$$

$$\frac{dv^{(0)}}{d\mu} = 0 \qquad \text{...(2.27)}$$

$$\frac{dv^{(1)}}{d\mu} = \frac{1}{g^{2(0)}}\left[\mu\,\frac{dg^{(0)}}{d\mu} - Bg^{(0)}\right] \qquad \text{...(2.28)}$$

$$\frac{dv^{(2)}}{d\mu} = \frac{1}{g^{2(0)}}\left[\mu\,\frac{dg^{(1)}}{d\mu} - (2g^{(0)}g^{(1)} - g^{2(0)})\,\frac{dv^{(1)}}{d\mu} - Bg^{(1)}\right] \qquad \text{...(2.29)}$$

$$\frac{dg^{(0)}}{d\mu} = Cv^{(0)} \qquad \text{...(2.30)}$$

$$\frac{dg^{(1)}}{d\mu} = C\nu^{(1)} + \mu \frac{d\nu^{(1)}}{d\mu} \qquad ...(2.31)$$

$$\frac{dg^{(2)}}{d\mu} = C\nu^{(2)} + \mu \frac{d\nu^{(2)}}{d\mu} \qquad ...(2.32)$$

where $B = \frac{\delta}{1+\delta}$, $\lambda = \alpha^{-1}(1-\alpha)$ and $C = \frac{\lambda}{1+\delta}$

Integrating eqns. (2.24)–(2.32) with boundary condition (2.23) we finally obtain the solution as

$$\xi^{(0)} = 1 \qquad ...(2.33)$$

$$\xi^{(1)} = \frac{1}{(1+\delta)\,C}\,[\ln z - \ln K] \qquad ...(2.34)$$

$$\xi^{(2)} = \ln z\,[\bar{D} + \bar{A}\ln z] + \frac{1}{z}[\bar{C} + \bar{B}\ln z] + \frac{\bar{E}}{z^2} + \bar{F} \qquad ...(2.35)$$

$$\nu^{(0)} = 1 \qquad ...(2.36)$$

$$\nu^{(1)} = A_1 \ln z + \frac{B_1}{z} + C_1 \qquad ...(2.37)$$

$$\nu^{(2)} = \ln z\,[A_2 \ln z + C_2] + \frac{\ln z}{z^2}[B_2 z + C_2] + \frac{1}{z^3}[D_2 z^2 + E_2 z + F_2] + H_2 \qquad ...(2.38)$$

$$g^{(0)} = Cz \qquad ...(2.39)$$

$$g^{(1)} = \ln z\,[D_1 z + E_1] + F_1 z - \frac{G_1}{z} + H_1 \qquad ...(2.40)$$

$$g^{(2)} = \ln z\,[(A_3 z + C_3)\ln z + (B_3 z + D_3)] + \frac{\ln z}{z^2} \times [B_2 z^2 + (G_2 - E_3)\,z - G_2 K] + \frac{1}{z^3}[D_2 z^3 + (E_2 - G_3)\,z^2 + (F_2 - H_3)\,z - F_2 K] + F_3 z + I_3 \qquad ...(2.41)$$

where $z = \mu + K$, $k = \frac{1-C}{C}$.

We have omitted explicit expressions for the constants A_1, A_2, ... for the sake of brevity.

From eqns. (2.18) and (2.19) we obtain

$$\frac{d\nu}{d\mu} = \frac{\beta[\alpha^{-1}(1-\alpha)\,\mu\nu - g\delta]}{(1+\delta)\,[(1-\beta)\,g^2 - \beta\mu^2]}. \qquad ...(2.42)$$

The equation (2.42) possesses several singular points. The solution is single valued and physically meaningful when $\frac{dv}{d\mu}$ is finite in the flow field and for this the numerator and the denominator in eqn. (2.42) must vanish simultaneously at the singular point. The numerator and the denominator vanish along the path

$$v = 1, g = \mu \sqrt{\beta/(1+\beta)} \qquad ...(2.43)$$

This gives an analytic expression for α, the similarity exponent, as

$$\alpha = \left[1 + \delta\sqrt{\frac{\beta}{1-\beta}}\right]^{-1}. \qquad ...(2.44)$$

