ON SOME INEQUALITIES FOR FOURIER SERIES

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In this note we obtain certain generalizations of some of the theorems of Izumi and Izumi (1968). In our theorems we replace the special class L^p of functions by a more general class L_Φ where Φ satisfies certain conditions.

§1. A non-decreasing continuous and real-valued function Φ defined on the non-negative half-line and vanishing only at the origin is called an Orlicz function (*OF*). Function $\Phi \in OF$ is said to satisfy Δ_α $(\alpha > 0)$ condition for large u if there are constants $C > 0$ and $u_0 \geqslant 0$ such that $\Phi(\alpha u), \leqslant C\,\Phi(u)$, $u \geqslant u_0$, for every $\alpha > 1$. A convex Orlicz function Φ satisfying the conditions

$$\lim_{u\to 0} \frac{\Phi(u)}{u} = 0 \quad \text{and} \quad \lim_{u\to\infty} \frac{\Phi(u)}{u} = \infty$$

is called a Young function (*YF*). Function Φ belongs to *YF* iff it admits a representation

$$\Phi(u) = \int_0^u \varphi(x)\,dx$$

where $\varphi(x)$ $(x \geqslant 0)$ is positive, $\varphi(0) = 0$, continuous on the right, non-decreasing and $\lim_{x\to\infty} \varphi(x) = \infty$.

We have for such functions the relation

$$\frac{\Phi(u)}{u} \leqslant \varphi(u) \leqslant \frac{\Phi(2u)}{u}.$$

We denote by M the class of Orlicz functions Φ which satisfy the following condition of Mulholland (1932):

"There exist a convex function Λ, $\lambda > 1$ and $0 < k < 1$, such that the inequality $\Lambda(u) \leqslant \Phi^k(u) \leqslant \lambda\Lambda(u)$ holds for all u."

We write $G(x) = \int_{x/2}^{x} \frac{f(t)}{t}\, dt.$

§2. Izumi and Izumi (1968) proved among others the following theorems :

Theorem A — Let $p > 1$ and $s > -1$ and let f be a non-negative, non-increasing and integrable function on $(0, \pi)$. If $x^s f^p(x)$ is integrable, then we have

$$\int_0^{\pi} x\, G^p(x)\, dx \leqslant A^* \int_0^{\pi} x^s (f(x/2) - f(x))^p\, dx + A\left(\int_{\pi/2}^{\pi} f(x)\, dx\right)^p.$$

Theorem B — Let $p > 1$ and $s < -1$ and let f be non-negative and integrable on $(0, \pi)$. If $x^s f^p(x)$ is integrable, then

$$\int_0^{\pi} x^s\, G^p(x)\, dx \leqslant \left(\frac{p}{-s-1}\right)^p \int_0^{\pi} x^s \mid f(x/2) - f(x) \mid^p dx.$$

Theorem C — Let $p > 1$ and $s \neq -1$ and let f be non-negative, non-constant and integrable on $(0, \infty)$. If $x^s f^p(x) \in L(0, \infty)$, then

$$\int_0^{\infty} x^s G^p(x)\, dx \leqslant \left|\frac{p}{-s-1}\right|^p \int_0^{\infty} x^s \mid f(x/2) - f(x) \mid^p dx$$

Theorem D — Let $p > 1$ and $\{a_n\}$ be a monotonic non-increasing sequence tending to zero and

$$\frac{1}{n} \sum_{m=0}^{n-1} a_m \leqslant 2 \sum_{m=n}^{2n}{}^{*} a_m \text{ for all } n \geqslant 1,$$

where Σ^* denotes the sum whose first and last terms are halved. Suppose that

$$f(x) \sim \frac{a_0}{2} + \sum_{1}^{\infty} a_n \cos nx = \sum_{n=0}^{\infty}{}^{*} a_n \cos nx \text{ and that } f(x) \text{ is non-negative.}$$

(I) If f is L^p integrable and decreases monotonically, then

$$\sum_{n=1}^{\infty} n^{p-2}\, a_n^p \leqslant A \int_0^{\pi} (f(x/2) - f(x))^p\, dx + A\left(\int_{\pi/2}^{\pi} f(x)\, dx\right)^p.$$

*A denotes a positive constant, not necessarily the same at each occurrence.

(II) More generally, if $-1 < s < p - 1$ and $x^s f^p(x)$ is integrable on $(0, \pi)$, then

$$\sum_{n=1}^{\infty} n^{p-s-2} a_n^p \leqslant A \int_0^{\pi} x^s \mid f(x/2) - f(x) \mid^p dx + A \frac{\left(\int_{\pi/2}^{\pi} f(x)\, dx \right)^p}{\left(\int_0^{\pi/2} f(x)\, dx \right)^{p^2-p}}.$$

(III) If $s < -1$ and $x^s f^p(x)$ is integrable on $(0, \pi)$, then

$$\sum_{n=1}^{\infty} n^{p-s-2} a_n^p \leqslant A \int_0^{\pi} x^s \mid f(x/2) - f(x) \mid^p dx.$$

The object of this note is to obtain certain generalizations of the above theorems. In our theorems we replace the special class L^p of functions by a more general class L_Φ, where Φ satisfies certain conditions. In Theorem 1 we have shown that condition of monotonocity in Theorem A is redundant. A similar remark is also applicable to Theorem D (Part I).

§3. We prove the following theorems :

Theorem 1 — Let $\Phi \in \Delta_\alpha \cap YF$ and f be non-negative and integrable on $(0, \pi)$. If $x^s \Phi(f(x))$ is integrable and $s > -1$, then we have

$$\int_0^{\pi} x^s \Phi(G(x))\, dx \leqslant A \int_0^{\pi} x^s \Phi(\mid f(x) - f(x/2) \mid)\, dx + A \Phi\left(\int_{\pi/2}^{\pi} f(x)\, dx \right).$$

Theorem 2 — Let $\Phi \in \Delta_\alpha \cap YF$ and f be non-negative. If $x^s \Phi(f(x))$ is integrable and $s < -1$, then

$$\int_0^{\pi} x^s \Phi(G(x))\, dx \leqslant A \int_0^{\pi} x^s \Phi(\mid f(x) - f(x/2) \mid)\, dx.$$

Theorem 3 — Let $\Phi \in \Delta_\alpha \cap YF$, $s \neq -1$ and f be non-negative, non-constant and integrable on $(0, \infty)$. If $x^s \Phi(f(x)) \in L(0, \infty)$, then

$$\int_0^{\infty} x^s \Phi(G(x))\, dx \leqslant A \int_0^{\infty} x^s \Phi(\mid f(x) - f(x/2) \mid)\, dx.$$

Theorem 4 — Let $\Phi \in \Delta_\alpha \cap YF \cap M$. Let $\{a_n\}$ satisfy the conditions of Theorem D. Suppose that $f(x) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx = \sum_{n=0}^{\infty}{}^{*} a_n \cos nx$ and that $f(x)$ is non-negative.

(i) If $x^s \Phi(f(x))$ is integrable and $-1 < s < \beta - 1$, $\beta > 1$, then

$$\sum_{n=1}^{\infty} n^{-s-2} \Phi(n a_n) \leqslant A \int_0^{\pi} x^s \Phi(\,|\, f(x) - f(x/2) \,|\,)\, dx + A\, \Phi\left(\int_{\pi/2}^{\pi} f(x)\, dx \right).$$

(ii) If $s < -1$ and $x^s \Phi(f(x))$ is integrable on $(0, \pi)$, then

$$\sum_{n=1}^{\infty} n^{-s-2} \Phi(n a_n) \leqslant A \int_0^{\pi} x^s \Phi(\,|\, f(x) - f(x/2) \,|\,)\, dx.$$

§4. We require the following lemmas for the proof of our theorems :

Lemma 1 (Askey and Wainger 1966) — Let $\{a_n\}$ be a monotonically decreasing sequence tending to zero and

$$\frac{1}{n} \sum_{m=0}^{n-1} a_m \leqslant 2 \sum_{m=n}^{2n}{}^{*} a_m, \quad \text{for all } n \geqslant 1$$

where Σ^* denotes the sum whose first and last terms are halved. If

$$f(x) \sim \sum_{n=0}^{\infty}{}^{*} a_n \cos nx$$

is non-negative and integrable, then the nth Fourier coefficient r_n of the even function Q defined by

$$Q(x) = \int_{\pi/2}^{x} \frac{f(t)}{2 \tan t/2}\, dt, \; 0 < x < \pi$$

is non-negative and further

$$\exists \theta : 0 < \theta < 1, r_n \geqslant \frac{\theta}{n} \sum_{m=n}^{2n}{}^{*} a_n \text{ for all } n \geqslant 1.$$