3. Results and Discussions

We have found approximate analytic solution, given by eqns. (2.33) to (2.41), of a strong shock approaching the surface of a star when the flow behind the shock is homothermal. The solution has been obtained in a closed form up to second order terms in β, the density ratio across the strong shock. To the zeroth order approximation, the particle velocity behind the shock is everywhere the same as the shock velocity while the density and pressure are linear functions of μ. The first order terms contribute much more to velocity than to the density. C and K are positive and thus the reduced density and pressure increase behind the shock as μ increases. The Eulerian distance x, to the zero order approximation, is the same as shock distance and hence the contribution from the first order term is significant. Equations (2.33) to (2.35) express the Eulerian similarity variable ξ in terms of the Lagrangian similarity variable μ. The first and second order terms in the solution are too complicated to lend themselves to visual estimation.

Table I gives values of the similarity exponent α for different values of $\delta = 3.25$, 2, 1, 0.5, the density exponent and $\gamma = 5/3$, $7/5$ and $6/5$, the ratio of specific heats as calculated from expression (2.44). The values of α for $\gamma = 5/3$ and $\delta = 3.25$ is particularly important in the context of stellar envelopes in radiative equilibrium. Table II gives the value of α obtained by numerical integration (Sachdev and Ashraf 1971). The difference between the values of α, obtained from eqn. (2.44) and the

TABLE I

Values of α obtained from analytic expression

γ/δ	3.25	2	1	1/2
5/3	0.3476	0.4641	0.6339	0.7760
7/5	0.4075	0.5278	0.6909	0.8172
6/5	0.4931	0.6125	0.7597	0.8634

numerical integration, reduces considerably for all γ when δ decreases. Table III gives the values of α for the adiabatic flow, studied by Sakurai (1960), Zel'dovich and Raizer (1967), Nadezhin and Frank-Kamanetskii (1964). We find from Tables I and III that the value of α for the isothermal case, for all γ and δ that we have considered, is less than for the adiabatic case, showing that the shock velocity $\dot{X} \propto \bar{X}^{(1-\alpha)/\alpha}$ in the isothermal case becomes much greater near the surface of the star compared to that in the adiabatic case. The reduced functions g and π are not bounded at $\mu = \infty$ which corresponds to either $t = 0$, $x \neq 0$ or the point far behind the shock for $t < 0$. Thus in the isothermal flow it is not possible to consider the flow behind the shock when it reaches the surface of the star or the flow thereafter. In fact the self-similar flow is valid in a small region near the surface (Zel'dovich and Raizer 1967) and our solution gives fairly good results in this limited region.

Figure 1 depicts the distribution of reduced functions v, g and π showing the velocity, density and pressure behind the shock respectively at any time for $\gamma = 6/5$. The corresponding exact values are shown by dotted curves. Our analytic solution is in good agreement with the exact solution. Sachdev and Ashraf (1971) have compared the isothermal flow with the adiabatic flow. Figures 2 and 3 show the distribution of v, g and π for $\gamma = 7/5$ and $\gamma = 5/3$ respectively for the isothermal flows only.

TABLE II

Values of α obtained from numerical integration

γ/δ	3.25	2	1	1/2
5/3	0.3136	0.4346	0.6146	0.7670
7/5	0.3690	0.4949	0.6720	0.809
6/5	0.4568	0.5835	0.7442	0.8572

TABLE III

Values of α for adiabatic flow

γ/δ	3.25	2	1	1/2
5/3	0.590	0.6966	0.8174	0.8976
7/5		0.7177	0.8318	0.9062
6/5		0.7514	0.8544	0.9195