Lemma 2 (Zygmund 1968) — Let $f(x)$ be a non-negative function defined for $x \geqslant 0$ and let $r > 1$, $s < r - 1$. Then if $x^s f^r(x) \in L(0, \infty)$ so is

$x^s \left(\frac{F(x)}{x}\right)^r$, where $F(x) = \int_0^x f(t)\,dt$. Moreover,

$$\int_0^\infty x^s \left(\frac{F(x)}{x}\right)^r dx \leqslant \left(\frac{r}{r-s-1}\right)^r \int_0^\infty x^s f^r(x)\,dx.$$

Lemma 3* — If $\Phi \in M$, $s < \alpha - 1$, $\alpha > 1$ and $x^s\,\Phi(f(x)) \in L(0, \infty)$, then

$$\int_0^\infty x^s\,\Phi\left(\frac{F(x)}{x}\right) dx \leqslant A \int_0^\infty x^s\,\Phi(f(x))\,dx$$

where $F(x) = \int_0^x f(t)\,dt$ and $f(t) \geqslant 0$ for $t \geqslant 0$.

Proof of Lemma 3 : By virtue of the assumption and Jensen's inequality, there exist an $\alpha > 1$ and a convex function $\Lambda(x)$ such that

$$\Phi^{1/\alpha}\left(\frac{F(x)}{x}\right) \leqslant \lambda \Lambda\left(\frac{F(x)}{x}\right) \leqslant \lambda \frac{1}{x}\int_0^x \Lambda(f(t))\,dt \leqslant \lambda \frac{1}{x}\int_0^x \Phi^{1/\alpha}(f(t))\,dt.$$

Now we have

$$\int_0^\infty x^s \Phi\left(\frac{F(x)}{x}\right) dx \leqslant \lambda^\alpha \int_0^\infty x^s \left(\frac{1}{x}\int_0^x \Phi^{1/\alpha} f(t)\,dt\right)^\alpha dx$$

$$\leqslant A \int_0^\infty x^s\,\Phi(f(x))\,dx, \quad \text{by Lemma 2.}$$

This proves Lemma 3.

§5. *Proof of Theorem* 1 — Integrating by parts, we have

$$\int_0^\pi x^s\,\Phi(G(x))\,dx = \left[\frac{x^{s+1}\,\Phi(G(x))}{s+1}\right]_0^\pi - \frac{1}{s+1}\int_0^\pi x^{s+1}\,\varphi(G(x)) \times \left(\frac{f(x)}{x} - \frac{\frac{1}{2}f(x/2)}{x/2}\right) dx$$

$$\leqslant \frac{\pi^{s+1}\,\Phi(G(\pi))}{s+1} + \frac{1}{s+1}\int_0^\pi x^s\,\varphi(G(x))\,|\,f(x) - f(x/2)\,|\,dx.$$

*Woyczyński (1970) proved this Lemma for $s \leqslant 0$ only. Following the same technique here we have extended the result to the case $s < \alpha - 1$, $\alpha > 1$.

Taking T to be arbitrarily large positive number, we have since $\varphi(x)$ is non-decreasing

$$\begin{aligned}\varphi(G(x)) \mid f(x) - f(x/2) \mid &= T^{-1} \{T \varphi(G(x)) \mid f(x) - f(x/2) \mid\} \\ &= T^{-1} \operatorname{Max} \{G(x) \varphi(G(x)), T \mid f(x) - f(x/2) \mid \varphi(T \mid f(x) - f(x/2) \mid)\} \\ &\leqslant T^{-1} \Phi(2G(x)) + T^{-1} \Phi(2T \mid f(x) - f(x/2) \mid) \\ &\leqslant AT^{-1} \Phi(G(x)) + AT^{-1} \Phi(\mid f(x) - f(x/2) \mid).\end{aligned}$$

Thus we have

$$\int_0^\pi x^s \Phi(G(x))\, dx \leqslant A\, \Phi(G(\pi)) + \frac{1}{s+1} \int_0^\pi x^s \varphi(G(x)) \times \mid f(x) - f(x/2) \mid dx$$

$$\leqslant A \Phi\left(\int_{\pi/2}^\pi \frac{f(t)}{t}\, dt \right) + \frac{A}{T(s+1)} \int_0^\pi x^s \Phi(G(x))\, dx + \frac{A}{T(s+1)} \int_0^\pi x^s \Phi(\mid f(x) - f(x/2) \mid)\, dx$$

$$\text{or} \quad \left(1 - \frac{A}{T(s+1)}\right) \int_0^\pi x^s \Phi(G(x))\, dx \leqslant A \Phi\left(\frac{2}{\pi} \int_{\pi/2}^\pi f(t)\, dt \right) + \frac{A}{T(s+1)} \int_0^\pi x^s \Phi(\mid f(x) - f(x/2) \mid)\, dx$$

$$\text{or} \quad \int_0^\pi x^s \Phi(G(X))\, dx \leqslant A \Phi\left(\int_{\pi/2}^\pi f(t)\, dt \right) + A \int_0^\pi x^s \Phi(\mid f(x) - f(x/2) \mid)\, dx$$

by choosing $T > \dfrac{A}{s+1}$.

This completes the proof of Theorem 1.

§6. *Proof of Theorem* 2 — First of all we have by Jensen's inequality

$$\Phi(G(x)) = \Phi\left(\int_{x/2}^x \frac{f(t)}{t} dt \right) \leqslant \Phi\left(\frac{2}{x} \int_{x/2}^x f(t)\, dt \right)$$

$$\leqslant \frac{2}{x} \int_{x/2}^x \Phi(f(t))\, dt \leqslant Ax^{-s-1} \int_{x/2}^x t^s\, \Phi(f(t))\, dt = o(1),\ x \to 0. \qquad \ldots(6.1)$$

Now integrating by parts, we have

$$\int_0^\pi x^s\,\Phi(G(x))\,dx = \left[\frac{x^{s+1}\,\Phi(G(x))}{x+1}\right]_0^\pi - \frac{1}{s+1}\int_0^\pi x^{s+1}\,\varphi(G(x))\left(\frac{f(x)}{x} - \frac{\frac{1}{2}f(x/2)}{x/2}\right)dx$$

$$= \frac{\pi^{s+1}\,\Phi(G(\pi))}{s+1} - \frac{1}{s+1}\int_0^\pi x^s\,\varphi(G(x))\,(f(x)-f(x/2))\,dx.$$

First term on the right-hand side is negative since $s < -1$. Hence we have

$$\int_0^\pi x^s\,\Phi(G(x))\,dx \leqslant -\frac{1}{s+1}\int_0^\pi x^s\,\varphi(G(x))\mid f(x)-f(x/2)\mid dx.$$

Taking T to be arbitrarily large positive integer, we have since $\varphi(x)$ is non-decreasing

$$\begin{aligned}\varphi(G(x))\mid f(x)-f(x/2)\mid &= T^{-1}\{T\,\varphi(G(x))\mid f(x)-f(x/2)\mid\}\\ &\leqslant T^{-1}\,\text{Max}\{G(x)\,\varphi(G(x)),\, T\mid f(x)-f(x/2)\mid \varphi(T\mid f(x)-f(x/2)\mid)\}\\ &\leqslant T^{-1}\,\Phi(2G(x) + T^{-1}\,\Phi(2T\mid f(x)-f(x/2)\mid)\\ &\leqslant AT^{-1}\,\Phi(G(x)) + AT^{-1}\,\Phi(\mid f(x)-f(x/2)\mid).\end{aligned}$$

Thus we have

$$\int_0^\pi x^s\,\Phi(G(x))\,dx \leqslant -\frac{1}{s+1}\int_0^\pi x^s\,\varphi(G(x))\mid f(x)-f(x/2)\mid dx$$

$$\leqslant \frac{-A}{T(s+1)}\int_0^\pi x^s\,\Phi(G(x))\,dx - \frac{A}{T(s+1)}\int_0^\pi \Phi(\mid f(x)-f(x/2)\mid)\,dx$$

or, $$\left(1+\frac{A}{(s+1)T}\right)\int_0^\pi x^s\,\Phi(G(x))\,dx \leqslant -\frac{A}{(s+1)T}\int_0^\pi x^s\Phi(\mid f(x)-f(x/2)\mid)\,dx$$

or $$\int_0^\pi x^s\,\Phi(G(x))\,dx \leqslant A\int_0^\pi x^s\,\Phi(\mid f(x)-f(x/2)\mid)\,dx$$

by choosing $T > -\dfrac{A}{s+1}$.

This completes the proof of Theorem 2.