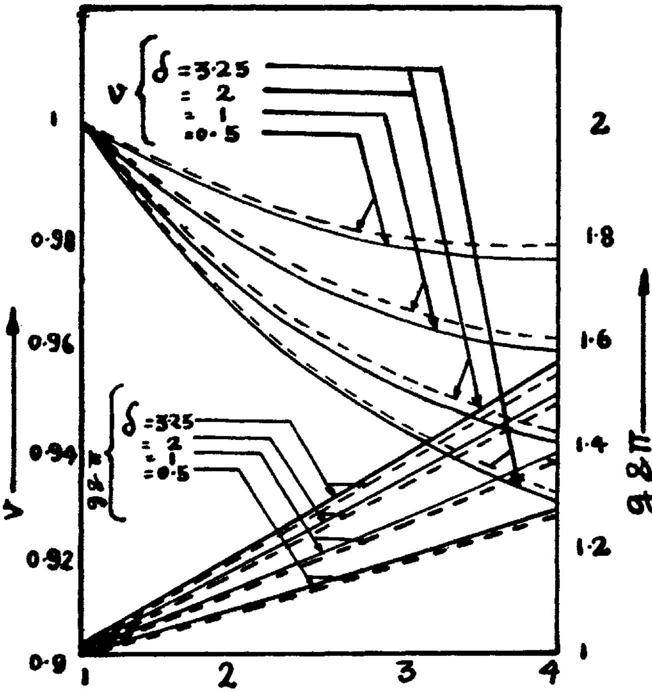

FIG. 1. Flow variable distributions behind the shock wave for $\gamma = 6/5$.

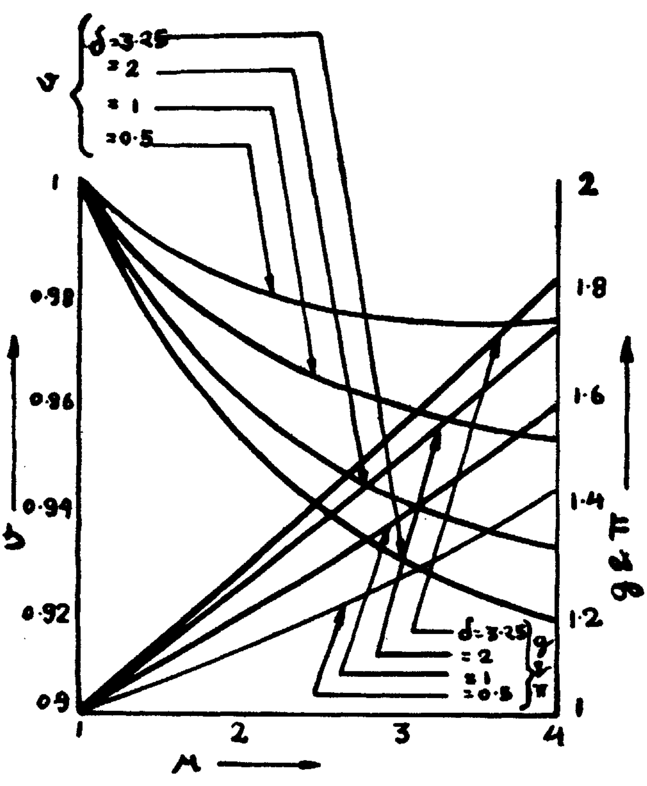

FIG. 2. Flow variable distributions behind the shock wave for $\gamma = 7/5$.

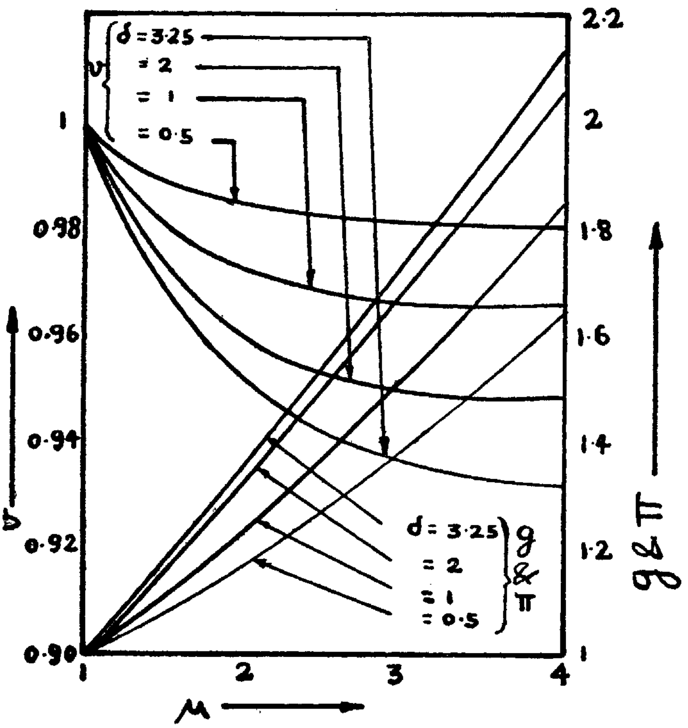

FIG. 3. Flow variable distributions behind the shock wave for $\gamma = 5/3$.

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