§7. *Proof of Theorem* 3 — It is easy to verify that the relation (6.1) holds also when $x \to \infty$ and then following the lines of proof of Theorem 2, we have

$$\int_0^\infty x^s\, \Phi(G(x))\, dx = -\frac{1}{s+1} \int_0^\infty x^s\, \varphi(G(x))\, (f(x) - f(x/2))\, dx$$

$$\leqslant \frac{1}{|\, s+1\,|} \int_0^\infty x^s\, \varphi(G(x))\, |\, f(x) - f(x/2)\, |\, dx$$

$$\leqslant \frac{A}{T\,|\, s+1\,|} \int_0^\infty x^s\, \Phi(G(x))\, dx + \frac{A}{T\,|\, s+1\,|}$$

$$\times \int_0^\infty x^s \Phi(\, |\, f(x) - f(x/2)\, |\,)\, dx.$$

or $$\int_0^\infty x^s\, \Phi(G(x))\, dx \leqslant A \int_0^\infty x^s\, \Phi(\, |\, f(x) - f(x/2)\, |\,)\, dx.$$

This proves Theorem 3.

§8. *Proof of Theorem* 4 — Let r_n be the Fourier coefficients of $Q(x)$ of Lemma 1. Then by virtue of this lemma, $r_n \geqslant 0$ for all $n \geqslant 1$. We put

$$Q_1(x) = \int_0^x Q(t)\, dt, \quad Q_2(x) = \int_0^x Q_1(t)\, dt \quad \text{for } x \geqslant 0.$$

Then $$Q_2(x) = \frac{r_0 x^2}{4} + \sum_{j=1}^{\infty} r_j j^{-2} (1 - \cos jx) \geqslant 2 \sum_{j=1}^{[n/2]} r_j j^{-2} \sin^2 \frac{jx}{2}$$

$$\geqslant 2 \sum_{j=1}^{[n/2]} r_j j^{-2}\, \frac{4}{\pi^2}\, \frac{j^2 x^2}{4}, \quad \frac{\pi}{n+1} \leqslant x \leqslant \pi/n,$$

$$= \frac{2}{\pi^2}\, x^2 \sum_{j=1}^{[n/2]} r_j \geqslant \frac{2}{\pi^2}\, \theta\, x^2 \sum_{j=1}^{[n/2]} \frac{1}{j} \sum_{k=j}^{2j}{}^{*} a_k, \quad 0 < \theta < 1,$$

$$\geqslant Ax^2 \sum_{j=1}^{[n/2]} a_{2j} \geqslant Ax^2 n a_n,$$

or, we have

$$na_n \leqslant \frac{1}{A}\,\frac{Q_2(x)}{x^2},\ \frac{\pi}{n+1} \leqslant x \leqslant \frac{\pi}{n},\ n \geqslant 1.$$

Now we have

$$\sum_{n=1}^{\infty} n^{-s-2}\,\Phi(na_n) \leqslant A \sum_{n=1}^{\infty} \int_{\pi/(n+1)}^{\pi/n} x^s\,\Phi\left(\frac{Q_2(x)}{Ax^2}\right)dx$$

$$\leqslant A \int_0^{\pi} x^s\,\Phi\left(\frac{Q_2(x)}{x^2}\right)dx$$

$$= A \int_0^{\pi} x^s\,\Phi\left(\frac{1}{x^2}\int_0^x Q_1(t)\,dt\right)dx \leqslant A \int_0^{\pi} x^s\,\Phi\left(\frac{Q_1(x)}{x}\right)dx$$

$$\leqslant A \int_0^{\pi} x^s\,\Phi(Q(x))\,dx \quad \text{(by Lemma 3).}$$

Now by Theorem 1 we have

$$\sum_{n=1}^{\infty} n^{-s-2}\,\Phi(na_n) \leqslant A \int_0^{\pi} x^s\,\Phi(G(x))\,dx$$

$$\leqslant A \int_0^{\pi} x^s\,\Phi(\,|\,f(x) - f(x/2)\,|\,)\,dx + A\Phi\left(\int_{\pi/2}^{\pi} f(x)\,dx\right).$$

This proves part (i) of Theorem 4.

Now if we suppose that $s < -1$ and use Theorem 2, then we get

$$\sum_{n=1}^{\infty} n^{-s-2}\,\Phi(na_n) \leqslant A \int_0^{\pi} x^s\,\Phi(G(x))\,dx$$

$$\leqslant A \int_0^{\pi} x^s\,\Phi(\,|\,f(x) - f(x/2)\,|\,)\,dx$$

This proves part (ii) of Theorem 4.

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